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Fredholm determinants, differential equations and matrix models. (English summary)

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FEATURED REVIEW.the pioneering work in the late 50's and early 60's of E. P. Wigner, F. J. Dyson and others, the eigenvalues of random matrices have been used to model the statistical properties of the energy levels of classically chaotic quantum systems. Basic quantities in the statistical analysis of the energy levels are the probabilities $E(n;J)$ that there are exactly n levels in the interval (or union of intervals) J. In particular, the distribution of the spacing between consecutive levels can be obtained from $E(0;J)$ by differentiation. For a large class of orthogonal polynomial random matrix models of $N\times N$ Hermitian matrices with unitary symmetry, as well as scaling limits of these models, the formula

$$E(n;J)=\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}\det(1-\lambda K)|_{\lambda=1},$$

where K is the integral operator on J with kernel

$$K(x,y)=\frac{\varphi(x)\psi(y)-\psi(x)\varphi(y)}{x-y}, \tag{1}$$

holds [see, e.g., M. L. Mehta, *Random matrices*, second edition, Academic Press, Boston, MA, 1991; MR1083764 (92f:82002)].

The paper under review is concerned with Fredholm determinants of integral operators having kernel of this form with emphasis on the determinants thought of as functions of the endpoints of $J=\bigcup_{j=1}^m(a_{2j-1},a_{2j})$. In the special case of the bulk scaling limit of the Gaussian unitary ensemble (GUE)

$$K(x,y)=\frac{\sin(x-y)}{x-y},$$

a characterisation of this type was first given by M. Jimbo et al. [Phys. D **1** (1980), no. 1, 80–158;

MR0573370 (84k:82037)]. (The GUE is the probability space of $N \times N$ Hermitian matrices with independent, complex Gaussian, mean zero elements. The bulk scaling limit is the limit $N \to \infty$ where distance is rescaled to make the mean spacing between consecutive eigenvalues in the bulk of the spectrum one.) Furthermore, in the case of a single interval of length s these PDE's were shown to reduce to a Painlevé equation of the fifth kind for the quantity

$$\sigma(s;\lambda) = -s\frac{d}{ds}\log\det(1-\lambda K).$$

Tracy and Widom significantly generalize the results of Jimbo et al. [op. cit.] to any Fredholm determinant with kernel of the type (1) for which φ and ψ satisfy a linear differential equation of the form

$$\frac{d}{dx}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \Omega(x)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}, \tag{2}$$

where $\Omega(x)$ is a 2×2 traceless matrix with rational entries. They show that the (φ,ψ) pairs arising from orthogonal polynomial Hermitian matrix models with weight functions $w(x) = \exp(-V(x))$, $-\infty < x < \infty$, $w(x) = x^\alpha \exp(-V(x))$, $0 < x < \infty$, and $w(x) = (1-x)^\alpha(1+x)^\beta \exp(-V(x))$, $-1 < x < 1$, V polynomial, satisfy (2) with explicit formulas given for the matrix elements of Ω. (The simplest cases are GUE with $V(x) = x^2$, Laguerre with $V(x) = x$, and Jacobi with $V(x) = 1$, respectively.) Furthermore, they show that the (φ,ψ) pairs in the "edge scaling limits" of GUE, Laguerre and Jacobi and the "double scaling limit" of 2D matrix models of quantum gravity satisfy (2). Thus their PDE's apply to a large class of problems of current interest. In particular, they obtain the distribution function for the largest eigenvalue of finite N GUE in terms of Painlevé IV and the limiting distribution for the scaled largest eigenvalue in terms of Painlevé II. (Further details of this last case are in the earlier paper [C. A. Tracy and H. Widom, Comm. Math. Phys. **159** (1994), no. 1, 151–174; see the preceding second review].)

There is also an exponential variant of the kernel in which the denominator in (1) is replaced by $e^{bx} - e^{by}$, where b is an arbitrary complex number. The authors find an analogous system of differential equations in this setting. If $b = i$ then one can interpret this operator as acting on the unit circle in the complex plane. As an application of this the authors write down a system of PDE's for Dyson's circular ensemble of $N \times N$ unitary matrices, and then an ODE if J is an arc of the circle.

Another significant feature of the present work, in addition to the generality of the final results, is the accessibility of the derivation to the non-expert. Indeed, the derivation is self-contained and is based mostly on some simple operator formulas. These features make a detailed study of the work—something I strongly recommend to those with interests in random matrices or applications of nonlinear equations—very feasible.

Reviewed by *Peter J. Forrester*