Math 205A

# Homework #1

**1.4.3.** (This refers to Chapter 1, §4, problem #3 of Stein & Shakarchi) With $\omega = s\,e^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where $n$ is a positive integer. How many solutions are there?

**1.4.7** The family of mappings introduced here plays an important role in complex analysis. These mapping, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

1. Let $z$, $w$ be two complex numbers such that $\overline{z}w \neq 1$. Prove that [see hints in textbook!]

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1.$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1.$$

2. Prove that for a fixed $w$ in the unit disc $\mathbb{D}$, the mapping

$$F : z \rightarrow \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(a) $F$ maps the unit disc to itself (that is, $F : \mathbb{D} \rightarrow \mathbb{D}$), and is holomorphic.

(b) $F$ interchanges 0 and $w$, namely $F(0) = w$ and $F(w) = 0$.

(c) $|F(z)| = 1$ if $|z| = 1$.

(d) $F : \mathbb{D} \rightarrow \mathbb{D}$ is bijective.

**1.4.9.** Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Use these equations to show that the function defined by

$$\log z = \log r + i\theta \quad \text{where } z = r\,e^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$ and has derivative $1/z$.