Math 185A, HW#8, Due Monday, March 12, 2012
Elliptic Functions

Throughout this problem set, ω_1 and ω_2 are complex numbers with $\Im(\omega_2/\omega_1) > 0$ and Ω is the lattice

$$\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

We set $\tau = \omega_2/\omega_1$ and denote by $\wp = \wp(z) = \wp(z, \Omega)$ the Weierstrass elliptic function. Denote by Λ the fundamental parallelogram consisting of the four line segments $[0, \omega_1]$, $[\omega_1, \omega_1 + \omega_2]$, $[\omega_1 + \omega_2, \omega_2]$, $[\omega_2, 0]$ oriented in the counter clockwise direction. And $\mathbb{H}$ denotes the upper half-plane,

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

Type I Homework:

#1. Let β_1 and β_2 denote two points lying inside Λ; thus $\beta_2 - \beta_1 \notin \Lambda$. Suppose that $f = f(z)$ is an elliptic function with two (and only two) simple poles located at $z = \beta_1$ and $z = \beta_2$ with residues α_1 and α_2, respectively. Find an expression for $f(z)$ in terms of $\wp$.

Hint: Consider the function

$$\frac{A}{\wp(z - \beta) + B} + C \tag{1}$$

where A, B, C and β do not depend upon z. How should these constants be chosen so as to construct $f(z)$?

#2. Show that

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$$

where we assume $2z \notin \Omega$. Hint: Start with the addition theorem

$$\wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_2) - \wp'(z_1)}{\wp(z_2) - \wp(z_1)}\right)^2 - \wp(z_1) - \wp(z_2)$$

#3. If we are given the curve

$$x^2 + y^2 = 1 \tag{2}$$

we can parametrize this curve by the use of trig functions; namely, if set $x = \cos z$ and $y = \sin z$ then for all complex z

$$\cos^2 z + \sin^2 z = 1.$$

Suppose instead of (2) we have the curve

$$x^3 + y^3 = 1. \tag{3}$$

Can we find functions that parametrize (3)? Show that if we set

$$x = \frac{a + b\,\wp'(z)}{\wp(z)} \quad \text{and} \quad y = \frac{a - b\,\wp'(z)}{\wp(z)}$$

then for certain values of a and b these parametrizations satisfy (3). Find a and b in terms of g_2 and g_3.

Type II Homework:

#1. Recall that

$$G_{2k}(\tau) = \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2k}}, \quad k = 2, 3, 4, \ldots$$

1. Show that

   $$G_{2k}(-\frac{1}{\tau}) = \tau^{2k} G_{2k}(\tau), \quad \Im(\tau) > 0. \tag{4}$$

   and

   $$G_{2k}(\tau + 1) = G_{2k}(\tau) \tag{5}$$

2. Using (4) and (5) show that

   $$G_6(i) = 0$$

   and

   $$G_4(\rho) = 0$$

   where $\rho = e^{2\pi i/3}$. Hint: $-\frac{1}{\rho} = \rho + 1$.

3. Show that $G_4(i)$ and $G_6(\rho)$ are real numbers.

4. Consider the lattice

   $$\Omega_i = \{m + ni : m, n \in \mathbb{Z}\}$$

   Draw a picture of this lattice. From the previous part $g_3 = 0$. Thus the cubic equation associated to this lattice is is

   $$y^2 = 4x^3 - g_2 x$$

5. Consider the lattice
$$\Omega_\rho = \{m + n\rho : m, n \in \mathbb{Z}\}$$
Draw a picture of this lattice. From the previous part $g_2 = 0$. Thus the cubic equation associated to this lattice is
$$y^2 = 4x^3 - g_3.$$

6. Does there exist a point $\tau_0 \in \mathbb{H}$ where $g_2(\tau_0) = 0$ and $g_3(\tau_0) = 0$? If yes, find the point τ_0, if no give a reason.