Symmetry group for a completely symmetric vertex model†

Matthew P Richey and Craig A Tracy
Department of Mathematics, University of California, Davis, CA 95616, USA

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Abstract. The symmetry group for the partition function of the completely X symmetric vertex model is determined. Here X is an arbitrary finite Abelian group.

1. Introduction

Let Λ be a square lattice with M rows and N columns. We assume periodic boundary conditions. Let X be any finite Abelian group. A configuration ω is defined by assigning elements of X to the bonds of Λ; let ΩX be the set of all such configurations ω. To each vertex with bond states x, y, u, v ∈ X is assigned a Boltzmann weight Sxyuv. The weight of a configuration ω ∈ ΩX is defined to be the product over all lattice sites in Λ of the individual Boltzmann weights. Thus the partition function ZX is given by

\[ Z_X = \sum_{\omega \in \Omega_X} \prod_{x,y \in \Lambda} S_{x,y}^{\omega_{xy}}. \]  

(1.1)

Following Belavin (1981) and Chudnovsky and Chudnovsky (1981), we say such a vertex model is completely X symmetric if the Boltzmann weights satisfy

\[ S_{xy}^{uv} = 0 \quad \text{unless } x + y = u + v \]  

(1.2a)

and

\[ S_{x+y+z}^{u+v+z} = S_{x,y}^{u,v} \quad \text{for every } x, y, u, v, z \in X \]  

(1.2b)

where we have written the group law of X in additive notation. For the case X = Z/2Z = Z2 the above vertex model is Baxter's symmetric eight-vertex model (Baxter 1972, 1982). If |X| denotes the order of X, then this X symmetric model depends upon |X|^2 independent parameters (one of these being an overall normalisation). We write ZΩ[S] to denote the dependence of ZΩ upon the Boltzmann weights Sxyuv.

For X = Z2 Fan and Wu (1970) (see also Fan 1972, Johnson et al 1973) discovered that ZΩ has certain symmetries. To state their results, we write S as

\[ S = \sum_{j=0}^{3} w_j \sigma^j \otimes \sigma^j \]  

(1.3)

where \( \sigma^0 \) is the 2 × 2 identity matrix and \( \sigma^j, j = 1, 2, 3 \), are the Pauli spin matrices. Then Fan and Wu proved

\[ Z_\Omega[\pm w_0, \pm w_1, \pm w_2, \pm w_3] = Z_\Omega[w_0, w_1, w_2, w_3] \]  

(1.4)

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where \((i_0, i_1, i_2, i_3)\) is any permutation of \((0, 1, 2, 3)\) and \(\pm\) denotes any choice of signs. This symmetry is important in Baxter’s solution of the symmetric eight-vertex model (Baxter 1972, 1982). It is the purpose of this paper to determine the symmetry group for \(Z_n[S]\) when \(X\) is any finite Abelian group and \(S\) is completely \(X\) symmetric.

As was first emphasised by Belavin (1981) and Chudnovsky and Chudnovsky (1981) (see also Bovier 1983, Cherednik 1982), the Heisenberg group is important in any discussion of the \(X\) symmetric model. To establish notation and to make the paper self-contained, we collect in § 2 those results we need concerning the Heisenberg group. Our principal mathematical references are Mumford (1983) and Siegel (1971). We have included a self-contained proof of the well known result in lemma 2.1. Presumably lemma 2.2 is also well known but we could not find any explicit references. In § 3 we determine the symmetry properties of \(Z_n[S]\) following the methods of Richey and Tracy (1986) where the case \(X = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n\) was first considered. These authors in turn built on the work of Fan and Wu (1970), Fan (1972), Wegner (1971, 1973) and Johnson et al (1973). In particular, the idea to use the similarity transformation in the proof of theorem 3.1 can be traced back to these early papers. Here, we have shown the generality of this method and have clarified the role the Heisenberg group plays in the symmetries. In § 4 we discuss the connection between theorem 3.1 for \(X = \mathbb{Z}_n\) and previous work (Richey and Tracy 1986). The results obtained here require no restrictions on the Boltzmann weights \(S_n^{\alpha}\). Normally restrictions on multistate vertex models arise when one demands commutativity of the transfer matrices. Further discussion of the relationship between symmetries and commutativity of transfer matrices can be found, for the case \(X = \mathbb{Z}_n\), in Richey and Tracy (1986).

2. Heisenberg group for a finite Abelian group

As is well known any finite Abelian group \(X\), with \(|X| = n\), is isomorphic to
\[
X = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r} \tag{2.1}
\]

where \(n = n_1 \ldots n_r\), \(n_i\) a prime power. Also distinct choices of the \(n_i\) result in distinct (non-isomorphic) finite Abelian groups. If \(X = \mathbb{Z}_n\), then the decomposition \(n = n_1 \ldots n_r\) is the prime decomposition of \(n\) with each of the primes distinct. In this case, the isomorphism \(\mathbb{Z}_n \rightarrow \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}\) is commonly referred to as the Chinese remainder theorem (CRT).

