J. Phys. A: Math. Theor. **42** (2009) 425002 (6pp) doi:10.1088/1751-8113/42/42/425002

On the distribution of a second-class particle in the asymmetric simple exclusion process

Craig A Tracy[1] **and Harold Widom**[2]

[1] Department of Mathematics, University of California, Davis, CA 95616, USA
[2] Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

E-mail: tracy@math.ucdavis.edu and widom@ucsc.edu

Received 25 July 2009
Published 5 October 2009
Online at stacks.iop.org/JPhysA/42/425002

Abstract

We give an exact expression for the distribution of the position $X(t)$ of a single second-class particle in the asymmetric simple exclusion process (ASEP) where initially the second-class particle is located at the origin and the first-class particles occupy the sites $\mathbb{Z}^+ = \{1, 2, \ldots\}$.

PACS number: 02.50.Ey
Mathematics Subject Classification: 60K35, 82C22

1. Introduction

The asymmetric simple exclusion process (ASEP) [2, 3] is one of the simplest models of nonequilibrium statistical mechanics and has been called the 'default stochastic model for transport phenomena' [8]. A useful concept in exclusion processes is that of a *second-class particle*.[3]

> Imagine that the particles in the system are each called either first class or second class. The evolution is the same as before, except that if a second-class particle attempts to go to a site occupied by a first-class particle, it is not allowed to do so, while if a first-class particle attempts to move to a site occupied by a second-class particle, the two particles exchange positions. In other words, a first-class particle has priority over a second-class particle. This rule has no effect on whether or not a given site is occupied at a given time. The advantage, though, is that viewed by itself, the collection of first-class particles is Markovian, and has the same law as the exclusion process. The collection of second-class particles is clearly not Markovian. However, the collection of first- and second-class particles is Markovian, and again evolves like an exclusion process.

[3] The following quote is taken from Liggett [3].

 © 2009 IOP Publishing Ltd Printed in the UK

Here we consider ASEP on the integer lattice $\mathbb{Z}$ with jumps one step to the right with rate p and jumps one step to the left with rate $q = 1 - p$. We assume a leftward drift, i.e. $q > p$. We further assume that the system has one second-class particle initially located at the origin and first-class particles initially located at sites in

$$Y = \{0 < y_1 < y_2 < \cdots\} \subset \mathbb{Z}^+.$$

With the above initial condition, we denote by $X(t)$ the position of the second-class particle at time t. The purpose of this note is to give an exact expression for the probability that the second-class particle is at position x at time t, i.e. $\mathbb{P}_Y(X(t) = x)$. (The subscript Y denotes the sites of the initial configuration of the first-class particles.) Our main result is for $Y = \mathbb{Z}^+$ and is given below in (9) and in a slightly different form in (11).

2. A basic lemma

The single second-class particle located at $X(t)$ can be viewed as the (single) discrepancy under *basic coupling* between two asymmetric simple exclusion processes η_t and ζ_t, where $\zeta_t(X(t)) = 1$ and $\eta_t(X(t)) = 0$ and initially $\{x : \zeta_0(x) = 1\} = Y' = \{0\} \cup Y$ and $\{x : \eta_0(x) = 1\} = Y$ [2, 3].

We first learned the following identity from H Spohn [5] but presumably it has a long history:

$$\mathbb{P}_Y(X(t) = x) = \mathbb{P}_{Y'}(\zeta_t(x) = 1) - \mathbb{P}_Y(\eta_t(x) = 1)\,. \tag{1}$$

For the convenience of the reader, we give a short proof of (1). Let ζ_t and η_t be as above evolving together under the basic coupling [2, 3]. Recall that the coupled processes satisfy $\eta_t \leqslant \zeta_t$ for all $t > 0$ since they satisfy this inequality at $t = 0$ [2, 3].[4] Define

$$\mathcal{J}_\eta(x,t) := \sum_{z\leqslant x} \eta_t(z) = \text{number of particles in configuration } \eta_t \text{ with positions } \leqslant x,$$

$$\mathcal{J}_\zeta(x,t) := \sum_{z\leqslant x} \zeta_t(z) = \text{number of particles in configuration } \zeta_t \text{ with positions } \leqslant x,$$

$$\mathcal{I}(x,t) = \begin{cases} 1 & \text{if} \quad X(t) \leqslant x, \\ 0 & \text{if} \quad X(t) > x. \end{cases}$$

