First Passage of a One-Dimensional Random Walker

The Problem: Consider a walker on the integer lattice

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

At each time step the walker takes either a unit step to the left with probability q or a unit step to the right with probability p ($p + q = 1$). The problem is to find the expected time for the *first passage* of the walker to the site $+1$. In terms of gambling, you win a dollar with probability p and lose a dollar with probability q. You are asked to find the expected time that your net earning *first* reaches $+1$.

The Solution: Introduce the independent random variables

$$X_j = \begin{cases} +1 & \text{if } j^{th} \text{ step is to the right,} \\ -1 & \text{if } j^{th} \text{ step is to the left,} \end{cases}$$

where $j = 1, 2, \ldots$. Define

$$S_n = X_1 + X_2 + \cdots + X_n \tag{1}$$

so that S_n is the position of the walker at time n. (In terms of the gambling interpretation, S_n is the net earnings after n bets.) We set $S_0 = 0$. We are interested in the event $\mathcal{F}_n$:

$$S_1 \leq 0,\ S_2 \leq 0,\ \ldots, S_{n-1} \leq 0,\ S_n = 1.$$

This event is the first passage of the walker to $+1$ at time n. Let φ_n be the probability of the event $\mathcal{F}_n$. For small n one can simply list the possibilities (clearly $\mathcal{F}_{2n} = \phi$ since the walker must walk an odd number of steps to reach $+1$ for the first time):

$n = 1$ $\mathcal{F}_1 = \{H\}$ so that $\varphi_1 = p$,

$n = 3$ $\mathcal{F}_3 = \{THH\}$ so that $\varphi_3 = p^2 q$,

$n = 5$ $\mathcal{F}_5 = \{THTHH, TTHHH\}$ so that $\varphi_5 = 2p^3q^2$.

(We have used the Head-Tail notation to denote the ± 1.)

We introduce the generating function

$$\Phi(t) = \sum_{n=0}^{\infty} \varphi_n t^n \,. \tag{2}$$

We already know that

$$\Phi(t) = pt + p^2 q t^3 + 2p^3 q^2 t^5 + \cdots .$$

We now give a method that will determine $\Phi(t)$.

Recall that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are any two disjoint events so that $\Omega = \mathcal{E}_1 \cup \mathcal{E}_2$, then for any random variable $X : \Omega \to \mathbb{R}$

$$E(X) = E(X|\mathcal{E}_1)\mathrm{P}(\mathcal{E}_1) + E(X|\mathcal{E}_2)\mathrm{P}(\mathcal{E}_2) \tag{3}$$

where $E(X|\mathcal{E}_j)$ is the conditional expectation.

Introduce the random variable N defined to be the first subscript n for which $S_n = +1$. Precisely, given $\omega \in \Omega$, let $N(\omega)$ equal the first integer n for which $S_n(\omega) = +1$. If no such n exists, set $N(\omega) = +\infty$. Observe that $N(\omega) = +1$ for all $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_1 = 1$. (In words, all walks that start with a first toss of head give $N = 1$.) Then

$$E(t^N) = \sum_{n=0}^{\infty} t^n \mathrm{P}(N = n) = \sum_{n=0}^{\infty} t^n \varphi_n = \Phi(t).$$

We now use (3) with $\mathcal{E}_1 = \{X_1 = 1\}$ and $\mathcal{E}_2 = \{X_1 = -1\}$:

$$E(t^N) = E(t^N|X_1 = 1)\mathrm{P}(X_1 = 1) + E(t^N|X_1 = -1)\mathrm{P}(X_1 = -1). \tag{4}$$

Given $X_1 = 1$, we know $N(\omega) = 1$ so that

$$E(t^N|X_1 = 1) = t.$$

This together with $p = \mathrm{P}(X_1 = 1)$, $q = \mathrm{P}(X_1 = -1)$ implies that (4) reduces to

$$E(t^N) = pt + qE(t^N|X_1 = -1). \tag{5}$$

If $X_1 = -1$, introduce the two times N_1 and N_2 where N_1 is the number of trials required to increase the partial sums from -1 to 0 and N_2 is the number of subsequent trials to increase the partial sums from 0 to +1. In random walker terminology, given that we first step to the left, we let N_1 equal the number of steps required for the first return to the origin 0. Then N_2 equals the number of subsequent steps required to go from the origin 0 to 1. N_1 and N_2 are independent since they depend upon different sets of X_j which are independent. They are also independent of X_1 since they depend only upon the subsequent steps. Further observe that the distributions of N_1 and N_2 are the same as the distribution of N. (The first passage time from to 0 starting at -1 has the same distribution as the first passage time to +1 starting at 0, etc.) The random variable N is then

$$N = 1 + N_1 + N_2$$

(the one is the first step to the left). Hence

$$\begin{aligned}
E(t^N|X_1 = -1) &= E\left(t^{1+N_1+N_2}|X_1 = -1\right) \\
&= E(t\,t^{N_1}\,t^{N_2}|X_1 = -1) \\
&= tE(t^{N_1}\,t^{N_2}|X_1 = -1) \\
&= tE(t^{N_1}|X_1 = -1)E(t^{N_2}|X_1 = -1) \text{ by independence} \\
&= tE(t^{N_1})E(t^{N_2}) \text{ by independence} \\
&= t\Phi(t)^2 \text{ by equality of distribution functions.}
\end{aligned}$$

