

Homework #4: Due Friday, February 10

Type I Problem:

By use of Cauchy's theorem show that

$$\int_0^{\infty} \sin(x^2) dx = \int_0^{\infty} \cos(x^2) dx = \frac{\sqrt{2\pi}}{4}.$$

Hint: Apply Cauchy's theorem to $f(z) = e^{-z^2}$ where the contour Γ is the path consisting of the three smooth curves: (1) the line segment $[0, R]$, (2) the arc $z = Re^{i\theta}$, $0 \leq \theta \leq \pi/4$, (3) the line segment from $Re^{i\pi/4}$ to 0. Then let $R \rightarrow \infty$.

Type II Problems:

1. Show that if $\xi \in \mathbb{R}$, then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx.$$

Hint: First suppose $\xi > 0$. Then apply Cauchy's theorem to $f(z) = e^{-\pi z^2}$ where the contour Γ is the rectangle whose four corners are located at $-R$, R , $R + i\xi$ and $-R + i\xi$. For $\xi < 0$ consider the rectangle obtained by reflecting in the x -axis.

2. Show by use of Cauchy's theorem that

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.$$

Hint: Apply Cauchy's theorem to $f(z) = (1 - e^{iz})/z^2$ where the contour Γ consists of four smooth curves: (1) line segment $[-R, -\varepsilon]$; (2) semi-circle $z = \varepsilon e^{i\theta}$, $-\pi \leq \theta \leq 0$; (3) line segment $[\varepsilon, R]$ and (4) semi-circle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Then let $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$.