Math 205A

Homework #3

1. Problem #20, page 107 of Stein & Shakarchi
2. Problem #3, page 109 of Stein & Shakarchi
3. **Mellin transform**:

   Suppose f is a continuous function of the real variable x satisfying

   $$f(x) = \mathrm{O}(x^{-\alpha}),\ x \to 0 \ \text{ and } \ f(x) = \mathrm{O}(x^{-\beta}),\ x \to \infty.$$

   We define the *Mellin transform* of f to be

   $$\widehat{f}(s) = \int_0^\infty x^{s-1} f(x)\, dx \tag{1}$$

   (a) Show that $\widehat{f}$ is holomorphic in the strip $\alpha < \Re(s) < \beta$.

   (b) Just as for Fourier integrals[1] there is an inversion formula for the Mellin transform:

   $$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s} \widehat{f}(s)\, ds,\ \ \alpha < \sigma < \beta. \tag{2}$$

   Show (2) follows from the Fourier inversion formula. Hint: Make the change of variable $x = e^X$ in (1).

   (c) The rest of this problem is an expansion of Problem #2, page 68, of Stein & Shakarchi. We shall prove later, and you may assume as given, that the Hurwitz zeta-function, defined by

   $$\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s},\ \sigma := \Re(s) > 1, 0 < a \leq 1, \tag{3}$$

   has an analytic continuation to a holomorphic function in the open region $\mathbb{C} - \{1\}$ with a simple pole at $s = 1$ with Laurent expansion

   $$\zeta(s, a) = \frac{1}{s-1} + \gamma_0(a) + \mathrm{O}(s-1),\ s \to 1,$$

[1] If you have not seen this, read Stein & Shakarchi, pages 111–117. You may assume as given the Fourier inversion formula given as Theorem 2.2, page 115 of Stein & Shakarchi.

where $\gamma_0(a)$ is explicitly known. (It depends upon a but not s.) Note that for $a = 1$ the Hurwitz zeta-function reduces to the Riemann zeta-function $\zeta(s)$. You may also wish to read some properties of the gamma function $\Gamma(s)$.

Define

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}, \tag{4}$$

and set $z = \exp(2\pi i p/q)e^{-t}$, $t > 0$, and $f(t; p, q) = F(z)$. Thus as $t \to 0$ we approach the unit circle $|z| = 1$ along the rational angle $2\pi p/q$.

(d) Show that the Mellin transform of $f(t; p, q)$ is

$$\widehat{f}(s; p, q) = \frac{\Gamma(s)}{q^{2s}} \zeta^2(s) + \frac{\Gamma(s)}{q^{2s}} \sum_{r_1, r_2 = 1}^{q-1} \exp(\frac{2\pi i p}{q} r_1 r_2) \zeta(s, r_1/q) \zeta(s, r_2/q)$$

(e) Using the Mellin inversion formula followed by a deformation of the contour, show that

$$f(t; 0, 1) = \frac{\log(1/t)}{t} + \frac{\gamma}{t} + \frac{1}{4} - \frac{t}{144} + \mathrm{O}(t^2),\ t \to 0^+,$$

where γ is Euler's constant. Remark: I found MATHEMATICA useful in computing various residues.

(f) For the rational angle $\alpha = 2\pi p/q$ show that as $t \to 0^+$ that

$$f(t; p, q) = c(p, q) \frac{\log(1/t)}{t} + \mathrm{O}(\frac{1}{t})$$

where

$$c(p, q) = \frac{1}{q^2} + \frac{1}{q^2} \sum_{r_1, r_2 = 1}^{q-1} \exp(2\pi i \frac{p}{q} r_1 r_2)$$

(g) Use these results to conclude that $F(z)$ cannot be analytically continued past the unit circle.