Math 185A, HW#6, Due February 17, 2012

Type I Homework: Define

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt,\ \ \Re(z) > 0. \tag{1}$$

1. Show the integral converges for $\Re(z) > 0$. (If you have trouble proving this, see §15.1 of your textbook. Follow that proof.)

2. Show that $\Gamma(z)$ is holomorphic in the right half-plane $\Re(z) > 0$.

   Extended Hint: This does not follow by a direct application of the theorem in §9.7 because the interval of integration is not a finite interval. There are extensions of the theorem in §9.7 (for example, Theorem 15.2, page 278 of your textbook.) Here we outline how to prove the above statement directly.

   (a) Let δ and M be any two positive numbers, $\delta < M$.

   $$S_{\delta,M} = \{z \in \mathbb{C} : \delta < \Re(z) < M\}$$

   We prove that Γ is holomorphic in each strip $S_{\delta,M}$. Since $0 < \delta < M$ are arbitrary, we will have proved that $\Gamma(z)$ is holomorphic in $\Re(z) > 0$.

   (b) Define

   $$\Gamma_\varepsilon(z) = \int_\varepsilon^{1/\varepsilon} t^{z-1}e^{-t}\,dt,\ \ \varepsilon > 0.$$

   Prove that $\Gamma_\varepsilon(z)$ is holomorphic in $S_{\delta,M}$.

   (c) If we can show as $\varepsilon \to 0$ that $\Gamma_\varepsilon \to \Gamma$ *uniformly* on compact subsets of the strip $S_{\delta,M}$, then we know from the theorem in §9.8 (page 138) the limit is also holomorphic. Thus one must obtain a uniform estimate of

   $$\Big|\Gamma(z) - \Gamma_\varepsilon(z)\Big| = \Big|\int_0^\varepsilon t^{z-1}e^{-t}\,dt + \int_{1/\varepsilon}^\infty t^{z-1}e^{-t}\,dt\Big|$$

   Prove this uniform convergence; and hence, the final result.

3. For $\Re(z) > 0$, prove

   $$\Gamma(z+1) = z\,\Gamma(z)\ \text{ and }\ \Gamma(1) = 1.$$

   and hence; for n a nonnegative integer, $\Gamma(n+1) = n!$.

Type II Homework:

1. Define

$$f(z) = \int_0^1 e^{-zt^2}\, dt.$$

   (a) Prove that $f = f(z)$ is an entire function of z.

   (b) Find the power series of f expanded about the point $z = 0$.

   Hint: §9.7, page 137, of your textbook.

2. Let ω_1 and ω_2 denote two complex numbers with $\Im(\frac{\omega_2}{\omega_1}) > 0$. Consider the parallelogram (contour) P formed by the four points 0, ω_1, $\omega_1 + \omega_2$, ω_2. More generally, consider the lattice of points

$$\Omega_{\omega_1,\omega_2} := \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

   We are familiar with holomorphic functions that are periodic with respect to one period. For example

$$f(z) = \sin z$$

   satisfies $f(z + 2\pi) = f(z)$ for all $z \in \mathbb{C}$. Suppose we ask for *holomorphic* functions that are periodic with respect to two periods ω_1 and ω_2; that is, we look at the family $\mathcal{F}$ of holomorphic functions that satisfy

$$f(z + \omega_1) = f(z) \quad \text{and} \quad f(z + \omega_2) = f(z) \quad \text{for all} \quad z \in \mathbb{C}. \tag{2}$$

   (a) Show that if f satisfies (2), then for any $\omega \in \Omega_{\omega_1,\omega_2}$ we have

$$f(z + \omega) = f(z).$$

   (b) Show that the class $\mathcal{F}$ of such doubly periodic holomorphic functions consists only of constants. Hint: Liouville's theorem.

   (c) Part (b) shows that if we want non-constant doubly periodic functions, they cannot be holomorphic inside the parallelogram P. Suppose we assume the class of doubly periodic functions $\mathcal{F}$ are not holomorphic inside the parallelogram P. Further assume $f \in \mathcal{F}$ are continuous on the parallelogram P. Show that for such functions f

$$\int_P f(z)\, dz = 0.$$

   Note: This does not follow from Cauchy's theorem since we are not assuming f is holomorphic inside P.