Craig A. Tracy Winter 2017

Mathematics 22B
The Final Examination, March 24, 2017

Instructions: **Work all four problems in your bluebook.** Only the bluebook will be collected.

$\star$ **Useful Information You May Assume as Given** $\star$

- For $a > 0$ and $b \in \mathbb{C}$
$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2+2bx}\, dx = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{b^2/(2a)}.$$
- The *heat kernel* for the line $\mathbb{R}$ is given by
$$K(x,y;t) = \frac{1}{\sqrt{4\pi t}}\, \mathrm{e}^{(x-y)^2/(4t)},\ \ x, y \in \mathbb{R},\ t > 0.$$
- For all integers $m, n = 1, 2, 3, \ldots$ and all $L > 0$
$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{1}$$

#1. (40 pts.) Suppose $u(x,t)$, $x \in \mathbb{R}$, $t > 0$, satisfies the heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
with initial condition
$$u(x,0) = \mathrm{e}^{-\alpha x^2},\ \ \alpha > 0,\ x \in \mathbb{R}.$$
Find *explicitly* the value of u at $x = 0$ for all $t > 0$; that is, find $u(0,t)$.

#2. (40 pts.) *Quantum Harmonic Oscillator:* We showed in class that if
$$H = -\frac{d^2}{dx^2} + x^2,$$
then the orthonormal eigenfunctions of the time-independent Schrödinger equation
$$H\psi_n = \varepsilon_n \psi_n,\ \ n = 0, 1, 2, \ldots$$
are given by
$$\psi_n(x) = \left[\sqrt{\pi}\, n!\, 2^n\right]^{-1/2} H_n(x)\, \mathrm{e}^{-x^2/2},\ \ n = 0, 1, 2, \ldots$$
where $H_n(x)$ are the Hermite polynomials and $\varepsilon_n = 2n + 1$.

It was proved (and you may assume as given) that for $n = 0, 1, 2, \ldots$
$$x\,\psi_n(x) = \sqrt{\frac{n}{2}}\psi_{n-1}(x) + \sqrt{\frac{n+1}{2}}\psi_{n+1}(x), \tag{2}$$
$$\frac{d}{dx}\psi_n(x) = \sqrt{\frac{n}{2}}\psi_{n-1}(x) - \sqrt{\frac{n+1}{2}}\psi_{n+1}(x). \tag{3}$$
Here $\psi_{-1}(x) \equiv 0$.

- Find an expression for $x^2\,\psi_n(x)$, $n = 0, 1, 2, \ldots$ that is the analogue of (2). Hint: Apply x to (2) and then use (2) again.
- Compute for all $n = 0, 1, 2, \ldots$

$$\left(x^4\psi_n, \psi_n\right) := \int_{-\infty}^{\infty} x^4\,\left(\psi_n(x)\right)^2\,dx.$$

Hint: First note this is equivalent to computing $\left(x^2\psi_n, x^2\psi_n\right)$ (why?). Now use your result from the previous part and the fact that $\{\psi_n\}_{n\geq 0}$ is an *orthonormal basis.*

#3. (40 pts.) There are three tanks that hold water. Tank I flows into Tank II at rate $\lambda_1 > 0$. Tank II flows into Tank III at rate $\lambda_2 > 0$ and Tank III flows into Tank I at rate $\lambda_3 > 0$. If $x_1(t)$ denotes the amount of water in Tank I at time t, $x_2(t)$ the amount of water in Tank II at time t and similarly for $x_3(t)$, then the system is modeled (in dimensionless units) by the system of differential equations

$$\begin{aligned}
\frac{dx_1}{dt} &= -\lambda_1\,x_1(t) + \lambda_3\,x_3(t), \\
\frac{dx_2}{dt} &= \lambda_1\,x_1(t) - \lambda_2\,x_2(t), \\
\frac{dx_3}{dt} &= \lambda_2\,x_2(t) - \lambda_3\,x_3(t).
\end{aligned}$$

- Define the column vector

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \in \mathbb{R}^3.$$

Find a 3×3 matrix A so that the matrix equation

$$\frac{dx}{dt} = Ax$$

is equivalent to the system of three equations above.