By counting

$$\mathcal{J}_\zeta(x,t) = \mathcal{J}_\eta(x,t) + \mathcal{I}(x,t). \tag{2}$$

Since

$$\mathbb{E}_{Y'}(\mathcal{J}_\zeta(x,t)) = \sum_{z\leqslant x} \mathbb{E}_{Y'}(\zeta_t(z)) = \sum_{z\leqslant x} \mathbb{P}_{Y'}(\zeta_t(z) = 1),$$

$$\mathbb{E}_Y(\mathcal{J}_\eta(x,t)) = \sum_{z\leqslant x} \mathbb{E}_Y(\eta_t(z)) = \sum_{z\leqslant x} \mathbb{P}_Y(\eta_t(z) = 1),$$

$$\mathbb{E}_Y(\mathcal{I}(x,t)) = \mathbb{P}_Y(X(t) \leqslant x) = \sum_{z\leqslant x} \mathbb{P}_Y(X(t) = z),$$

the expectation of (2) gives

$$\sum_{z\leqslant x} \mathbb{P}(X(t) = z) = \sum_{z\leqslant x} \mathbb{P}(\zeta_t(z) = 1) - \sum_{z\leqslant x} \mathbb{P}(\eta_t(z) = 1)$$

from which (1) follows.

[4] Given two configurations $\eta, \zeta \in \{0,1\}^{\mathbb{Z}}$ we say $\eta \leqslant \zeta$ if $\eta(x) \leqslant \zeta(x)$ for all $x \in \mathbb{Z}$.

3. Probability for a site to be occupied in ASEP

For ASEP with particles initially at Y we denote by $x_m(t)$ the position of the mth left-most particle at time t (so $x_m(0) = y_m$). In theorem 5.2 of [6] the authors gave an exact expression for $\mathbb{P}_Y(x_m(t) = x)$. To state this result we first recall the definition of the τ-binomial coefficients. For $0 \leqslant \tau := p/q < 1$ we define for each $n \in \mathbb{Z}^+$

$$[n] = \frac{1-\tau^n}{1-\tau}, \qquad [n]! = [n][n-1]\cdots[1], \qquad [0]! := 1,$$
$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \qquad 0 \leqslant k \leqslant n,$$

and if $k > n$ we set $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$. Equation (5.12) of [6] can be written in the following way:[5,6]

$$\mathbb{P}_Y(x_m(t) = x) = \sum_{k=1}^{|Y|} \sum_{\substack{S \subset Y \\ |S|=k}} c_{m,k} \tau^{\sigma(S,Y)} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} I(x,k,\xi) \prod_{i=1}^{k} \xi_i^{-s_i}\, \mathrm{d}^k\xi, \tag{3}$$

where, if $S := \{s_1, \ldots, s_k\}$ then

$$c_{m,k} = q^{k(k-1)/2}(-1)^{m+1}\tau^{m(m-1)/2}\tau^{-km}\begin{bmatrix} k-1 \\ k-m \end{bmatrix},$$
$$\sigma(S,Y) = \#\{(s,y) : s \in S,\ y \in Y, \text{ and } y \leqslant s\}$$
$$= \text{sum of the positions of the elements of } S \text{ in } Y,$$
$$I(x,k,\xi) = \prod_{1 \leqslant i<j \leqslant k} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i}\left(1 - \prod_{i=1}^{k}\xi_i\right)\prod_{i=1}^{k}\frac{\xi_i^{x-1}\mathrm{e}^{\varepsilon(\xi_i)t}}{1-\xi_i},$$
$$\varepsilon(\xi) = \frac{p}{\xi} + q\xi - 1$$

and $\mathcal{C}_R$ is a circle of radius R centered at the origin with $R \gg 1$ so that all (finite) singularities of the integrand are enclosed by $\mathcal{C}_R$. Observe that $c_{m,k} = 0$ when $m > k$.