Using this last expression for $E(t^N|X_1 = -1)$ in (5) we obtain

$$\Phi(t) = pt + qt\,\Phi(t)^2\,. \tag{6}$$

This is a quadratic equation for Φ. Solving (6) gives

$$\Phi(t) = \frac{1 - \sqrt{1 - 4pq\,t^2}}{2qt}\,. \tag{7}$$

We took the minus solution since this gives $\Phi(0) = 0$ as we know must be the case. Using Mathematica we power series this about the origin to obtain the first few probabilities φ_n:

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In[1]:= phi[t_]:=(1-Sqrt[1-4*p*q*t^2])/(2*q*t);

In[2]:= Series[phi[t],{t,0,10}]
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                   2      3         3   2   5         4   3   7          5   4   9          11
Out[2]= p t + p   q t   + 2 p   q   t   + 5 p   q   t   + 14 p   q   t   + O[t]
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To be more systematic we use the Taylor expansion of

$$f(x) = \sqrt{1-x}\,.$$

Elementary calculus gives $f(0)=1$, $f'(0)=-1/2$, $f''(0)=-1/2^2$ and for $n\geq 3$[1]

$$f^{(n)}(0) = -\frac{1}{2^n}(2n-3)!!$$

so that

$$\sqrt{1-x} = 1 + \sum_{n=1}^{\infty} \frac{(2n-3)!!}{2^n n!}\, x^n$$

where we are interpreting $(-1)!! = +1$. Substituting this Taylor expansion into (7) (with $x = 4pqt^2$) we find that $\varphi_{2n} = 0$ and

$$\varphi_{2n-1} = \frac{(2n-3)!!}{n!}\, 2^{n-1} p^n q^{n-1}\,.$$

For $p = q = 1/2$ these probabilities become

$$\varphi_{2n-1} = \frac{(2n-3)!!}{n!\,2^n}\,.$$

To get some feeling for these numbers let p_{2n-1} denote the probability that the first passage time occurs in time less than or equal to $2n-1$. Clearly,

$$p_{2n-1} = \varphi_1 + \varphi_3 + \cdots + \varphi_{2n-1}.$$

In Table 1 we give some numerical values of p_{2n-1}. One sees that there is approximately a 7.8% chance that after 100 bets ones net earnings has never reached +1. Even after 10,000 bets there is approximately a 1% chance that the net earnings are never positive.

[1]Recall the double factorial notation: $(2n+1)!! = (2n+1)(2n-1)\cdots 5\cdot 3\cdot 1$.

$2n-1$	p_{2n-1}
1	0.5000
3	0.6250
5	0.6875
7	0.7266
9	0.7539
11	0.7744
101	0.9212
1001	0.9748
100001	0.9920

Table 1: Probabilities p_{2n-1}.

The probability that the site +1 is visited by our walker is $\Phi(1)$. (The sum of the probabilities that it is visited for the first time at step n.) Substituting $t = 1$ into (6) and doing some algebra[2] gives

$$\Phi(1) = \frac{1 - |p-q|}{2q},$$

and hence

$$\Phi(1) = \begin{cases} p/q & \text{if } p < q, \\ 1 & \text{if } p \geq q. \end{cases}$$

That is, if $p < q$ the probability that the walker stays to the left of +1 is $(1 - p/q)$ but if $p \geq q$ then with probability one the walker will visit the site +1. What is the expected waiting time in this second instance?

To compute $E(N)$ we first note that

$$E(N) = \sum_{n\geq 1} n\varphi_n = \Phi'(1).$$

Taking the derivative of $\Phi(t)$ and then setting $t = 1$ gives

$$\Phi(1) = \frac{1 - |p-q|}{2q\,|p-q|}\,.$$

[2]Use fact that if $p + q = 1$, then $1 - 4pq = (p-q)^2$.

For $p > q$ this reduces to $1/(p-q)$ but is infinite for $p = q = 1/2$.

Hence a walker with equal chance of going to the left or the right (or in gambling terminology, for a fair game) will, with probability one, visit the site $+1$ but the expected waiting time for this first passage will be infinite. At first this seems paradoxical, but we must remember that the expected value being infinite means that the sum

$$\sum_{n\geq 1} n\varphi_n$$

diverges; the probability one statement says the sum

$$\sum_{n\geq 1} \varphi_n$$

converges to $+1$. This means that the probabilities φ_n, though going to zero for $n \to \infty$, are not going to zero very fast; indeed, slow enough so that the series $\sum n\varphi_n$ diverges. What does this mean in terms of simulations and/or experiments? Suppose you run the computer until the first time the walker visits $+1$ and you record the number of steps for this first passage. The result says with probability one the computer will stop[3] and record a first passage time. However, if you average the various first passage times you will observe that for larger and larger averages (more experiments), the average grows and does not approach a limit.

[3]Note that if $p < q$ then with probability $1 - p/q$ the computer will not turn off.