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A simple proof of Ramanujan's summation of the ${}_1\psi_1$

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Abstract. A simple proof by functional equations is given for Ramanujan's ${}_1\psi_1$ sum. Ramanujan's sum is a useful extension of Jacobi's triple product formula, and has recently become important in the treatment of certain orthogonal polynomials defined by basic hypergeometric series.

In [5; p. 222, eq. (12.12.2)] G. H. Hardy alludes to Ramanujan's "... remarkable formula with many parameters.":

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n x^n}{(b;q)_n} \equiv {}_1\psi_1\binom{a;q,x}{b} \tag{1}$$
$$= \frac{(b/a;q)_\infty (q;q)_\infty (q/ax;q)_\infty (ax;q)_\infty}{(b;q)_\infty (b/ax;q)_\infty (q/a;q)_\infty (x;q)_\infty},$$

where

$$\left|\frac{b}{a}\right| < |x| < 1, \qquad |q| < 1, \qquad (a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n),$$

and

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty.$$

There are four published proofs of this result ([1], [2], [4] and [7]). Those in [1], [2] and [7] rely on somewhat tricky rearrangement of series and on the q-analog of Gauss's summation [10; p. 97, eq. (3.3.2.5)]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n (c/ab)^n}{(c;q)_n (q;a)_n} = \frac{(c/a;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (c/ab;q)_\infty}, \tag{2}$$

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where $|c|<\min(1,|ab|)$. The other proof uses the q-analogue of the binomial series [10; p. 92, eq. (3.2.2.11)]:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}t^n=\frac{(at;q)_\infty}{(t;q)_\infty},\qquad |t|<1,\qquad |q|<1,\tag{3}$$

but it is far from simple. Since Ramanujan's summation (1) has recently become important in the treatment of certain orthogonal polynomials defined by basic hypergeometric series [3], it has become worthwhile to present an almost trivial proof of (1). Another very simple proof has been found by M. Ismail [6].

Proof of (1). We begin by noting that for $|q|<1$. $f(b)\equiv {}_1\psi_1\binom{a;q,x}{b}$ is an analytic function of b inside $|b|<\min(1,|ax|)$, since

$$f(b)=\sum_{n=0}^{\infty}\frac{(a;q)_nx^n}{(b;q)_n}+\sum_{n=1}^{\infty}\frac{(1-b/q^n)\cdots(1-b/q)x^{-n}}{(1-a/q^n)\cdots(1-a/q)}.\tag{4}$$

Furthermore,

$$\begin{aligned}{}_1\psi_1\binom{a;q,x}{b}-a\ {}_1\psi_1\binom{a;q,qx}{b}&=\sum_{n=-\infty}^{\infty}\frac{(a;q)_{n+1}x^n}{(b;q)_n}\\&=x^{-1}\left(1-\frac{b}{q}\right)\sum_{n=-\infty}^{\infty}\frac{(a;q)_{n+1}x^{n+1}}{\left(\frac{b}{q};q\right)_{n+1}}\\&=x^{-1}\left(1-\frac{b}{q}\right){}_1\psi_1\binom{a;q,x}{b/q}.\end{aligned}$$

Hence

$$\begin{aligned}f(bq)-x^{-1}(1-b)f(b)&=a\sum_{n=-\infty}^{\infty}\frac{(a,q)_nq^nx^n}{(bq;q)_n}\\&=-ab^{-1}\sum_{n=-\infty}^{\infty}\frac{(a;q)_n(1-bq^n-1)x^n}{(bq;q)_n}\\&=-ab^{-1}(1-b)f(b)+ab^{-1}f(bq),\end{aligned}\tag{6}$$

and so

$$\left(1-\frac{a}{b}\right)f(bq)=(1-b)(x^{-1}-ab^{-1})f(b),$$

or

$$f(b)=\frac{(1-b/a)}{(1-b)(1-b/ax)}f(bq). \tag{7}$$

If we iterate (7) $n-1$ times, we find that

$$f(b)=\frac{(b/a;q)_n}{(b;q)_n(b/ax;q)_n}f(bq^n), \tag{8}$$

and, since $f(b)$ is analytic in the neighborhood of 0 given by $|b|<|ax|$, we obtain in the limit as $n\to\infty$,

$$f(b)=\frac{(b/a;q)_\infty f(0)}{(b;q)_\infty(b/ax;q)_\infty}. \tag{9}$$

Now we observe from (4) and (3) that

$$f(q)=\sum_{n=0}^{\infty}\frac{(a;q)_n x^n}{(q;q)_n}=\frac{(ax,q)_\infty}{(x;q)_\infty}. \tag{10}$$