Since

$$\mathbb{P}_Y(\eta_t(x) = 1) = \sum_{m=1}^{|Y|} \mathbb{P}_Y(x_m(t) = x), \tag{4}$$

we sum the right-hand side of (3) over all $m \leqslant k$. To carry out this sum recall the τ-binomial theorem

$$\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}(-1)^j z^j \tau^{j(j-1)/2} = (1-z)(1-z\tau)\cdots(1-z\tau^{n-1}).$$

Using this a simple calculation shows

$$\sum_{m=1}^{k}(-1)^{m+1}\tau^{m(m-1)/2}\tau^{-km}\begin{bmatrix} k-1 \\ k-m \end{bmatrix} = (-1)^{k+1}\tau^{-k(k+1)/2}\prod_{j=1}^{k-1}(1-\tau^j).$$

[5] We make some changes in the notation in (5.12) of [6]. The (p,q)-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ of [6] equals $q^{k(n-k)}$-times the τ-binomial coefficient $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ defined above. The second change is a little more subtle. The sum in (5.12) is over all finite subsets $S \subset \{1, 2, \ldots, |Y|\}$ with $|S| \geqslant m$. If $S = \{s_1, \ldots, s_k\}$ the subset $Y_S := \{y_{s_1}, \ldots, y_{s_k}\}$ and the factor $\prod_{i\in S}\xi_i^{-y_i}$ appears in the integrand of (5.12). Thus, we can equivalently sum over all finite subsets $S \subset Y$ where now the factor $\prod_{1\leqslant i\leqslant k}\xi_i^{-s_i}$ appears in the integrand. The factor $\sigma(S) = \sum_{i\in S}\mathrm{i}$ of (5.12) becomes $\sigma(S,Y)$ given above.

[6] All contour integrals are to be given a factor of $1/2\pi\mathrm{i}$.

Thus,

$$\mathbb{P}_Y\,(\eta_t(x)=1)=\sum_{k=1}^{|Y|}(-1)^{k+1}q^{k(k-1)/2}\tau^{-k(k+1)/2}\prod_{j=1}^{k-1}(1-\tau^j)$$
$$\times\sum_{\substack{S\subset Y\\|S|=k}}\tau^{\sigma(S,Y)}\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}I(x,k,\xi)\prod_{i=1}^{k}\xi_i^{-s_i}\,\mathrm{d}^k\xi. \quad (5)$$

Remark 1. The above formula holds for either $|Y|$ finite or infinite. For $|Y|=N$, the integral of order N in (5) is obtained from the summand $S=Y$. Since $\sigma(Y,Y)=N(N+1)/2$, we get for the coefficient of this integral

$$(-1)^{N+1}q^{N(N-1)/2}\prod_{j=1}^{N-1}(1-\tau^j)=(-1)^{N+1}\prod_{j=1}^{N-1}(q^j-p^j). \quad (6)$$

4. Probability for a site to be occupied by a second-class particle

As above, suppose that our initial configuration consists of a second-class particle at site 0 and first-class particles at sites in Y. As above, set $Y'=Y\cup\{0\}$. The process ζ_t has initially its particles at sites in Y'. We apply formula (5) to the initial configurations Y' and Y and by (1) we subtract to obtain $\mathbb{P}_Y(X(t)=x)$. If $|Y'|=N$ there is one N-dimensional integral that comes from the expansion of $\mathbb{P}_{Y'}(\zeta_t(x)=1)$ when $S=Y'$. The coefficient of the integral of highest order equals (6).

