 Instructions: Work all four problems in your bluebook. Only the bluebook will be collected.

 Useful Information You May Assume as Given

- For \( a > 0 \) and \( b \in \mathbb{C} \)
  \[
  \int_{-\infty}^{\infty} e^{-ax^2+2bx} \, dx = \sqrt{\frac{\pi}{a}} e^{b^2/(2a)}.
  \]

- The heat kernel for the line \( \mathbb{R} \) is given by
  \[
  K(x, y; t) = \frac{1}{\sqrt{4\pi t}} e^{(x-y)^2/(4t)}, \quad x, y \in \mathbb{R}, \ t > 0.
  \]

- For all integers \( m, n = 1, 2, 3, \ldots \) and all \( L > 0 \)
  \[
  \frac{2}{L} \int_0^L \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{L} x\right) \, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
  \tag{1}
  \]

#1. (40 pts.) Suppose \( u(x, t), x \in \mathbb{R}, \ t > 0, \) satisfies the heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

with initial condition

\[
u(x, 0) = e^{-\alpha x^2}, \quad \alpha > 0, \ x \in \mathbb{R}.
\]

Find explicitly the value of \( u \) at \( x = 0 \) for all \( t > 0 \); that is, find \( u(0, t) \).

#2. (40 pts.) Quantum Harmonic Oscillator: We showed in class that if

\[
H = -\frac{d^2}{dx^2} + x^2,
\]

then the orthonormal eigenfunctions of the time-independent Schrödinger equation

\[
H \psi_n = \varepsilon_n \psi_n, \quad n = 0, 1, 2, \ldots
\]

are given by

\[
\psi_n(x) = \left[\sqrt{\pi} n! 2^n\right]^{-1/2} H_n(x) e^{-x^2/2}, \quad n = 0, 1, 2, \ldots
\]

where \( H_n(x) \) are the Hermite polynomials and \( \varepsilon_n = 2n + 1 \).

It was proved (and you may assume as given) that for \( n = 0, 1, 2, \ldots \)

\[
\begin{align*}
x \psi_n(x) &= \sqrt{\frac{n}{2}} \psi_{n-1}(x) + \sqrt{\frac{n+1}{2}} \psi_{n+1}(x), \\
\frac{d}{dx} \psi_n(x) &= \sqrt{\frac{n}{2}} \psi_{n-1}(x) - \sqrt{\frac{n+1}{2}} \psi_{n+1}(x).
\end{align*}
\tag{2}
\tag{3}
\]

Here \( \psi_{-1}(x) \equiv 0. \)
Find an expression for $x^2 \psi_n(x)$, $n = 0, 1, 2, \ldots$ that is the analogue of (2). Hint: Apply $x$ to (2) and then use (2) again.

Compute for all $n = 0, 1, 2, \ldots$

$$ (x^4 \psi_n, \psi_n) := \int_{-\infty}^{\infty} x^4 (\psi_n(x))^2 \, dx. $$

Hint: First note this is equivalent to computing $(x^2 \psi_n, x^2 \psi_n)$ (why?). Now use your result from the previous part and the fact that $\{\psi_n\}_{n \geq 0}$ is an orthonormal basis.

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**#3. (40 pts.)** There are three tanks that hold water. Tank $I$ flows into Tank $II$ at rate $\lambda_1 > 0$. Tank II flows into Tank $III$ at rate $\lambda_2 > 0$ and Tank III flows into Tank $I$ at rate $\lambda_3 > 0$. If $x_1(t)$ denotes the amount of water in Tank $I$ at time $t$, $x_2(t)$ the amount of water in Tank $II$ at time $t$ and similarly for $x_3(t)$, then the system is modeled (in dimensionless units) by the system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= -\lambda_1 x_1(t) + \lambda_3 x_3(t), \\
\frac{dx_2}{dt} &= \lambda_1 x_1(t) - \lambda_2 x_2(t), \\
\frac{dx_3}{dt} &= \lambda_2 x_2(t) - \lambda_3 x_3(t).
\end{align*}
\]

- Define the column vector

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \in \mathbb{R}^3.
\]

Find a $3 \times 3$ matrix $A$ so that the matrix equation

\[
\frac{dx}{dt} = Ax
\]

is equivalent to the system of three equations above.