An element \(\alpha'\) of \(X\) may be identified with an \(r\)-tuple \(\alpha' = (\alpha'_{i_1}, \ldots, \alpha'_{i_r})\), \(\alpha'_{i_i} \in \mathbb{Z}_{n_i} = \{0, 1, \ldots, n_i - 1\}\). Let \(\mathbb{Z}_n^2 = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_1}\) and \(G_n = \mathbb{Z}_n^2 \times \ldots \times \mathbb{Z}_n^2\). Thus an element \(\alpha \in G_n\) can be written as
\[
\alpha = (\alpha', \alpha'') \quad \alpha', \alpha'' \in X = \mathbb{Z}_{n_1} \oplus \ldots \oplus \mathbb{Z}_{n_r}
\]
or
\[
\alpha = (\alpha_1, \ldots, \alpha_r) \quad \alpha_i = (\alpha'_{i_1}, \alpha'_{i_2}) \in \mathbb{Z}_n^2 \tag{2.2}
\]
or
\[
\alpha = (\alpha'_1, \ldots, \alpha'_i, \alpha''_i, \ldots, \alpha''_r) \quad \alpha'_i, \alpha''_i \in \mathbb{Z}_{n_i}.
\]
The method of expressing \(\alpha\) will depend upon the context.

We now define a Heisenberg group associated with \(X\). This could be defined abstractly, but we prefer to give first the ‘coordinate version’ as it proves useful in the
statistical mechanics. For an arbitrary positive integer \( n_k (>1) \) we define operators \( g_k \) and \( h_k \) by

\[
\begin{align*}
g_k e_j &= \omega_k^j e_j & j &= 0, 1, \ldots, n_k - 1 \\
h_k e_j &= e_{j+1}
\end{align*}
\]

(2.3)

where \( \{e_j\}_{j=0}^{n_k} \) is the standard basis of \( \mathbb{C}^{n_k} \) and \( \omega_k = \exp(2\pi i/n_k) \). Let

\[
I_{\alpha_k} = h_k^{\alpha_k^*} g_k^{\alpha_k^*} & \quad \alpha_k = (\alpha_k^*, \alpha_k^*) \in \mathbb{Z}^{2}_{n_k}
\]

(2.4)

then

\[
I_{\alpha_k} I_{\beta_k} = \omega_k^{(\beta_k^*, \alpha_k^*)} I_{\beta_k} I_{\alpha_k}
\]

(2.5)

with \( (\beta_k, \alpha_k) = \beta_k^* \alpha_k^* - \beta_k^* \alpha_k^* \). Let \( H_{n_k} \) be the group of operators generated by \( I_{\alpha_k} \), \( \alpha_k \in \mathbb{Z}^{2}_{n_k} \). Then \( H_{n_k} \) is the Heisenberg group associated with \( \mathbb{Z}_{n_k} \). For \( X \) identified as in (2.1) we define the associated Heisenberg group \( H_X \) by

\[
H_X = H_{n_1} \otimes \ldots \otimes H_{n_r}.
\]

(2.6)

We write

\[
I_\alpha = I_{\alpha_1} \otimes \ldots \otimes I_{\alpha_r} \quad \alpha = (\alpha_1, \ldots, \alpha_r), \alpha_i \in \mathbb{Z}^{2}_{n_i}
\]

(2.7)

and observe for all \( \alpha, \beta \in X \times X \)

\[
I_\alpha I_\beta = \omega^{k(\beta, \alpha)} I_{\alpha + \beta} = \omega^{B(\beta, \alpha)} I_\beta I_\alpha
\]

(2.8)

where

\[
k(\beta, \alpha) = \sum_{i=1}^r \frac{n_i}{n} \alpha_i^* \beta_i^*
\]

(2.9a)

\[
B(\beta, \alpha) = k(\beta, \alpha) - k(\alpha, \beta)
\]

\[
= \sum_{i=1}^r \frac{n}{n_i} (\beta_i^* \alpha_i^* - \beta_i^* \alpha_i^*)
\]

(2.9b)

and \( \omega = \exp(2\pi i/n) \).

