Math 135A: HW Assignment #9
Continuous Random Variables

1. Let $X$ and $Y$ be independent random variables uniformly distributed on $[0, 1]$. Define $Z = \max(X, Y)$. Find the distribution $F_Z$, the density $f_Z$ and $E(Z)$. Now do the general case: Let $X_1, X_2, \ldots, X_n$ be independent random variables uniformly distributed on $[0, 1]$. Find the distribution of $Z = \max(X_1, X_2, \ldots, X_n)$. Find $E(Z)$.

2. Define $= \min(X, Y)$ where $X$ and $Y$ are as in #1 and answer the same questions as in #1.

3. A point is picked uniformly at random on the surface of the unit sphere. Writing $\Theta$ and $\Phi$ for its longitude and latitude, find the conditional density functions of $\Theta$ given $\Phi$, and of $\Phi$ given $\Theta$. (Note that in physics textbooks exactly the opposite notation is used, i.e. the latitude is $\Theta$.)

4. James Clerk Maxwell (1831–1879), who is best known for his theory of electromagnetism, was the first to apply the methods of probability to the motion of molecules in a gas. In this problem we explore Maxwell’s kinetic theory of (dilute) gases. As you work through this problem you might enjoy the website

http://www.chm.davidson.edu/ChemistryApplets/index.html#KineticMolecularTheory

where there are Java applets illustrating the kinetic theory of gases.

For a typical macroscopic container, the number of gas molecules is of order of Avogadro’s constant ($\approx 6 \times 10^{23}$); and hence, a dynamical description in which the position and velocity of each molecule is specified is not feasible. Assuming an ideal gas, in a container of volume $V$ with $N$ molecules, is in thermodynamic equilibrium at temperature $T$, Maxwell assumed that there exists a probability density function $f = f(\vec{v}), \vec{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$, such that the number of molecules with velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$ is

$$N f(\vec{v}) dv_x dv_y dv_z.$$

Since $f$ is assumed to be a probability density,

$$N \int_{\mathbb{R}^3} f(\vec{v}) dv_x dv_y dv_z = N.$$

Maxwell further assumed that in every Cartesian coordinate system the three components of the velocity are mutually independent random variables with zero expectations. He showed that assumption, together
with a thermodynamic requirement relating to the equation of state of an ideal gas, that $f(\vec{v}) = f_x(v_x)f_y(v_y)f_z(v_z)$ where

$$f_j(v_j) = \left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{mv_j^2}{2k_B T}\right), \ j = x, y, z.$$  

Here $m$ is the mass of the molecule and $k_B$ is what is (now) called Boltzmann’s constant. ($k_B = 1.3806 \times 10^{-23}$ Joules/degrees Kelvin.)

(a) Show that the probability density for the speed, $v := ||\vec{v}||$, is

$$f_s(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right).$$

(b) Show that the average speed is

$$\mathbb{E}(v) = \left(\frac{8k_B T}{\pi m}\right)^{1/2}.$$ 

Use this to find the average speed of $\text{H}_2$ and $\text{O}_2$ at temperature 300 K.

(c) Calculate the velocity necessary for $\text{H}_2$, $\text{O}_2$ and $\text{CO}_2$ to escape the pull of the earth’s gravity. What fraction of these molecules at 300 K have sufficient velocity to escape the gravitational field of the earth?

(d) Show that the average kinetic energy is

$$\mathbb{E}(\frac{1}{2}mv^2) = \frac{3}{2} k_B T.$$ 

(e) The average energy of the gas is

$$\mathbb{E}(\mathcal{E}) = \mathbb{E}(\frac{N}{2}mv^2) = \frac{3}{2} Nk_B T$$

from above. Find var($\mathcal{E}$).

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\[1\] Note the conversion factor: $1.660539 \times 10^{-27}$ kg/amu. Here amu stands for atomic mass unit. Its reciprocal, called Avogadro’s number, has the value $6.022142 \times 10^{23}$ kg/amu. Hydrogen has 1.00797 amu, Carbon 12.01115 amu, and Oxygen 15.9994 amu.