- For the special case $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 1$, the eigenvalues of the matrix $A$ found in the previous part are $0$, $-2 + i$ and $-2 - i$. We now use our linear algebra algorithms to compute $\exp(tA)$. To make the results easier to read, define $f_e(t) = e^{-2t} \cos t$ and $f_s(t) = e^{-2t} \sin t$.

Then we have

\[
\exp(tA) = \begin{pmatrix}
\frac{2}{5} + \frac{2}{5} f_e(t) + \frac{1}{5} f_s(t) & \frac{2}{5} - \frac{2}{5} f_e(t) - \frac{4}{5} f_s(t) & \frac{2}{5} - \frac{2}{5} f_e(t) + \frac{4}{5} f_s(t) \\
\frac{1}{5} - \frac{1}{5} f_e(t) + \frac{3}{5} f_s(t) & \frac{1}{5} + \frac{1}{5} f_e(t) - \frac{2}{5} f_s(t) & \frac{1}{5} - \frac{1}{5} f_e(t) - \frac{2}{5} f_s(t) \\
\frac{2}{5} - \frac{2}{5} f_e(t) - \frac{4}{5} f_s(t) & \frac{2}{5} - \frac{2}{5} f_e(t) + \frac{4}{5} f_s(t) & \frac{2}{5} + \frac{4}{5} f_e(t) + \frac{2}{5} f_s(t)
\end{pmatrix}.
\]

(4)

- If the initial condition is

\[
x(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (50\% \text{ of the water is initially in Tank } I \text{ and } 50\% \text{ in Tank } II)
\]

find $x(t)$. Express your answer in terms of $f_e(t)$ and $f_s(t)$. Hint: You should be able to find $x(t)$ with minimal computations.
#4. (80 pts.) Consider the one-dimensional heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.
\]

In class we solved this equation on the real line \( \mathbb{R} \), the half-line \( \mathbb{R}^+ \) and the circle \( S \). In this problem you are asked to solve the heat equation on the line segment \([0, L]\) subject to the boundary conditions

\[
\begin{align*}
u(0, t) &= 0 & \text{and} & & u(L, t) = 0 & \text{for all} & & t > 0 \\
u(x, 0) &= f(x) & \text{for} & & 0 < x < L.
\end{align*}
\]  

(6)

with initial condition

\[
u(x, 0) = f(x) \quad \text{for} \quad 0 < x < L.
\]  

(7)

We assume \( f(x) \) is a continuous function with \( f(0) = f(L) = 0 \).

• If we assume a solution of the form (separate variables)

\[
u(x, t) = X(x)T(t),
\]

find ODEs that \( X(x) \) and \( T(t) \) must satisfy. Solve these ODEs. (The arithmetic is a bit easier if you call the separation constant \( -k^2 \).)

• Using the solutions found in the first part, apply the boundary conditions (6) and find the allowed values of the constant \( k \). This should give you a sequence of solutions \( u_n(x, t) = X_n(x)T_n(t) \).

• Show that the solution \( u(x, t) \) satisfying the boundary conditions (6) and the initial condition (7) can be written as

\[
u(x, t) = \int_0^L K_L(x, y; t) f(y) \, dy \quad \text{(8)}
\]

where

\[
K_L(x, y; t) = \frac{2}{L} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{n\pi}{L} y\right) e^{-\left(\frac{\pi^2 n^2}{L^2}\right)t} \quad \text{(9)}
\]

• We write (8) symbolically as an operator \( \mathcal{K}_t \) acting on functions \( f \) by

\[
(\mathcal{K}_t f)(x) := \int_0^L K_L(x, y; t) f(y) \, dy.
\]

Show that for \( t > 0 \) and \( s > 0 \) that

\[
\mathcal{K}_{t+s} = \mathcal{K}_t \mathcal{K}_s,
\]

that is, show

\[
(\mathcal{K}_{t+s} f)(x) = (\mathcal{K}_t \mathcal{K}_s f)(x) \quad \text{(10)}
\]

for all \( f \) satisfying the above given conditions.

Hint: First show that (10) is equivalent to showing

\[
K_L(x, y; t + s) = \int_0^L K_L(x, z; t) K_L(z, y; s) \, dz. \quad \text{(11)}
\]

Then show (11) follows from (9). See also helpful information (1).