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Level-spacing distributions and the Airy kernel

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Scaling level-spacing distribution functions in the "bulk of the spectrum" in random matrix models of $N \times N$ hermitian matrices and then going to the limit $N \to \infty$, leads to the Fredholm determinant of the *sine kernel* $\sin \pi(x-y)/\pi(x-y)$. Similarly a double scaling limit at the "edge of the spectrum" leads to the *Airy kernel* $[\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)]/(x-y)$. We announce analogies for this Airy kernel of the following properties of the sine kernel: the completely integrable system of PDE's found by Jimbo, Miwa, Môri and Sato; the expression, in the case of a single interval, of the Fredholm determinant in terms of a Painlevé transcendent; the existence of a commuting differential operator; and the fact that this operator can be used in the derivation of asymptotics, for general n, of the probability that an interval contains precisely n eigenvalues.

1. Introduction

In this note we announce new results for the *level spacing distribution functions* obtained from scaling random matrix models of $N \times N$ hermitian matrices at the edge of the support of the (tree-level) eigenvalue densities when the parameters of the potential V are not "finely tuned." This universality class is already present in the gaussian unitary ensemble. It is known [1,2] that these distribution functions are expressible in terms of a Fredholm determinant of an integral operator K whose kernel involves Airy functions.

There are striking analogies between the properties of this *Airy kernel*

$$K(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y},$$

where $A(x) = \sqrt{\lambda}\,\mathrm{Ai}(x)$, and the *sine kernel*

$$\frac{\lambda}{\pi}\,\frac{\sin \pi(x-y)}{x-y},$$

whose associated Fredholm determinant describes the classical level spacing distribution functions first studied by Wigner, Dyson, Mehta, and others [1]. (In retrospect we should not have been surprised by this; the two kernels are, after all, both scaled limits of the same family of kernels.) We describe below three of these which we have found. The first is the analogue of the completely integrable system of PDE's of Jimbo, Miwa, Môri, and Sato [3] when the underlying domain is a union of intervals. The second is the fact that in the case of the semi-infinite interval (s, ∞) (the analogue of a single finite interval for the sine kernel) the Fredholm determinant is closely related to a Painlevé transcendent of the second kind (the fifth transcendent arises for the sine kernel [3]). And the third is the existence of a second order differential operator commuting with the Airy operator K. (The existence of such a differential operator in the sine kernel case has been known for some time [4,5].) This last fact leads to an explicit asymptotic formula, as the interval (s, ∞) expands, for the probability that it contains precisely n eigenvalues ($n = 1, 2, \ldots$) (the

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analogue of results in ref. [6]).

2. The system of PDE's

We set

$$I = \bigcup_{j=1}^{m} (a_{2j-1}, a_{2j})$$

and write $D(I;\lambda)$ for the Fredholm determinant of K acting on I. We think of this as a function of $a = (a_1, \ldots, a_{2m})$. Then

$$\mathrm{d}_a \log D(I;\lambda) = -\sum_{j=1}^{2m} (-1)^j R(a_j, a_j)\, \mathrm{d}a_j \qquad (1)$$

where $R(x,y)$ is the kernel of the operator $K(1-K)^{-1}$. We introduce quantities

$$q_j = (1-K)^{-1} A(a_j), \quad p_j = (1-K)^{-1} A'(a_j),$$

(which are the analogues of the quantities $r_{\pm j}$ of ref. [3]; see also ref. [7]) as well as two further quantities

$$u = \big(A, (1-K)^{-1} A\big), \quad v = \big(A, (1-K)^{-1} A'\big)$$

where the inner products refer to the domain I. Then the equations read

$$\frac{\partial q_j}{\partial a_k} = (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k}\, q_k \quad (j \neq k),$$

$$\frac{\partial p_j}{\partial a_k} = (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k}\, p_k \quad (j \neq k),$$

$$\frac{\partial q_j}{\partial a_j} = -\sum_{k\neq j} (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k}\, q_k + p_j - q_j u,$$

$$\frac{\partial p_j}{\partial a_j} = -\sum_{k\neq j} (-1)^k \frac{q_j p_k - p_j q_k}{a_j - a_k}\, p_k + a_j q_j + p_j u - 2 q_j v,$$

$$\frac{\partial u}{\partial a_j} = (-1)^j q_j^2,$$

$$\frac{\partial v}{\partial a_j} = (-1)^j p_j q_j.$$

Moreover the quantities $R(a_j, a_j)$ appearing in (1) are given by

$$R(a_j, a_j) = \sum_{k\neq j} (-1)^k \frac{(q_j p_k - p_j q_k)^2}{a_j - a_k} + p_j^2 - a_j q_j^2 - 2 p_j q_j u + 2 q_j^2 v.$$

These equations are derived very much in the spirit of ref. [8]; see also ref. [7].