- For the special case $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 1$, the eigenvalues of the matrix A found in the previous part are 0, $-2 + \mathrm{i}$ and $-2 - \mathrm{i}$. We now use our linear algebra algorithms to compute $\exp(tA)$. To make the results easier to read, define

$$f_c(t) = \mathrm{e}^{-2t}\cos t \quad\text{and}\quad f_s(t) = \mathrm{e}^{-2t}\sin t.$$

Then we have

$$\exp(tA) = \begin{pmatrix} \frac{2}{5} + \frac{3}{5}f_c(t) + \frac{1}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) - \frac{4}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) + \frac{1}{5}f_s(t) \\ \frac{1}{5} - \frac{1}{5}f_c(t) + \frac{3}{5}f_s(t) & \frac{1}{5} + \frac{4}{5}f_c(t) - \frac{2}{5}f_s(t) & \frac{1}{5} - \frac{1}{5}f_c(t) - \frac{2}{5}f_s(t) \\ \frac{2}{5} - \frac{2}{5}f_c(t) - \frac{4}{5}f_s(t) & \frac{2}{5} - \frac{2}{5}f_c(t) + \frac{6}{5}f_s(t) & \frac{2}{5} + \frac{3}{5}f_c(t) + \frac{1}{5}f_s(t) \end{pmatrix}. \tag{4}$$

- If the initial condition is

$$x(0) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{(50\% of the water is initially in Tank } I \text{ and 50\% in Tank } II\text{)} \tag{5}$$

find $x(t)$. Express your answer in terms of $f_c(t)$ and $f_s(t)$. Hint: You should be able to find $x(t)$ with minimal computations.

#4. (80 pts.) Consider the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

In class we solved this equation on the real line $\mathbb{R}$, the half-line $\mathbb{R}^+$ and the circle S. In this problem you are asked to solve the heat equation on the line segment $[0, L]$ subject to the boundary conditions

$$u(0,t) = 0 \;\text{ and }\; u(L,t) = 0 \;\text{ for all }\; t > 0 \tag{6}$$

with initial condition

$$u(x,0) = f(x) \;\text{ for }\; 0 < x < L. \tag{7}$$

We assume $f(x)$ is a continuous function with $f(0) = f(L) = 0$.

- If we assume a solution of the form (separate variables)

  $$u(x,t) = X(x)T(t),$$

  find ODEs that $X(x)$ and $T(t)$ must satisfy. Solve these ODEs. (The arithmetic is a bit easier if you call the separation constant $-k^2$.)
- Using the solutions found in the first part, apply the boundary conditions (6) and find the allowed values of the constant k. This should give you a sequence of solutions $u_n(x,t) = X_n(x)T_n(t)$.
- Show that the solution $u(x,t)$ satisfying the boundary conditions (6) and the initial condition (7) can be written as

  $$u(x,t) = \int_0^L K_L(x,y;t) f(y)\, dy \tag{8}$$

  where

  $$K_L(x,y;t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)\, \sin(\frac{n\pi}{L}y)\; \mathrm{e}^{-(\pi^2 n^2/L^2)t} \tag{9}$$

- We write (8) symbolically as an operator $\mathbb{K}_t$ acting on functions f by

  $$(\mathbb{K}_t f)(x) := \int_0^L K_L(x,y;t) f(y)\, dy.$$

  Show that for $t > 0$ and $s > 0$ that

  $$\mathbb{K}_{t+s} = \mathbb{K}_t \mathbb{K}_s,$$

  that is, show

  $$(\mathbb{K}_{t+s} f)(x) = (\mathbb{K}_t \mathbb{K}_s f)(x) \tag{10}$$

  for all f satisfying the above given conditions.
  Hint: First show that (10) is equivalent to showing

  $$K_L(x,y;t+s) = \int_0^L K_L(x,z;t) K_L(z,y;s)\, dz. \tag{11}$$

  Then show (11) follows from (9). See also helpful information (1).

End of Examination