Math 205A, HW#8, Due Friday, March 9, 2012
Elliptic Functions

Throughout this problem set, ω_1 and ω_2 are complex numbers with $\Im(\omega_2/\omega_1) > 0$ and Ω is the lattice

$$\Omega = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}.$$

We set $\tau = \omega_2/\omega_1$ and denote by $\wp = \wp(z) = \wp(z, \Omega)$ the Weierstrass elliptic function. Denote by Λ the fundamental parallelogram consisting of the four line segments $[0, \omega_1]$, $[\omega_1, \omega_1+\omega_2]$, $[\omega_1+\omega_2, \omega_2]$, $[\omega_2, 0]$ oriented in the counter clockwise direction. And $\mathbb{H}$ denotes the upper half-plane,

$$\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}.$$

1. Let β_1 and β_2 denote two points lying inside Λ; thus $\beta_2 - \beta_1 \notin \Lambda$. Suppose that $f = f(z)$ is an elliptic function with two (and only two) simple poles located at $z = \beta_1$ and $z = \beta_2$ with residues α_1 and α_2, respectively. Find an expression for $f(z)$ in terms of $\wp$.

   Hint: Consider the function
   $$\frac{A}{\wp(z-\beta) + B} + C \tag{1}$$
   where A, B, C and β do not depend upon z. How should these constants be chosen so as to construct $f(z)$?

2. Show that
   $$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$$
   where we assume $2z \notin \Omega$.

3. If we are given the curve
   $$x^2 + y^2 = 1 \tag{2}$$
   we can parametrize this curve by the use of trig functions; namely, if set $x = \cos z$ and $y = \sin z$ then for all complex z
   $$\cos^2 z + \sin^2 z = 1.$$
   Suppose instead of (2) we have the curve
   $$x^3 + y^3 = 1. \tag{3}$$
   Can we find functions that parametrize (3)? Show that if we set
   $$x = \frac{a + b\,\wp'(z)}{\wp(z)} \quad \text{and} \quad y = \frac{a - b\,\wp'(z)}{\wp(z)}$$

then for certain values of a and b these parametrizations satisfy (3). Find a and b in terms of g_2 and g_3.

4. Recall that

$$G_{2k}(\tau) = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^{2k}}, \quad k = 2, 3, 4, \ldots$$

   (a) Show that

$$G_{2k}(-\frac{1}{\tau}) = \tau^{2k} G_{2k}(\tau), \quad \Im(\tau) > 0. \tag{4}$$

   and

$$G_{2k}(\tau+1) = G_{2k}(\tau) \tag{5}$$

   (b) Using (4) and (5) show that

$$G_6(i) = 0$$

   and

$$G_4(\rho) = 0$$

   where $\rho = e^{2\pi i/3}$. Hint: $-\frac{1}{\rho} = \rho + 1$.

   (c) Show that $G_4(1, i)$ and $G_6(1, \rho)$ are real numbers.

   (d)* Show that

$$g_3(\rho) = \frac{4\pi^3}{729}\left[\frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})}\right]^6 = \frac{\Gamma(\frac{1}{3})^{18}}{(2\pi)^6} = 820.8244371\ldots$$