Math 205A

Homework #6—Due Feb 17, 2012

#1. Prove that if f is an entire function that satisfies

$$\sup_{|z|=R} \left|f(z)\right| \leq AR^k + B$$

for all $R > 0$, and for some positive integer $k \geq 0$ and some constants $A, B >$), then f is a polynomial of degree $\leq k$.

#2. Prove that for $|z| < 1$

$$(1+z)\,(1+z^2)\,(1+z^4)\,(1+z^8)\cdots = \prod_{k=0}^{\infty}(1+z^{2^k}) = \frac{1}{1-z}.$$

#3.

1. Please read the the product formula (3), page 142 of Stein & Shakarchi for $\sin \pi z/\pi$. (This will not be covered in the Monday-Wednesday lectures.)

2. Show that the sine product formula gives Wallis's formula (1655):

$$\pi = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots .$$

3. Let B_k denote the Bernoulli numbers, i.e.

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}\, z^k.$$

   Prove that

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k} B_{2k}}{2(2k)!}$$

   where $\zeta(s)$ is the Riemann zeta-function. For example,

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450}, \ldots .$$

   Hint:

(a) From the sine product formula first establish

$$e^u - 1 = e^{u/2} u \prod_{k=1}^{\infty} (1 + \frac{u^2}{4\pi^2 k^2})$$

(b) By logarithmic differentiation of this product followed by some algebraic manipulations show that

$$\frac{u}{e^u - 1} = 1 - \frac{u}{2} + \sum_{k=1}^{\infty} \frac{2u^2}{u^2 + 4\pi^2 k^2}$$

(c) Now obtain a power series in u of this last sum.

(d) Remark: There exist no simple formulas for $\zeta(2k+1)$, $k = 1, 2, \ldots$.