We now consider the special case of *step initial condition*, that is, $Y=\mathbb{Z}^+$, and use corollary (5.13) of [6] to obtain a more compact expression for $\mathbb{P}_{\mathbb{Z}^+}(x_m(t)=x)$. To find $\mathbb{P}_{\mathbb{Z}^+}(\eta_t(x)=1)$ we again apply (4) but use (5.13) of [6]. As above we interchange the sums over k and m, use the τ-binomial theorem ([4], page 26), to conclude

$$\mathbb{P}_{\mathbb{Z}^+}\,(\eta_t(x)=1)=-\sum_{k\geqslant 1}\frac{q^{k^2}}{k!}\prod_{j=1}^{k-1}(1-\tau^j)\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}\tilde{J}_k(x,\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_k, \quad (7)$$

where

$$\tilde{J}_k(x,\xi)=\prod_{i\neq j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\left(1-\prod_i\xi_i\right)\prod_i\frac{\xi_i^{x-1}\,\mathrm{e}^{\varepsilon(\xi_i)t}}{(1-\xi_i)(q\xi_i-p)}.$$

We can get the corresponding formula for $Y'=\mathbb{Z}^+\cup\{0\}$ by observing that there is a one–one correspondence between subsets $S'\subset Y'$ and subsets $S\subset Y$ given by $S=S'+1$. Then $\sigma(S',\,Y')=\sigma(S,\,Y)$ and, with obvious notation, $\prod\xi_i^{-s_i'}=\prod\xi_i\cdot\prod\xi_i^{-s_i}$. It follows that for the difference $\mathbb{P}_{Y'}(\zeta_t(x)=1)-\mathbb{P}_Y(\eta_t(x)=1)$ we multiply the integrand $\tilde{J}_k(x,\xi)$ in (7) by $\prod\xi_i-1$.

Thus,

$$\mathbb{P}_{\mathbb{Z}^+}(X(t)=x)=\sum_{k\geqslant 1}\frac{q^{k^2}}{k!}\prod_{j=1}^{k-1}(1-\tau^j)\int_{\mathcal{C}_R}\cdots\int_{\mathcal{C}_R}\tilde{\tilde{J}}_k(x,\xi)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_k, \quad (8)$$

where

$$\tilde{\tilde{J}}_k(x,\xi)=\prod_{i\neq j}\frac{\xi_j-\xi_i}{p+q\xi_i\xi_j-\xi_i}\left(1-\prod_i\xi_i\right)^2\prod_i\frac{\xi_i^{x-1}\,\mathrm{e}^{\varepsilon(\xi_i)t}}{(1-\xi_i)(q\xi_i-p)}.$$

From this it follows that the distribution function is (on $\mathcal{C}_R$, $|\xi^{-1}| \ll 1$)

$$\mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) = \sum_{k \geqslant 1} \frac{q^{k^2}}{k!} \prod_{j=1}^{k-1} (1 - \tau^j) \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} J_k(x, \xi)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k, \tag{9}$$

where

$$J_k(x, \xi) = \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \left(\prod_i \xi_i - 1 \right) \prod_i \frac{\xi_i^x\, \mathrm{e}^{\varepsilon(\xi_i)t}}{(1 - \xi_i)(q\xi_i - p)}.$$

Since

$$\frac{1}{p + q\xi\xi' - \xi} = \frac{1}{\xi(\xi' - 1)} + \mathrm{O}(\tau), \qquad \tau \to 0,$$

the TASEP limit of $J_k(x, \xi)$ is

$$J_k^{\mathrm{TASEP}}(x, \xi) := \lim_{\tau \to 0} J_k(x, \xi) = \prod_{i \neq j} (\xi_j - \xi_i) \left(\prod \xi_i - 1 \right) \prod_i \frac{\xi_i^x\, \mathrm{e}^{\varepsilon(\xi_i)t}}{(\xi_i(1 - \xi_i))^k},$$

where now $\varepsilon(\xi) = \xi - 1$, and hence,

$$\lim_{\tau \to 0} \mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) = \sum_{k \geqslant 1} \frac{1}{k!} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} J_k^{\mathrm{TASEP}}(x, \xi)\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k. \tag{10}$$

