Math 135A: Homework Assignment #6
The Method of Generating Functions

Let $X$ be a random variable assuming nonnegative integer values. For $0 < s < 1$ define

$$G_X(s) = E(s^X) = \sum_{n=0}^{\infty} s^n P(X = n). \quad (1)$$

$G_X$ is called the generating function of the random variable $X$. Here $E(\cdot)$ denotes expectation. Similarly, define

$$E_X(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} P(X = n), \quad (2)$$

the exponential generating function associated to the random variable $X$.

1. In each of the following cases, compute $G_X$.

   (a) If $X$ is a Bernoulli random variable with $P(X = 1) = p$ and $P(X = 0) = q = 1 - p$

   (b) If $X$ is geometrically distributed with parameter $q$, i.e. $P(X = n) = q^n (1 - q)$, $n = 0, 1, 2, \ldots$

   (c) If $X$ is Poisson distributed with parameter $\lambda$, i.e.

       $$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}.$$

2. If $X$ has generating function $G_X$, prove that

   $$E(X) = G'(1)$$

   where prime denotes differentiation with respect to $s$; and, of course, $E(\cdot)$ denotes expectation. Find a formula for $\text{var}(X)$ in terms of $G_X$ and its derivatives.

3. (a) If $X$ and $Y$ are independent random variables, prove

   $$G_{X+Y}(s) = G_X(s) G_Y(s)$$

   (b) Using the above property, find the generating function of

   $$S_n := X_1 + \cdots + X_n$$

   where $X_j$ are independent, identically distributed Bernoulli random variables.
4. (a) Prove: If $X_1, X_2, \ldots$ is a sequence of independent and identically distributed random variables with common generating function $G_X$, and $N(\geq 0$ is a random variable which is independent of the $X_i$ and has generating function $G_N$, then

$$S := X_1 + X_2 + \cdots + X_N$$

has generating given by

$$G_S(s) = G_N(G_X(s))$$

(b) A hen lays $N$ eggs, where $N$ is Poisson distributed with parameter $\lambda$. Each egg hatches with probability $p$, independently of all other eggs. Let $K$ be the number of chicks. Then $K = X_1 + \cdots + X_N$. Prove

$$G_K(s) = e^{\lambda p(s-1)}$$

and hence; $E(K) = \lambda p$. 

2