Ramanujan Summation Formula

Let

$$f(z;\mu,q) := \sum_{k=-\infty}^{\infty} \frac{q^k}{1-\mu q^k}\, z^k,\ z \neq 0. \tag{1}$$

We assume $0 < q < 1$, $|\mu| < 1$ and $\mu \neq q^k$, $k = 0, 1, 2, \ldots$.

1. As a function of z, show that f is holomorphic in the disk
$$1 < |z| < q^{-1}.$$
2. Show that f, again as a function of z, extends analytically to all $z \neq 0$ except for poles at $z = q^k$, $k \in \mathbb{Z}$.
3. The Ramanujan function ψ, traditionally denoted by ${}_1\psi_1$, is defined as
$$\psi(x;a,b) := \sum_{k=-\infty}^{\infty} \frac{(a;q)_k}{(b;q)_k}\, x^k$$
where
$$|\frac{b}{a}| < |x| < 1,\ |q| < 1 \text{ and } (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$
Setting
$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$
the Ramanujan summation formula is
$$\psi(x;a,b) = \frac{(b/a;q)_\infty (q;q)_\infty (q/ax;q)_\infty (ax;q)_\infty}{(b;q)_\infty (b/ax;q)_\infty (q/a;q)_\infty (x;q)_\infty} \tag{2}$$
Prove (2). You may want to consult (and fill in the details of) the paper[1]

   G. E. Andrews and R. Askey, *A simple proof of Ramanujan's summation of the* ${}_1\psi_1$, Aequationes Mathematicae **18** (1978), 333–337.

   Show, by a judicious choice of the parameters a, b and x, that Ramanujan's formula (2) implies that (1) has the product representation
$$\mu f(z;\mu,q) = \frac{1-\mu z}{(1-z)(1-\mu)} \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-zq^n)(1-z^{-1}q^n)} \prod_{n=1}^{\infty} \frac{(1-\mu z q^n)(1-(\mu z)^{-1}q^n)}{(1-\mu q^n)(1-\mu^{-1}q^n)} \tag{3}$$
From this conclude that in both variables z and μ, f has simple poles at q^n, $n \in \mathbb{Z}$.

[1] For a recent review of topics centered on the Ramanujan summation formula, the article by S. Ole Warnaar is recommended—see Math 205A homepage for link.