Homework #6—Due March 4, 2013

#1. Problem #9, page 202 of Stein & Shakarchi. Hint: The distributed notes might prove useful.

#2. Problem #1a, page 203 of Stein & Shakarchi.

Remarks: For \( F(s) = \zeta(s) \), the formula proved becomes

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 \, dt = \zeta(2\sigma), \quad \sigma > 1.
\]

Since \( \zeta(s) \) has a pole at \( s = 1 \) it is not clear what happens at \( \sigma = 1 \). One can prove\(^1\) that

\[
\int_{1}^{T} \left| \zeta(1 + it) \right|^2 \, dt = \zeta(2)T + O\left(\log^2 T\right)
\]

It is a much deeper result that

\[
\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \, dt = T \log T + O\left(T \log^{1/2} T\right).
\]

In 1926 A. E. Ingham proved

\[
\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \, dt = (2\pi^2)^{-1} T \log^4 T + O\left(T \log^3 T\right).
\]

It is conjectured that

\[
\int_{1}^{T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} \, dt \sim a_k T \left(\log T\right)^{k^2}
\]

where there are explicit conjectures for the coefficients \( a_k \).

#3. Let \( \omega_1, \omega_2 \in \mathbb{C}, \Im(\omega_2/\omega_1) > 0 \) and set

\[
\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.
\]

Define
\[ \sigma(z) = \sigma(z, \Omega) := z \prod_{0 \neq \omega \in \Omega} E_2\left( \frac{z}{\omega} \right) \] (1)
where \( E_2(z) \) is the Weierstrass canonical factor
\[ E_2(z) = (1 - z)e^{z + z^2/2}. \]

For (1) to be well-defined we must show the product converges.

1. Let \( U = \{(u, v) \in \mathbb{C}^2 : \Im(u/v) > 0\} \). Let \( K \subset U \) be compact and let \( \alpha > 2 \). Prove there exists a bound \( M > 0 \) such that
\[ \sum_{0 \neq \omega \in \Omega} |\omega|^{-\alpha} \leq M \] (2)
for all \( (\omega_1, \omega_2) \in K \).

Useful linear algebra lemma: Let \( u, v \) be real variables and consider the quadratic form
\[ Q(u, v) = a u^2 + 2 b u v + c v^2 \]
Suppose that \( Q(\cdot, \cdot) \) is positive definite; that is,
\[ Q(u, v) > 0 \text{ for all } u, v, (u, v) \neq (0, 0). \]

Then there exist positive constants \( \lambda_1 \) and \( \lambda_2 \) (depending upon \( a, b \) and \( c \) but not \( u \) and \( v \)) such that
\[ \lambda_1(u^2 + v^2) \leq Q(u, v) \leq \lambda_2(u^2 + v^2). \]
If you wish to use this lemma, first prove it. Using this lemma prove there exist positive constants \( c_1 \) and \( c_2 \) such that
\[ c_1 \sqrt{m^2 + n^2} \leq |m \omega_1 + n \omega_2| \leq c_2 \sqrt{m^2 + n^2}. \] (3)
Use (3) to prove (2).

2. Show the convergence of (1) follows now from (2) and conclude \( \sigma(z) \) is an entire function of \( z \) whose zero set is \( \Omega \). The entire function \( \sigma(z, \Omega) \) is called the Weierstrass sigma-function.
3. Set
\[ \wp (z) = \wp (z; \omega_1, \omega_2) := - \frac{d^2}{dz^2} \log \sigma (z) \] (4)
and show that
\[ \wp (z) = \frac{1}{z^2} + \sum_{0 \neq \omega \in \Omega} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right). \] (5)

4. Show that \( \wp (z) \) is doubly periodic, i.e.
\[ \wp (z + \omega) = \wp (z), \ \omega \in \Omega. \]

The elliptic function \( \wp \) is called the Weierstrass \( \wp \)-function.

5. Remarks: The infinite product for \( \sigma (z) \) is similar to that of \( \sin z \) given by
\[ \sin z = z \prod_{0 \neq m \in \mathbb{Z}} E_1 \left( \frac{z}{m \pi} \right) \]
and \( \wp \) is similar to
\[ \csc^2 z = - \frac{d^2}{dz^2} \log \sin z = \frac{1}{z^2} + \sum_{0 \neq m \in \mathbb{Z}} \frac{1}{(z - m \pi)^2}. \]