HW Assignment 2

1. In §5.11 and in the class lecture notes, an expression was derived for the rate function $\Lambda^*(a)$. In particular, if S_n is a sum of iid Bernoulli random variables (probability of success $=p$), then ($b := 1 - a$, $q := 1 - p$) for $0 < a < 1$

$$\Lambda^*(a) = a \log \frac{a}{p} + b \log \frac{b}{q}\,, \tag{1}$$

so that as $n \to \infty$

$$\mathbb{P}\left(S_n \geq na\right) \sim e^{-n\Lambda^*(a)}, \;\; a > p$$

and similarly,

$$\mathbb{P}\left(S_n \leq na\right) \sim e^{-n\Lambda^*(a)}, \;\; a < p.$$

To denote the dependence upon the parameter p in $\Lambda^*(a)$, we write (1) as $\Lambda^*(a, p)$.

Here is an application to *statistical hypothesis testing.* Suppose we run a series of success-failure experiments and want to test between two possible values of p, the probability of success. That is we want to distinguish between Hypothesis 1: $p = p_1$ and Hypothesis 2: $p = p_2$. For definiteness, assume $p_1 < p_2$. Now pick a numer a, $p_1 < a < p_2$. If $S_n < na$ then we choose p_1 as the likely value of p; where as, if $S_n \geq na$ we choose $p_2 = p$. The question is how to choose the number a. If Hypothesis 1 is true there is a probability, P_1, that $S_n > na$; and similarly, there is a probability, P_2, that $S_n < na$ given that Hypothesis 2 is true. We know that

$$P_1 \approx e^{-n\Lambda^*(a,p_1)} \text{ and } P_2 \approx e^{-n\Lambda^*(a,p_2)}.$$

P_1 is the probability of erroneously deciding in favor of Hypothesis 2 when Hypothesis 1 is, in fact, true; and similarly, P_2 is the probability of deciding in favor of Hypothesis 1 when, in fact, Hypothesis 2 is true. Since there is no reason to make one probability larger than another, we choose a so that

$$P_1 = P_2.$$

Show that the choice of a is

$$a = \frac{\log(q_1/q_2)}{\log(p_2/p_1) + \log(q_1/q_2)} \tag{2}$$

Write

$$p_2 = p_1 + t, \;\; t > 0.$$

We assume that t is relatively small, i.e. $t \ll 1$. Let $a = a(p_1, p_2)$ be the solution to (2). Show that

$$a(p_1, p_2) = p_1 + \frac{1}{2}\,t + \frac{p_1 - q_1}{12 p_1 q_1}\,t^2 + \frac{1 - 2p_1 + 2p_1^2}{24 q_1^2 p_1^2}\,t^3 + \mathrm{O}(t^4).$$

And hence,

$$\Lambda^*(a(p_1, p_2), p_1) = \frac{1}{8p_1q_1}\, t^2 + \frac{p_1 - q_1}{16p_1^2q_1^2}\, t^3 + \mathrm{O}(t^4).$$

2. #1 in GS, page 175.

3. #3 in GS, page 175.

4. #5 in GS, page 175.