Math 135A: HW Assignment #7

Let $X$ be a random variable assuming nonnegative integer values. Recall that for $0 < s < 1$, the generating function of $X$ is

$$G_X(s) := \mathbb{E}(s^X) = \sum_{n=0}^{\infty} s^n \mathbb{P}(X = n).$$

(1)

Also recall that if $X$ and $Y$ are independent random variables, each assuming nonnegative integer values, that

$$G_{X+Y}(s) = G_X(s)G_Y(s).$$

1. Let $N$ be Poisson distributed with parameter $\lambda$. Show that, for any function $g : \mathbb{R} \to \mathbb{R}$ such that the expectations exist,

$$\mathbb{E}(Ng(N)) = \lambda \mathbb{E}(g(N + 1)).$$

2. Let $\{X_j\}$ be a sequence of mutually independent random variables with common distribution

$$\mathbb{P}(X_k = j) = p_j$$

Let $G_X(s)$ denote the common generating function of $X_j$. Consider the sum

$$S_N = X_1 + X_2 + \cdots + X_N$$

where the number $N$ of terms is a random variable independent of the $X_j$. Let $\mathbb{P}(N = n) = g_n$ be the distribution of $N$.

(a) Show that the generating function of $S_N$ is given by

$$G_{S_N}(s) = G_N(G_X(s))$$

where $G_N$ is the generating function of $N$.

(b) Using the preceding result to show that

$$\mathbb{E}(S_N) = \mathbb{E}(N)\mathbb{E}(X)$$
$$\text{var}(S_N) = \mathbb{E}(N)\text{var}(X) + \text{var}(N)\left(\mathbb{E}(X)\right)^2$$

(c) In case that $N$ has Poisson distribution with parameter $\lambda$, show that the preceding result implies

$$G_{S_N}(s) = e^{-\lambda + \lambda G_X(s)}$$
3. Recall that in the *coupon problem* we expressed the waiting time $T_n$ to collect all $n$ coupons as a sum of $n$ independent random variables $X_k$ where $X_k$ has geometric distribution with $p_k = (n - k)/n$. That is, 

$$T_n = X_0 + X_1 + \cdots + X_{n-1}$$

with

$$
\mathbb{P}(X_k = j) = q_k^{j-1} p_k, \quad p_k = \frac{n - k}{n}, \quad q_k = 1 - p_k, \quad j = 1, 2, 3, \ldots
$$

(a) If $G_{T_n}$ denotes the generating function of $T_n$, show that

$$G_{T_n}(s) = \frac{(n - 1)! s^n}{(n - s)(n - 2s)(n - 3s) \cdots (n - (n - 1)s)}$$

(b) For $n = 4$ (four coupons), show that for $j \geq 4$

$$
\mathbb{P}(T_4 = j) = \frac{27}{64} \left(\frac{3}{4}\right)^{j-4} - \frac{3}{8} \left(\frac{1}{2}\right)^{j-4} + \frac{3}{64} \left(\frac{1}{4}\right)^{j-4}
$$

Hint: First partial fraction

$$
\frac{1}{(4 - s)(4 - 2s)(4 - 3s)} = \frac{9/128}{1 - 3s/4} - \frac{1/16}{1 - s/2} + \frac{1/128}{1 - s/4}
$$

(c) Using Mathematica or some other software package, produce a plot of the probabilities $\mathbb{P}(T_{25} = j)$ for a reasonable range of $j$. 