1. Let \( \{X_n\}_{n \geq 1} \) be random variables such that the partial sums \( S_n = X_1 + \cdots + X_n \) determine a martingale. Show that \( \mathbb{E}(X_iX_j) = 0 \) if \( i \neq j \).

2. Let \( \{X_n\}_{n \geq 0} \) denote a Markov chain with countable state space \( S \) and transition matrix \( P \). Let \( f : S \to \mathbb{R} \) denote any bounded function on the state space \( S \). (Bounded means there exists a constant \( C \) so that \( |f(i)| < C \) for all \( i \in S \).) The transition matrix \( P \) acts on bounded functions \( f : S \to \mathbb{R} \) as follows:

\[
(Pf)(i) = \sum_{j \in S} p_{ij} f(j), \quad i \in S.
\]

If \( f : S \to \mathbb{R} \) is a bounded function, define the process

\[
M_n^f = f(X_n) - f(X_0) - \sum_{m=0}^{n-1} (P - I)f(X_m), \quad n = 1, 2, \ldots
\]

(One can set \( M_0^f = 0 \).) Show that the process \( \{M_n\} \) is a martingale with respect to \( \{X_n\} \). Recall you must show (i) \( \mathbb{E}(|M_n^f|) < \infty \) and the martingale property (ii) \( \mathbb{E}(M_{n+1}^f | X_0, X_1, \ldots, X_n) = M_n^f \).

3. Let \( S_n = a + \sum_{j=1}^{n} X_j \) (\( S_0 = a \)) be a simple symmetric random walk on \( \mathbb{Z} \). The walk stops at the earliest time \( T \) when it reaches either of the two positions \( 0 \) or \( K \) where \( 0 < a < K \). Show that

\[
M_n = \sum_{j=0}^{n} S_j - \frac{1}{3} S_n^3
\]

is a martingale and deduce that

\[
\mathbb{E}\left( \sum_{r=0}^{T} S_r \right) = \frac{1}{3}(K^2 - a^2)a + a.
\]

4. An urn (called Polya’s urn model) contains \( b \) black balls and \( r \) red balls. A ball is drawn at random. It is replaced and, moreover, \( c \) balls of the color drawn are added. Let

\[
Y_0 = \frac{b}{b + r}
\]

and let \( Y_n \) be the proportion of the black balls attained by the nth drawing.
(a) Show that \( \{Y_n\} \) is a martingale. (With respect to what?)

(b) What is \( \mathbb{E}(Y_n) \)?

(c) Does \( Y := \lim_{n \to \infty} Y_n \) exist? If so, why and in what sense (what sense of convergence). What can you say about \( \mathbb{E}(Y) \)?

(d) Let \( X_j = 1 \) if the \( j \)th drawing results in a black ball, and \( X_j = 0 \) if the result is a red ball. Let

\[
S_n = X_1 + \cdots + X_n
\]

Show that

\[
Y_n = \frac{b + cS_n}{b + r + nc}
\]

Conclude from this

\[
\lim_{n \to \infty} \frac{S_n}{n} = Y
\]