Math 185A, HW#7, Due March 2, 2012
Applications of the Residue Theorem

Type I Homework: Using the residue theorem, evaluate the following integrals (answers given)

1. $$\int_0^\infty \frac{x^a}{1+x^2}\,dx = \frac{\pi}{2}\sec(\frac{\pi a}{2}),\ \ 0 < a < 1.$$

2. $$\int_0^\infty \frac{dx}{1+x^5} = \frac{\pi}{5\sin(\pi/5)}$$

3. $$\int_0^\infty \frac{\log x}{1+x^5}\,dx = -\frac{\pi^2}{25}\,\frac{\cos(\pi/5)}{\sin^2(\pi/5)} = -\frac{\pi^2}{25}\left(1+\frac{3}{\sqrt{5}}\right)$$

Hints: For (1) model your proof on the evaluation of

$$\int_0^\infty \frac{x^{\lambda-1}}{1+x}\,dx,\ \ 0 < \lambda < 1$$

done in class. For (2) consider the contour Γ consisting of the line segment $[0, R]$, the arc $z = Re^{i\theta}$, $0 \le \theta \le 2\pi/5$, and the line segment $re^{2\pi i/5}$, $r \in [R, 0]$. For (3) make an appropriate modification of the contour in (2) to account for the log singularity at zero.

Type II Homework:

1. Let
$$\Omega = \left\{ m\omega_1 + n\omega_2 \middle| m, n \in \mathbb{Z} \right\},\ \ \Im(\frac{\omega_2}{\omega_1}) > 0,$$
denote the lattice generated by the two complex numbers ω_1 and ω_2. We showed that the only entire functions f that satisfy
$$f(z+\omega) = f(z),\ \ \omega \in \Omega \tag{1}$$
are constants. We then considered functions f that were meromorphic (the only singularities are pole singularities) and satisfied (1). Such functions are called *elliptic functions*. Assume f has no singularities lying on the lattice Ω. Let P denote the fundamental parallelogram formed from the four points $0, \omega_1, \omega_1 + \omega_2, \omega_2$. Show that the number of poles inside P (we count poles with multiplicities) must be greater than or equal to two. That is, there are no elliptic functions f that have a single simple pole inside the fundamental parallelogram P.

2. The total number of poles (counted according to their multiplicities) of an elliptic function is called its *order*. Show that every elliptic function of order m has m zeros in P. (Again for simplicity assume there are no zeros lying on the lattice Ω.)

3. (a) Show that

$$S(z) := \frac{\pi}{\sin \pi z}$$

is holomorphic in $\{z \in \mathbb{C} \big| z \notin \mathbb{Z}\}$ and has simple poles at $z = n \in \mathbb{Z}$ with residue $(-1)^n$.

(b) Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}$$

Hint: Let $f(z) = 1/(2z+1)^3$ and consider the function $S(z)f(z)$ over a *suitably chosen* contour Γ.