Expression (9) for the distribution function can be simplified somewhat. Define the kernel

$$K_{x,t}(\xi, \xi') = q \frac{(\xi')^x \mathrm{e}^{\varepsilon(\xi')t}}{p + q\xi\xi' - \xi},$$

and the associated operator $K_{x,t}$ on $L^2(\mathcal{C}_R)$ by

$$f(\xi) \longrightarrow \int_{\mathcal{C}_R} K_{x,t}(\xi, \xi') f(\xi')\, \mathrm{d}\xi', \quad \xi \in \mathcal{C}_R.$$

Then using the identity [7]

$$\det \left(\frac{1}{p + q\xi_i\xi_j - \xi_i} \right)_{1 \leqslant i,j \leqslant k} = (-1)^k (pq)^{k(k-1)/2} \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1 - \xi_i)(q\xi_i - p)}$$

we have

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x) &= \sum_{k \geqslant 1} \tau^{-k(k-1)/2} \prod_{j=1}^{k-1} (1 - \tau^j) \\ &\quad \times \frac{(-1)^k}{k!} \int_{\mathcal{C}_R} \cdots \int_{\mathcal{C}_R} \left[\det(K_{x+1,t}(\xi_i, \xi_j))_{1 \leqslant i,j \leqslant k} - \det(K_{x,t}(\xi_i, \xi_j))_{1 \leqslant i,j \leqslant k} \right] \\ &= \sum_{k \geqslant 1} \tau^{-k(k-1)/2} \prod_{j=1}^{k-1} (1 - \tau^j) \int_{\mathcal{C}_R} \frac{1}{\lambda^{k+1}} [\det(I - \lambda K_{x+1,t}) - \det(I - \lambda K_{x,t})]\, \mathrm{d}\lambda, \end{aligned} \tag{11}$$

where $\det(I - \lambda K_{x,t})$ is the Fredholm determinant and the last line follows from the Fredholm expansion.

Remark 2.

(i) One cannot interchange the sum and the integration in (11) as was possible in an analogous calculation in [7]. This is the case even though (9) converges absolutely for all $0 \leqslant \tau \leqslant 1$

(recall one may take $R \gg 1$). Thus, we do not have a representation of $\mathbb{P}_{\mathbb{Z}^+}(X(t) \leqslant x)$ as a single integral whose integrand involves the above Fredholm determinants as was the case in [7].

(ii) ASEP with first- and second-class particles is integrable in the sense that the Yang–Baxter equations are satisfied [1]. Using this integrable structure, it is possible to compute directly, i.e. without using the basic lemma (1), $\mathbb{P}_{\mathbb{Z}^+}(X(t) = x)$ using methods similar to that of [6]. We have carried this out to the extent that (6) was computed by this approach. However, this route is much more involved than the one presented here.

Acknowledgments

The authors wish to thank Professor Dr Herbert Spohn for communicating to them the basic lemma. The authors thank Professor Dr Gert-Martin Greul and the staff at the Mathematisches Forschungsinstitut Oberwolfach for their hospitality during the authors' stay under the Research in Pairs program. This work was supported by the National Science Foundation under grants DMS-0553379 (first author) and DMS-0552388 (second author).

References

[1] Alcaraz F C and Bariev R Z 2000 Exact solution of asymmetric diffusions with second-class particles of arbitrary size. *Braz. J. Phys.* **30** 13–26

[2] Liggett T M 2005 *Interacting Particle Systems* (Berlin: Springer) (Reprint of the 1985 Edition)

[3] Liggett T M 1999 *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* (Berlin: Springer)

[4] Macdonald I G 1995 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)

[5] Spohn H 2009 private communication

[6] Tracy C A and Widom H 2008 Integral formulas for the asymmetric simple exclusion process *Commun. Math. Phys.* **279** 815–44

[7] Tracy C A and Widom H 2008 A Fredholm determinant representation in ASEP *J. Stat. Phys.* **132** 291–300

[8] Yau H-T 2004 $(\log t)^{2/3}$ law of the two-dimensional asymmetric simple exclusion process *Ann. Math.* **159** 377–405