**Lemma 2.1.** The action of \( H_X \) on \( \mathbb{C}^n \) is irreducible.

**Proof.** Let \( x \) be an arbitrary non-zero element of \( \mathbb{C}^n \)

\[
x = \sum_{i_1, \ldots, i_r \in \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_r}} c_{i_1 \ldots i_r} e_{i_1} \otimes \ldots \otimes e_{i_r}.
\]

We will show that the action of \( H_X \) on \( x \) spans \( \mathbb{C}^n \). If we can show that \( e_0 \otimes \ldots \otimes e_0 \) is in the span of \( H_X \) on \( x \), then we are done since

\[
h_0^1 \otimes \ldots \otimes h_0^{i_r} e_0 \otimes \ldots \otimes e_0 = e_0 \otimes \ldots \otimes e_0.
\]

Since \( x \neq 0 \) one of the coefficients \( c_i = c_{i_1 \ldots i_r} \neq 0 \). Applying \( h_0^{-1} \otimes \ldots \otimes h_0^{-i_r} \) to \( x \) allows us to assume that \( c_0 = c_0 \ldots \neq 0 \). Now if

\[
x_\beta := g_0^{i_1} \otimes \ldots \otimes g_0^{i_r} x
\]

\[
= \sum_{i} c_i \omega_{\beta_1}^{i_1} \ldots \omega_{\beta_r}^{i_r} e_0 \otimes \ldots \otimes e_0
\]

then

\[
\sum_{\beta_1, \ldots, \beta_r} x_\beta = nc_0 e_0 \otimes \ldots \otimes e_0
\]

since \( \sum_{\beta_i} \omega_{\beta_i}^{i_0} = n_k \delta_{0, i_0} \). Since \( c_0 \neq 0 \), \( e_0 \) is in the subspace spanned by \( H_X \) on \( x \); and hence the lemma has been proved.
Let $E$ denote the $r \times r$ diagonal matrix with diagonal entries $n/n_i$ and define the $2r \times 2r$ matrix $J$ by

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}.$$  

(2.10)

Then in obvious matrix notation we can write (2.9) as

$$k(\alpha, \beta) = (\alpha' E \beta')$$  

(2.11a)

and

$$B(\alpha, \beta) = (\alpha' \beta)$$  

(2.11b)

where $\alpha = (\alpha', \alpha'')$, $\beta = (\beta', \beta'') \in G_n$ (and are written as column vectors).

It will be convenient to introduce the Siegel upper half-space $\mathcal{H}$, which consists of all $r \times r$ complex symmetric matrices whose imaginary parts are positive definite. Let $\Omega \in \mathcal{H}$, denote by $\Omega_1, \ldots, \Omega_r$ the columns of $\Omega$ and $e_1, \ldots, e_r$ the standard basis of $\mathbb{C}^r$. Then to each $\alpha \in G_n$ we associate a point on the lattice

$$L_{\Omega} = \Omega \mathbb{Z}^r + E \mathbb{Z}^r$$  

(2.12)

by

$$\alpha \mapsto \alpha_1 \Omega_1 + \ldots + \alpha_r \Omega_r + \alpha_1' (n/n_1) e_1 + \ldots + \alpha_r' (n/n_r) e_r,$$  

(2.13a)

or, more compactly,

$$\alpha \mapsto \Omega \alpha' + E \alpha''.$$  

(2.13b)

This mapping forms a natural association of elements $I_n$ of $H_X$ with lattice points of $L_{\Omega}$. In particular, changing generators of $H_X$ is equivalent to changing generators of $L_{\Omega}$. Consider the group of matrices $\Gamma_E$ defined by

$$\Gamma_E = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n : A, B, C, D, r \times r \text{ matrices with integer entries and } \gamma J \gamma = J \right\}.$$  

(2.14)

Note for $E = I$, $\Gamma_E = \text{Sp}(2r, \mathbb{Z})$, and for any $\gamma \in \Gamma_E$, $\det \gamma = \pm 1$. Let

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \Omega \\ E \end{pmatrix}$$

where $\gamma = (\begin{pmatrix} P \\ Q \end{pmatrix}) \in \Gamma_E$, then $(P, Q)$ are new generators of $L_{\Omega}$. To see this, we need only show that for any $\alpha = \Omega \alpha' + E \alpha'' \in L_{\Omega}$, there is a $\xi = (\xi', \xi'') \in G_n$ such that

$$\alpha = (\xi', \xi'')$$

$$= (\Omega' A + E' B) \xi' + (\Omega' C + E' D) \xi''$$

$$= \Omega (A \xi' + C \xi'') + E (B \xi' + D \xi'').$$

That is,

$$\begin{pmatrix} \alpha' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} \xi' \\ \xi'' \end{pmatrix} = \gamma \xi.$$

Since $\Gamma_E$ is a group, $\xi = \gamma^{-1} \alpha$ is an element of $G_n$. For $\gamma \in \Gamma_E$ let $\gamma_1, \ldots, \gamma_r, \delta_1, \ldots, \delta_r$ denote the $2r$ columns of $\gamma$. To each $\gamma_i, \delta_i$, we associate an element of $G_n$ by $\gamma_i = (\gamma_i, \gamma_i')$.
where \( \gamma' \) consists of the first \( r \) rows of \( \gamma \) and \( \gamma'' \) consists of the last \( r \) rows, and similarly for \( \delta_k = (\delta'_k, \delta''_k) \).

