Math 205A

Homework #5—Due Feb 10, 2012

Problem #1: Use of Mellin transform in asymptotics.

Suppose f is a continuous function of the real variable x satisfying

$$f(x) = \mathrm{O}(x^{-\alpha}),\ x \to 0 \ \text{ and } \ f(x) = \mathrm{O}(x^{-\beta}),\ x \to \infty.$$

We define the *Mellin transform* of f to be

$$\widehat{f}(s) = \int_0^\infty x^{s-1} f(x)\, dx \tag{1}$$

(a) Show that $\widehat{f}$ is holomorphic in the strip $\alpha < \Re(s) < \beta$.

Just as for Fourier integrals there is an inversion formula for the Mellin transform:

$$f(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} x^{-s}\widehat{f}(s)\, ds,\ \ \alpha < \sigma < \beta. \tag{2}$$

(b) Show (2) follows from the Fourier inversion formula. Hint: Make the change of variable $x = e^X$ in (1).

In an earlier exercise we examined the function

$$F(z) = \sum_{n=1}^\infty \frac{z^n}{1 - z^n}$$

in the vicinity of the unit circle. Let $z = e^{-t}$, $t \in \mathbb{R}^+$ and set $f(t) := F(z)$. Let $\widehat{f}$ denote the Mellin transform of f.

(c) Show that

$$\widehat{f}(s) = \Gamma(s)\zeta(s)^2$$

where $\Gamma(s)$ is the gamma function and $\zeta(s)$ is the Riemann zeta-function. Recall that we showed (Euler-Maclaurin summation notes) that $\zeta(s)$ has a simple pole at $s = 1$ and is holomorphic at all other points in the complex s-plane.

(d) Using the Mellin inversion formula followed by a deformation of the contour, show that

$$f(t) = \frac{\log(1/t)}{t} + \frac{\gamma}{t} + \frac{1}{4} - \frac{t}{144} + \mathrm{O}(t^2),\ t \to 0^+,$$

where γ is Euler's constant. Hint: Shift the contour into the left-half plane to the region $\Re(s) = -2$; thus having to compute the residues of $\widehat{f}(s)$ at $s = 1, 0, -1$. You may read ahead and assume any properties of $\Gamma(s)$ and $\zeta(s)$ needed to make the computation.[1] One particularly useful property of the gamma function is

$$\lim_{y\to\infty} \left|\Gamma(x+iy)\right| e^{\pi|y|/2}\,|y|^{1/2-x} = \sqrt{2\pi},\;\; x,y\in\mathbb{R}.$$

(e) Here's a problem I have not tried to work!! Can the above be generalized to $z = e^{2\pi i p/q}e^{-t}$? (Recall Problem #2, page 68 of textbook.)

(f) This method can be applied to a number of similar problems. For example,

$$f(x) := \sum_{k=2}^{\infty} (\log k) e^{-x\sqrt{k}},\; x > 0.$$

I get

$$f(x) = 4\,\frac{\log(1/x)}{x^2} + \frac{4(1-\gamma)}{x^2} + \frac{1}{2}\log(2\pi) + \mathrm{o}(1),\;\; x\to 0^+.$$

Problem #2: #20, page 107 of textbook.

[1] In doing residue calculations—like this one—I use Mathematica.