3. The ODE's

For the special case $I = (s, \infty)$ the above equations can be used to show that $q(s;\lambda)$ (the quantity q of the last section corresponding to the end-point s) satisfies

$$q'' = sq + 2q^3, \quad ({}' = \frac{\mathrm{d}}{\mathrm{d}s}) \qquad (2)$$

with $q(s;\lambda) \sim \sqrt{\lambda}\,\mathrm{Ai}(s)$ as $s \to \infty$. This equation is a special case of the $\mathrm{P_{II}}$ differential equation [9–12]. One can similarly derive for $R(s) := R(s,s)$, which in view of (1) equals

$$\frac{\mathrm{d}}{\mathrm{d}s} \log D(I;\lambda),$$

the third-order equation

$$\frac{1}{2}\left(\frac{R''}{R'}\right)' - \frac{R}{R'} + R' = 0. \qquad (3)$$

It is also the case that $R'(s) = -q(s;\lambda)^2$ and this gives the following simple formula for $D(I;\lambda)$ in terms of a $\mathrm{P_{II}}$ transcendent:

$$D(I;\lambda) = \exp\left(-\int_s^\infty (x-s)\, q(x;\lambda)^2\, \mathrm{d}x\right).$$

This is much simpler than the corresponding representation of $D(I;\lambda)$ for the sine kernel in terms of a $\mathrm{P_V}$ transcendent. The fact that $q(s;\lambda)$ satisfies (2) can also be obtained by combining some results in refs. [10,12]. Thus in this case of a semi-infinite interval our results have inverse scattering interpretations.

4. Asymptotics and commuting differential operators

Again we take $I = (s, \infty)$ and consider asymptotics as $s \to -\infty$. (Asymptotics as $s \to \infty$ can be obtained trivially from the Neumann series for $(1-K)^{-1}$.)

From the random matrix point of view the interesting quantities are

$$E(n;s) := \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \lambda^n} D(I;\lambda)\Big|_{\lambda=1} .$$

This is the probability that exactly n eigenvalues lie in I. Using both differential equations (2) and (3), plus the fact that $R' = -q^2$, we can obtain the asymptotics of R as $s \to -\infty$:

$$R(s) \sim \frac{1}{4}s^2 - \frac{1}{8s} + \frac{9}{64s^4} - \frac{189}{128s^7} + \cdots .$$

(We also use the fact [12] that $q(s;1)$ is asymptotic to $\sqrt{-s/2}$ as $s \to -\infty$.) Therefore as $s \to -\infty$

$$E(0;s) = D(I;1) \sim \frac{\tau_0}{(-s)^{1/8}} \exp\left(\tfrac{1}{12}s^3\right) \times \left(1 - \frac{3}{64s^3} + \frac{2025}{8192s^6} + \cdots\right),$$

where τ_0 is an undetermined constant. (The analogue of this formula for the sine kernel was obtained by Dyson [13].)

For asymptotics of $E(n;s)$ for general n we introduce

$$r(n;s) := \frac{E(n;s)}{E(0;s)} .$$

Successive differentiation of (2) with respect to λ, plus the known asymptotics of $q(s;1)$, allows us to find asymptotic expansions for the quantities

$$q_n(s) := \frac{\partial^n q}{\partial \lambda^n}\Big|_{\lambda=1}$$

(for the analogue in the sine kernel case see ref. [6]); and these in turn can be used to find expansions for $r(n;s)$. One drawback of this approach is that yet another undetermined constant factor enters the picture (in ref. [6] Toeplitz and Wiener–Hopf techniques, not available for the Airy kernel, fixed this constant). Another drawback is that when one expresses the $r(n;s)$ in terms of the $q_n(s)$ a large amount of cancellation takes place, with the result that even the first-order asymptotics of $r(n;s)$ are out of reach by this method when n is large.

There is, however, another approach (briefly indicated in ref. [6] and with details in ref. [8] for the sine kernel case). We have

$$r(n;s) = \sum_{i_1<\ldots<i_n} \frac{\lambda_{i_1}\cdots\lambda_{i_n}}{(1-\lambda_{i_1})\cdots(1-\lambda_{i_n})}, \tag{4}$$

where $\lambda_0 > \lambda_1 > \ldots$ are the eigenvalues of the integral operator K (with $\lambda = 1$). Now quite analogous to the fact that the operator with the sine kernel commutes with the differential operator for the prolate spheroidal wave functions refs. [4,5], is that the Airy operator commutes with the differential operator L given by

$$Lf(x) = [(x-s)f'(x)]' - x(x-s)f(x) .$$

An application of the WKB method, plus a trick (not at this writing rigorously justified), allows us to derive the following asymptotic formula for λ_i with i fixed:

$$1 - \lambda_i \sim \frac{\sqrt{\pi}}{i!} 2^{5i+3} t^{3i/2+3/4} \exp\left(-\tfrac{8}{3}t^{3/2}\right) \quad (s = -2t \to -\infty). \tag{5}$$

(The analogue of this for the sine kernel is in ref. [5].) From this it is seen that the term in (4) corresponding to $i_1 = 0$, $i_1 = 1, \ldots, i_n = n-1$ dominates each of the others. In fact this term dominates the sum of all the others, and so

$$r(n,-2t) \sim \frac{1!\,2!\cdots(n-1)!}{\pi^{n/2}\,2^{(5n^2+n)/2}}\, t^{-3n^2/4} \times \exp\left(\tfrac{8}{3}nt^{3/2}\right) . \tag{6}$$

Thus one can use (6) to fix the constant in $q_1(s)$ mentioned above.

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