For any \( \gamma \in \Gamma_E \) we define a family of Heisenberg elements
\[
I^\gamma = (I_{\eta_1})^{\alpha_1} \cdots (I_{\eta_n})^{\alpha_n}(I_{\eta_1})^\gamma \cdots (I_{\eta_n})^\gamma \quad \alpha \in G_n.
\]
(2.15)

It is easy to check that
\[
I_{\rho}^\gamma = \omega^\rho I_{\eta_\rho}, \quad \rho \in \mathbb{Z}.
\]
(2.16)

Then we have the following lemma.

**Lemma 2.2.** For each \( \gamma \in \Gamma_E \) there exists an invertible matrix \( U_\gamma : \mathbb{C}^n \to \mathbb{C}^n \) such that for all \( \alpha \in G_n \)
\[
I^\gamma = \lambda U_\gamma I_\alpha U_\gamma^{-1}
\]
where \( \lambda \) is some scalar depending on \( \alpha \).

**Proof.** Let \( \gamma \in \Gamma_E \) and denote \( \xi_k \) the \( 2r \times 1 \) column vector with all zeros except for a 1 at the \( k \)th row.

Then for all \( k, l = 1, \ldots, r \)
\[
B(\gamma_k, \gamma_l) = B(\gamma_{k+}, \gamma_{l+}) = 0
\]
\[
B(\delta_k, \delta_l) = B(\eta_{k+}, \eta_{l+}) = 0
\]
\[
B(\gamma_k, \delta_l) = B(\eta_k, \eta_{l+}) = (1/\eta_k) \delta_{k,l}.
\]

Thus
\[
I_{\gamma_k}I_{\gamma_l} = I_{\gamma_k}I_{\gamma_l}, \quad \text{if } k \neq l \tag{2.17a}
\]
\[
I_{\delta_k}I_{\delta_l} = I_{\delta_k}I_{\delta_l} \quad \text{if } k = l \tag{2.17b}
\]

and
\[
I_{\gamma_k}I_{\delta_l} = \begin{cases} I_{\delta_l}I_{\gamma_k} & \text{if } k \neq l \\ \omega_k^{\gamma_k}I_{\delta_l}I_{\gamma_k} & \text{if } k = l \end{cases}
\]

Now let \( x_0 \) be any common eigenvector of the \( I_{\delta_l}, l = 1, \ldots, r \):
\[
I_{\delta_l}x_0 = \lambda_l x_0.
\]

Define \( x_j = x_{j_1} \cdots x_{j_r} \) by
\[
x_j = (I_{\gamma_1})^{j_1} \cdots (I_{\gamma_n})^{j_n} x_0 \quad \text{if } j \in \mathbb{Z}_m.
\]

Then using (2.17) we see that
\[
I_{\delta_1}^{j_1} \cdots I_{\delta_r}^{j_r} x_{j_1} \cdots x_{j_r} = \lambda_1^{j_1} \cdots \lambda_r^{j_r} \omega^{\gamma_1+\cdots+(n/\eta_{j_r}) j_r} x_{j_1} \cdots x_{j_r}
\]
and
\[
I_{\gamma_1}^{j_1} \cdots I_{\gamma_n}^{j_n} x_{j_1} \cdots x_{j_r} = x_{j_1} \cdots x_{j_r}.
\]

Since the action of \( H_X \) on \( \mathbb{C}^n \) is irreducible, we conclude that the set \( \{x_{j_1} \cdots x_{j_r} : (j_1, \ldots, j_r) \in \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n\} \) is a basis for \( \mathbb{C}^n \). Let \( U_j \) be the change of basis matrix
\[
U_j e_j = x_j \quad j = (j_1, \ldots, j_r).
\]
Then
\[
U^{-1}_\gamma I_{n} U_\gamma e_j = U^{-1}_\gamma I_{n} x_j
= U^{-1}_\gamma \lambda_i x_i^{n/n_i} x_j
= \lambda_i r^{n_i} e_j
\]
and
\[
U^{-1}_\gamma I_{n} U_\gamma e_i = U^{-1}_\gamma I_{n} x_j
= U^{-1}_\gamma x_{j_1, \ldots, j_{k+1}}
= e_{j_1, \ldots, j_{k+1}}
\]
That is, \( \lambda_i^{-1} U^{-1}_\gamma I_{n} U_\gamma \) is \( I \otimes \ldots \otimes g \otimes \ldots \otimes I \) (\( g \) at the \( k \)th slot) and \( U^{-1}_\gamma I_{n} U_\gamma \) is \( I \otimes \ldots \otimes h \otimes \ldots \otimes I \) (\( h \) at the \( k \)th slot). Then an easy calculation shows that
\[
I_\lambda^\gamma = \lambda U_\gamma I_{n} U_\gamma^{-1}
\]
where \( \lambda = \lambda_1^{\alpha_1} \ldots \lambda_r^{\alpha_r} \).