This allows us to evaluate $f(0)$ by setting $b=q$ in (9):

$$\begin{aligned} f(0)&=\frac{(q;q)_\infty(q/ax;q)_\infty f(q)}{(q/a;q)_\infty}\\ &=\frac{(q;q)_\infty\left(\frac{q}{ax}q\right)_\infty(ax;q)_\infty}{(q/a;q)_\infty(x;q)_\infty}. \end{aligned} \tag{11}$$

Finally we may utilize (11) to eliminate $f(0)$ from (9):

$${}_1\psi_1\left(\begin{smallmatrix}a;q,x\\ b\end{smallmatrix}\right)=f(b)=\frac{(b/a;q)_\infty(q;q)_\infty(q/ax;q)_\infty(ax;q)_\infty}{(b;q)_\infty(b/ax;q)_\infty(q/a,q)_\infty(x;q)_\infty}, \tag{12}$$

as desired.

Note that Jacobi's triple product identity follows directly from (1) if we replace a by α^{-1}, x by $z\alpha$ and then set $\alpha=b=0$:

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}z^n=(q;q)_\infty(q/z;q)_\infty(z;q)_\infty. \tag{13}$$

I. J. Schoenberg has pointed out an interesting property of $(a;q)_n/(b;q)_n$ which follows from Ramanujan's sum. A sequence a_n, $n=0, \pm 1, \ldots$, is said to be totally positive if all subdeterminants of the doubly infinite matrix $A=(a_{i-j})_{-\infty<i,j<\infty}$ are nonnegative. Schoenberg [9] proved that a sequence a_n is totally positive if the bilateral generating function $f(z)=\sum_{-\infty}^{\infty} a_n z^n$ has the representation

$$f(z)=e^{cz+dz^{-1}} \prod_{i=1}^{\infty} \frac{(1+\alpha_i z)(1+\delta_i z^{-1})}{(1-\beta_i z)(1-\gamma_i z^{-1})}, \tag{14}$$

$$c, d, \alpha_i, \beta_i, \gamma_i, \delta_i \geq 0, \sum_{i=1}^{\infty} (\alpha_i+\beta_i+\gamma_i+\delta_i)<\infty,$$

in the interior of an annulus centered at the origin.

If $a<b<0$ in (1) then, the generating function has the form (14) and so

$$a_n=\frac{(a;q)_n}{(b;q)_n}=\prod_{k=0}^{\infty} \frac{(1-bq^{k+n})(1-aq^k)}{(1-aq^{k+n})(1-bq^k)}$$

is a totally positive sequence for $a<b\leqq 0$, $0<q<1$. Schoenberg [9] proved this when $b=0$. For an extended discussion of totally positive sequences see Karlin [8].

REFERENCES

[1] ANDREWS, G. E., *On Ramanujan's summation of* ${}_1\psi_1(a;b;z)$. Proc. Amer. Math. Soc. *22* (1969), 552–553.

[2] ANDREWS, G. E., *On a transformation of bilateral series with applications*. Proc. Amer. Math. Soc. *25* (1970), 554–558.

[3] ANDREWS, G. E. and ASKEY, R., *The classical and discrete orthogonal polynomials and their q-analogues*. To appear.

[4] HAHN, W., *Beiträge zur Theorie der Heineschen Reihen*. Math. Nach. *2* (1949), 340–379.

[5] HARDY, G. H., *Ramanujan*. Cambridge University Press, Cambridge, 1940. Reprinted: Chelsea, New York.

[6] ISMAIL, M., *A simple proof of Ramanujan's* ${}_1\psi_1$ *sum*. Proc. Amer. Math. Soc., *63* (1977), 185–186.

[7] JACKSON, M., *On Lerch's transcendant and the basic bilateral hypergeometric series* ${}_2\psi_2$. J. London Math. Soc *25* (1950), 189–196.

[8] KARLIN, S., *Total Positivity*, Volume One. Stanford University Press, Stanford, 1968.

[9] SCHOENBERG, I. J., *Some analytical aspects of the problem of smoothing.* Studies and Essays presented to R. Courant on his 60th Birthday, Interscience Publishers, New York, 1948, 351–370.

[10] SLATER, L. J., *Generalized hypergeometric functions.* Cambridge University Press, Cambridge 1966.

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