Math 205A

# Homework #2

1. Let $p(n)$ denote the number of partitions of the integer $n$. For example, $p(5) = 7$ since 5 can be partitioned in 7 ways:

   $5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$

   By definition, $p(0) = 1$. Prove Euler's identity

   $$P(z) := \sum_{n=0}^{\infty} p(n) z^n = \prod_{n=1}^{\infty} \frac{1}{1-z^n} \tag{1}$$

   What is the radius of convergence of (1)?

   Recall that for $k = 0, 1, 2, \ldots$

   $$\sigma_k(n) := \sum_{d|n} d^k$$

   (the sum of the $k$th power of the divisors of $n$). Prove that

   $$z \frac{d}{dz} \log P(z) = \sum_{n=1} \sigma_1(n) z^n$$

   Remark: The function

   $$\eta(z) := e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}),\ \ \Im(z) > 0,$$

   is called the *Dedekind $\eta$-function*. This function plays a decisive role in the theory of modular functions. Perhaps later in the class (or 205B) we will show for

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

   that $\eta(z)$ satisfies

   $$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon \sqrt{cz+d}\ \eta(z) \tag{2}$$

   where $\epsilon = \epsilon(a,b,c,d)$ and $|\epsilon| = 1$. It is *nontrivial* to prove (2).

2. #11, page 66 of Stein & Shakarchi.

3. #12, page 66 of Stein & Shakarchi.