3. Symmetries of the partition function

The Boltzmann weights \( S_{i,j}^{kl} \), \( i,j,k,l \in X \) define a completely \( X \) symmetric matrix \( S \) in the standard basis \( \{ e_i \otimes e_j \}_{i,j \in X} \) for \( C^n \times C^n \). Belavin (1981) and Chudnovsky and Chudnovsky (1981) have shown that a \( n \times n \) \( S \) is completely \( X \) symmetric (satisfies conditions (1.2)) if and only if
\[
S = \sum_{\alpha \in \ell_n} w_\alpha I_{n} \otimes I_{n}^{-1} \quad \text{(3.1)}
\]
The partition function, \( Z_\Lambda \), for a finite rectangular lattice \( \Lambda \) with cyclic boundary conditions may be written as
\[
Z_\Lambda = \text{Tr}(T^M)
M = \text{number of rows of } \Lambda
\]
where
\[
T_{\alpha,\beta} = \text{Tr}(L(\alpha_1, \beta_1) \ldots L(\alpha_N, \beta_N))
\]
\[
\alpha = (\alpha_1, \ldots, \alpha_N) \quad \beta = (\beta_1, \ldots, \beta_N) \quad \alpha_i, \beta_i \in X
\]
\( N \) is the number of columns of \( \Lambda \) and \( L(\alpha, \beta) \) is the \( n \times n \) matrix defined by
\[
(L(\alpha, \beta))_{\lambda,\mu} = S_{\alpha}^{\alpha,\beta} \quad \alpha, \beta, \lambda, \mu \in X
\]
Due to the invariance of the trace under similarity transformations, it is easy to see that if
\[
S \rightarrow U_1 \otimes U_2 S U_1^{-1} \otimes U_2^{-1} = \tilde{S}
\]
where \( U_1 \) and \( U_2 \) are any \( n \times n \) invertible matrices, then
\[
Z_\Lambda(\tilde{S}) = Z_\Lambda(S)
\]
We will now consider some special cases of similarity transformations; in particular, those associated with the Heisenberg group \( H_X \). First choose
\[
U_1 = I_\beta \quad U_2 = I = \text{identity}
\]
and consider
\[ \tilde{S} = I_\beta \otimes ISI_\beta^{-1} \otimes I = I \otimes I_\beta^{-1}SI \otimes I_\beta. \]

In terms of the \( w_\alpha \) we have using (2.8)
\[ I_\beta \otimes ISI_\beta^{-1} \otimes I = \sum \omega^{\beta(\alpha, \mu)} w_{\alpha} I_\alpha \otimes I_\alpha^{-1}. \]

Hence the similarity transformation on the \( S \) matrix \( S \mapsto I_\beta \otimes ISI_\beta^{-1} \otimes I \) is equivalent to the transformation on the \( w_\alpha \) coordinates
\[ w_\alpha \mapsto \omega^{\beta(\alpha, \mu)} w_\alpha. \]

The second type of similarity transformation we will consider arises from the choice of new generators for \( H_\Xi \). For \( \gamma \in \Gamma_E \), we define
\[ S^\gamma = \sum_{\ell \in C_\alpha} w_\ell I_\ell \otimes (I_\ell)^{-1} \]
where \( I_\ell \) is defined by (2.15). By lemma 2.2 we have
\[ S^\gamma = U_\gamma \otimes U_\gamma SU_\gamma^{-1} \otimes U_\gamma^{-1} \]
so that
\[ Z_A[S^\gamma] = Z_A[S]. \]

By (2.16)
\[ I_\ell \otimes (I_\ell)^{-1} = I_{\gamma \ell} \otimes I_{\gamma \ell}^{-1} \]
so that
\[ S^\gamma = \sum w_{\gamma \ell} I_{\gamma \ell} \otimes I_{\gamma \ell}^{-1} = \sum w_{\gamma \ell} I_\ell \otimes I_\ell^{-1}. \]

Hence the similarity transformation \( S \mapsto S^{-1} \) corresponds to the transformation \( w_\ell \mapsto w_{\gamma \ell} \). We now have the following theorem.

