Math 205A

Homework #7—Due March 2, 2012

#1. Define the *Hurwitz zeta-function* by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \Re(s) > 1, \ \ 0 < a \leq 1.$$

Prove that $\zeta(s,a)$ extends to a holomorphic function in $\mathbb{C} - \{1\}$ and has a simple pole of residue one at $s = 1$.

#2. Problem #17, page 156 of Stein & Shakarchi.

Note: On the RHS the term $E'(z)$ appearing in the denominator should be replaced by $E'(0)$.

#3. Let $\omega_1, \omega_2 \in \mathbb{C}$, $\Im(\omega_2/\omega_1) > 0$ and set

$$\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

Define

$$\sigma(z) = \sigma(z, \Omega) := z \prod_{0 \neq \omega \in \Omega} E_2(\frac{z}{\omega}) \tag{1}$$

where $E_2(z)$ is the Weierstrass canonical factor

$$E_2(z) = (1-z)e^{z + z^2/2}.$$

For (1) to be well-defined we must show the product converges.

1. Let $U = \{(u,v) \in \mathbb{C}^2 : \Im(u/v) > 0\}$. Let $K \subset U$ be compact and let $\alpha > 2$. Prove there exists a bound $M > 0$ such that

   $$\sum_{0 \neq \omega \in \Omega} |\omega|^{-\alpha} \leq M \tag{2}$$

   for all $(\omega_1, \omega_2) \in K$.

   Useful linear algebra lemma: Let u, v be real variables and consider the quadratic form

   $$Q(u,v) = a\,u^2 + 2\,b\,u\,v + c\,v^2$$

Suppose that $Q(\cdot,\cdot)$ is *positive definite*; that is,

$$Q(u,v) > 0 \text{ for all } u, v, \ (u,v) \neq (0,0).$$

Then there exist positive constants λ_1 and λ_2 (depending upon a, b and c but not u and v) such that

$$\lambda_1(u^2+v^2) \leq Q(u,v) \leq \lambda_2(u^2+v^2).$$

If you wish to use this lemma, first prove it. Using this lemma prove there exist positive constants c_1 and c_2 such that

$$c_1\sqrt{m^2+n^2} \leq \left|m\omega_1 + n\omega_2\right| \leq c_2\sqrt{m^2+n^2}. \tag{3}$$

Use (3) to prove (2).

2. Show the convergence of (1) follows now from (2) and conclude $\sigma(z)$ is an entire function of z whose zero set is Ω. The entire function $\sigma(z,\Omega)$ is called the *Weierstrass sigma-function*.

3. Set

$$\wp(z) = \wp(z;\omega_1,\omega_2) := -\frac{d^2}{dz^2}\log\sigma(z) \tag{4}$$

and show that

$$\wp(z) = \frac{1}{z^2} + \sum_{0\neq\omega\in\Omega}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right). \tag{5}$$

4. Show that $\wp(z)$ is doubly periodic, i.e.

$$\wp(z+\omega) = \wp(z), \ \ \omega \in \Omega.$$

The elliptic function $\wp$ is called the *Weierstrass $\wp$-function.* See page 266 of textbook.

5. Remarks: The infinite product for $\sigma(z)$ is similar to that of $\sin z$ given by

$$\sin z = z\prod_{0\neq m\in\mathbb{Z}} E_1\left(\frac{z}{m\pi}\right)$$

and $\wp$ is similar to

$$\csc^2 z = -\frac{d^2}{dz^2}\log\sin z = \frac{1}{z^2} + \sum_{0\neq m\in\mathbb{Z}}\frac{1}{(z-m\pi)^2}.$$