**Theorem 3.1.** Let \( S \) be any \( X \) symmetric matrix, \( Z_A[S] \) the corresponding partition function for a finite rectangular lattice with cyclic boundary conditions. Write
\[ S = \sum_{\ell \in C_\alpha} w_\ell I_\ell \otimes I_\ell^{-1}. \]

Then \( Z_A[S] \) is invariant under the transformations on \( S \) given by
\[ w_\ell \mapsto \omega^{\beta(\ell, \alpha)} w_\ell \quad \alpha \in \mathbb{Z} \times \mathbb{Z} \]
\[ w_{\gamma \ell} \mapsto w_{\gamma \ell} \quad \gamma \in \Gamma_E. \]

The \( \alpha \) only depends on its coset in \( (\mathbb{Z}/n_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/n_n \mathbb{Z})^2 \) and \( \gamma \) only depends on its coset in \( \Gamma_E \setminus \Gamma_E(n_1, \ldots, n_n) \) where
\[ \Gamma_E(n_1, \ldots, n_n) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_E : \begin{pmatrix} A_{kk} & B_{kk} \\ C_{kk} & D_{kk} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n_k, \text{ and } A_{ki} = B_{ki} \right\}. \]

\[ = C_{ki} = D_{ki} = 0 \mod n_k \text{ for } i \neq k \].

Also, the action of \( \Gamma_E \) normalises the action of \( \mathbb{Z} \times \mathbb{Z} \) showing that the symmetry group is
\[ \Gamma_E \rtimes (\mathbb{Z} \times \mathbb{Z}). \]
\textbf{Proof.} The invariance of the partition function has already been demonstrated. The statement that $\alpha$ only depends on its coset in $(\mathbb{Z}/n_1\mathbb{Z} \times \ldots \times \mathbb{Z}/n_r\mathbb{Z})^\gamma$ follows from if $\beta \in (n_1\mathbb{Z} \times \ldots \times n_r\mathbb{Z})^\gamma$, then from (2.9b)

$$B(\xi, \beta) = 0 \pmod{n}.$$  

If $\gamma \in \Gamma_E(n_1, \ldots, n_r)$, then

$$I_{\gamma_1} = I \otimes \ldots \otimes h_1 \otimes \ldots \otimes I$$  

$$I_{\gamma_2} = I \otimes \ldots \otimes g_1 \otimes \ldots \otimes I.$$  

Thus $I_{\gamma} = I_{\xi}$ for all $\xi$.

The fact that $\Gamma_E$ normalises the action of $\mathbb{Z}^r \times \mathbb{Z}^r$ follows from the combined action of $\gamma \in \Gamma_E$ and $\alpha \in (\mathbb{Z}^r \times \mathbb{Z}^r)$: for arbitrary $w_{\xi}$ we have

$$w_{\xi} \gamma \rightarrow w_{\gamma \cdot \gamma^{-1} \cdot \xi} \rightarrow \omega^{B(\gamma \cdot \gamma^{-1} \cdot \xi, \alpha)} w_{\gamma \cdot \gamma^{-1} \cdot \xi} \rightarrow \omega^{B(\gamma \cdot \gamma^{-1} \cdot \xi, \alpha)} w_{\gamma \cdot \gamma^{-1} \cdot \xi} \rightarrow \omega^{B(\gamma \cdot \gamma^{-1} \cdot \xi, w_{\xi})} w_{\xi}.$$  

Thus the combined action is equivalent to the action of just $\gamma \alpha = \alpha' \in \mathbb{Z}^r \times \mathbb{Z}^r$. This being the definition of the semidirect product we conclude the symmetry group is $\Gamma_E \times (\mathbb{Z}^r \times \mathbb{Z}^r)$.

To obtain the symmetries of Fan and Wu mentioned previously, we must consider one more construction. We choose any Heisenberg matrix $I_{\alpha}$ such that $I_{\alpha}^2$ is a multiple of the identity. This is only possible if $n$ is even, in which case $\alpha = (\alpha_1', \ldots, \alpha_r', \alpha_1^\prime, \ldots, \alpha_r^\prime)$ must be such that

$$\alpha_1', \alpha_1^\prime = 0 \pmod{n_1} \quad \text{if } n_1 \text{ is odd}$$  

$$\alpha_1', \alpha_1^\prime = \frac{1}{2} n_i \pmod{n_1} \quad \text{if } n_i \text{ is even}$$  

implying that $2 \alpha = (0, 0)$ and $I_{\alpha}^2 = \pm 1$ identity. Following the construction in Richey and Tracy (1986), one can show that $\mathbb{Z}[S]$ is invariant under the transformation

$$S \rightarrow I_{\alpha} \otimes I_{\alpha}^{-1} S$$  

(assuming that the number of rows and columns of the lattice is even). If $n = 2$, this, along with theorem 3.1, gives us the symmetries of Fan and Wu.

Using theorem 3.1, we now obtain an interesting case of the symmetries which reduces to the so-called 'weak graph duality' (Wegner 1971, 1974, Fan and Wu 1970, Fan 1972, Johnson et al 1973). This relates a high-temperature model (all weights almost equal) to a low-temperature one (one weight, $S_{ab}^{oi}$, dominant).

For any $\alpha \in G_{\alpha}$, $\alpha = (\alpha_1', \ldots, \alpha_r', \alpha_1^\prime, \ldots, \alpha_r^\prime) = (\alpha', \alpha^\prime)$, consider the transformation

$$\alpha = \begin{pmatrix} \alpha' \\ \alpha^\prime \end{pmatrix} \rightarrow \begin{pmatrix} \alpha'' \\ -\alpha' \end{pmatrix}.$$  

The matrix $\gamma \in \Gamma_E$ representing this transformation is

$$\gamma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{I } = r \times r \text{ identity matrix.}$$  

The corresponding transformation on the weights can be described as follows. Using $X$ symmetry, any weight $S_{ij}^{kl}$ can be written as

$$S_{ij}^{kl} = S_{ij}^{k-i, l-i} = S_{ab}^{oi}$$  

$$a = k - i \in X$$  

$$b = l - i \in X.$$
It is straightforward to confirm that
\[ S_{ij}^{kl} = S_{a,b}^{a',b'} = \sum_{a \in \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n} w_{(-a,a)}^{-(n/n_a)\alpha_i \alpha_j - -(n/n_a)\beta_i \beta_j}, \]
and the associated inversion formula
\[ w_{(a,b)} = \frac{1}{n} \sum_{\beta \in \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n} S^{-a,b}_{\alpha_i \beta_j} \omega^{-(n/n_a)\alpha_i \beta_j + \ldots + (n/n_a)\beta_j \alpha_i}. \]
Letting \((a', \alpha^n) \to (a^n, -\alpha')\) induces
\[ S_{a,b}^{a',b'} \sum_{\alpha} w_{(a,a')}^{-(n/n_a)\beta_i \beta_j - (n/n_a)\alpha_i \alpha_j} = \tilde{S}_{a,b}^{a',b'}. \]
Using the inversion formula
\[ \tilde{S}_{a,b}^{a',b'} = \frac{1}{n} \sum_{\alpha} S^{-a,b}_{\alpha_i \beta_j} \omega^{-(n/n_a)\alpha_i \beta_j + \ldots + (n/n_a)\beta_j \alpha_i - (n/n_a)\beta_j \alpha_i}, \]
\[ = \frac{1}{n} \sum_{(a,b) \in G_n} S^{-a,b}_{\alpha_i \beta_j} \omega^{B((a,b),(a,b))}. \]
If we let \(T \to \infty\), \(S_{ij}^{kl} \to 1\) \(i, j, k, l\); hence
\[ \tilde{S}_{a,b}^{a',b'} \to \frac{1}{n} \sum_{(a,b)} \omega^{B((a,b),(a,b))} \]
\[ = n\delta_0^a \ldots \delta_0^b \delta_0^{a'} \ldots \delta_0^{b'} \]
\[ = \begin{cases} n & \text{if } a = b = (0, \ldots, 0) \\ 0 & \text{otherwise.} \end{cases} \]
Hence the \(\Gamma_E\) transformation \((a', \alpha^n) \to (a^n, -\alpha')\) sends a high-temperature model to a low-temperature one, while preserving the partition function.

4. Discussion of the case \(X = \mathbb{Z}_n\)

If \(X = \mathbb{Z}_n = \mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n\), then we have \((n_i, n_j) = 1, i \neq j\), i.e. the primes are distinct. This case has been previously investigated (Richey and Tracy 1986) so we show that, as expected, the new formalism of the previous sections results in no new symmetries. The isomorphism \(\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n \to \mathbb{Z}_n\) is the Chinese remainder theorem. This theorem asserts that for any \(r\) -tuple \((a_1, \ldots, a_r) \in \mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n\), there is a unique \(\tilde{a}\) (mod \(n\)) such that \(\tilde{a} = a_i \mod (n_i), i = 1, \ldots, r\).

Let \(l = (n/n_l)(\text{mod } n_l), i = 1, \ldots, r\). Then since \((n_i, n/n_l) = 1\) we have \((n, l) = 1\) which implies \((n, l) = 1\). Moreover for any \(a, b \in G_n\) we have
\[ B(a, b) = l(\tilde{a}, \tilde{b})(\text{mod } n) \]
(4.1)
where \(\tilde{a} = (\tilde{a}, \ldots, \tilde{a})\), \(\tilde{b} = (\tilde{b}, \ldots, \tilde{b})\), \(\tilde{a} = a_i \text{ (mod } n_i)\), etc. To show this is true mod \(n\), we need only to demonstrate (4.1) holds mod \(n_i\), \(i = 1, \ldots, r\):
\[ B(a, b) = \sum_{k=1}^{r} \frac{n}{n_k} (a_k^{'b_k}' - a_k^{'b_k}) \]
\[ = (n/n_i)(a'_i \beta_i' - a_i \beta_i')(\text{mod } n_i) \quad \text{since } (n_i, n_j) = 1 \text{ for } i \neq j \]
\[ = l(\tilde{a}' \beta_i' - \tilde{a} \beta_i')(\text{mod } n_i). \]
Under the map \(\xi \in G_n \to \tilde{\xi} \in \mathbb{Z}_n \times \mathbb{Z}_n\) we can identify \(w_\xi\) with \(w_\tilde{\xi}\). We first see that the symmetry \(w_\xi \to \omega^{B(\tilde{\xi}, \tilde{\xi})} w_\xi, \xi, \tilde{\xi} \in G_n = (\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n)^2\) is equivalent to \(w_\xi \to \omega^{\tilde{\xi}, \tilde{\xi}} w_\xi\),
\(\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_n \times \mathbb{Z}_n\). Since \((l, n) = 1\), \(\omega^l\) is a primitive \(n\)th root of unity, and hence no new symmetries are obtained.

A more interesting situation arises when we consider the effect of changing generators in \(H_k\). This corresponds to choosing some \(\gamma \in \Gamma_E\). The action of \(\gamma\) on \((\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n)^2\) induces our action on \(\mathbb{Z}_n^2\) as follows:

\[
\begin{align*}
\alpha \in (\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n)^2 & \xrightarrow{\gamma} \gamma \alpha \in (\mathbb{Z}_n \oplus \ldots \oplus \mathbb{Z}_n)^2 \\
\xrightarrow{\text{CRT}} & \bar{\alpha} \in \mathbb{Z}_n^2 \\
& \xrightarrow{\gamma} \gamma \bar{\alpha} \in \mathbb{Z}_n^2.
\end{align*}
\]

Since the CRT and \(\gamma\) are linear, the map \(\bar{\alpha} \mapsto \gamma \bar{\alpha}\) is also linear and hence has a matrix representation \(\bar{\alpha} \mapsto \gamma \bar{\alpha} = A \bar{\alpha}\). We will show that \(A \in \text{SL}(2, \mathbb{Z}_n)\). This will suffice to show that the action \(w_{\alpha} \mapsto w_{\gamma \alpha} = w_{A \alpha}\) induces no new symmetries not discussed in Richey and Tracy (1986). Since \(A: \mathbb{Z}_n^2 \to \mathbb{Z}_n^2\), it is enough to show that \(\det(A) = 1 \pmod{n}\).

Now \(\gamma \in \Gamma_E\) if and only if \(B(\gamma \alpha, \gamma \beta) = B(\alpha, \beta)\) for all \(\alpha, \beta \in G_n\) and \(A \in \text{SL}(2, \mathbb{Z}_n)\) if and only if \(\langle A \bar{\alpha}, A \bar{\beta} \rangle = \langle \bar{\alpha}, \bar{\beta} \rangle\) for all \(\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_n^2\). In particular, for \(\bar{\alpha} = (1, 0)\) and \(\bar{\beta} = (0, 1)\)

\[
\langle A \bar{\alpha}, A \bar{\beta} \rangle = \det A
\]

but by (4.1)

\[
l(A \bar{\alpha}, B \bar{\beta}) = l(\gamma \bar{\alpha}, \gamma \bar{\beta}) = B(\gamma \alpha, \gamma \beta)
\]

\[
= B(\alpha, \beta) = \sum_{k=1}^{r} \frac{n}{n_k}
\]

since

\[
\alpha = (\alpha_1', \ldots, \alpha_r', \alpha_1'', \ldots, \alpha_r'')
\]

with \(\alpha_1' = 1\), \(\alpha_i'' = 0\) and \(\beta = (\beta_1', \ldots, \beta_r', \beta_1'', \ldots, \beta_r'')\) with \(\beta_1' = 0\), \(\beta_i'' = 1\). Hence we have

\[
l \det A = \sum_{k=1}^{r} \frac{n}{n_k}.
\]

Consider this equation mod \(n_i\) and recall that \(l = (n/n_i)(\text{mod } n_i)\) and \((n_i, n_j) = 1\), \(i \neq j\), we have

\[
\det A = 1 \pmod{n_i} \quad i = 1, \ldots, r
\]

which implies \(\det A = 1 \pmod{n}\).

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Symmetry group for a completely symmetric vertex model

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