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The weak coupling asymptotics to order γ of the ground state energy of the delta-function Fermi gas, derived heuristically in the literature, is here made rigorous. Further asymptotics are in principle computable. The analysis applies to the Gaudin integral equation, a method previously used by one of the authors for the asymptotics of large Toeplitz matrices. *Published by AIP Publishing.*
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I. INTRODUCTION

Since Hans Bethe's seminal 1931 paper, the ideas he introduced, now known under the rubric *Bethe Ansatz*, have had a wide impact in physics and mathematics. For an historical account see Batchelor[2] and for book-length treatments see Refs. 4, 8, and 17.

One of the most widely studied Bethe Ansatz solvable models is the quantum, many-body system in one-dimension with delta-function two-body interaction;[11] namely, with Hamiltonian

$$H_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{i<j} \delta(x_i - x_j),$$

where N is the number of particles and $2c$ is the coupling constant. A basic quantity is the ground state energy per particle in the thermodynamic limit: If $E_0(N,L)$ is the ground state energy for the finite system of N particles on a circle of length L, then in the limit $N \to \infty$, $L \to \infty$, such that $\rho := N/L$ is fixed, the ground state energy per particle is

$$\varepsilon_0 := \lim \frac{E_0(N,L)}{N}.$$

In their now classic paper, Lieb and Liniger[11] showed, for particles with Bose statistics and repulsive interaction ($c > 0$), that $e_B := \varepsilon_0/\rho^2$ is a function only of $\gamma := c/\rho$. To state their basic result, we first define the *Lieb-Liniger operator*

$$\mathcal{L}_\kappa f(x) := \frac{\kappa}{\pi} \int_{-1}^{1} \frac{f(y)}{(x-y)^2 + \kappa^2}\, dy, \quad -1 < x < 1. \tag{1}$$

Here $\kappa > 0$. If $f_B(x;\kappa)$ solves the Lieb-Liniger integral equation

$$f(x) - \mathcal{L}_\kappa f(x) = 1, \tag{2}$$

then $e_B(\gamma)$ is determined, by the elimination of κ, from the relations

$$\frac{\kappa}{\gamma} = \frac{1}{2\pi} \int_{-1}^{1} f_B(x;\kappa)\, dx, \quad e_B(\gamma) = \frac{1}{2\pi} \left(\frac{\gamma}{\kappa}\right)^3 \int_{-1}^{1} x^2 f_B(x;\kappa)\, dx.$$

Since $\mathcal{L}_\kappa$ tends to the identity operator as $\kappa \to 0$, the asymptotic expansion of $e_B(\gamma)$ as $\gamma \to 0$ is nontrivial. (The asymptotics for $\gamma \to \infty$ are straightforward.) It is "known"[4,11,14,19] that as $\gamma \to 0$,

$$e_B(\gamma) = \gamma - \frac{4}{3\pi}\gamma^{3/2} + \left[\frac{1}{6} - \frac{1}{\pi^2}\right]\gamma^2 + o(\gamma^2).$$

The first two terms in the above expansion have been rigorously justified[4] using the results of Hutson.[6]

A natural question to ask is how the problem changes when the particles obey Fermi statistics. The generalization of Bethe Ansatz to this case was solved by Gaudin[3,4] and Yang[22] (see also Refs. 13 and 16). In particular, for spin-1/2 particles with attractive interaction ($c < 0$) with total spin zero, the ground state energy per particle in the thermodynamic limit is given by[4]

$$\frac{\varepsilon_0}{\rho^2} = -\frac{\gamma^2}{4} + e_F(\gamma),$$

where $\gamma = |c|/\rho$ and the equation is now the *Gaudin integral equation*

$$f(x) + \mathcal{L}_\kappa f(x) = 1. \tag{3}$$

If $f_F(x; \kappa)$ solves this equation, then $e_F(\gamma)$ is determined by the elimination of κ from the equations

$$\frac{\kappa}{\gamma} = \frac{2}{\pi} \int_{-1}^{1} f_F(x; \kappa)\, dx, \quad e_F(\gamma) = \frac{2}{\pi}\left(\frac{\gamma}{\kappa}\right)^3 \int_{-1}^{1} x^2 f_F(x; \kappa)\, dx. \tag{4}$$

The integral equations (2) and (3) are well-known in the potential theory literature under the name *Love integral equation*.[12,15] Equation (3) also arises in the computation of the charge Q on each of the two coaxial conducting discs of radius one separated by a distance κ and each maintained at the same unit potential. For the Lieb-Liniger equation (2), the discs are maintained at opposite potentials ± 1. In both cases, the charge Q is given by a constant times the zeroth moment of f. For the case of equal potentials (the Fermi case), the charge is given by

$$Q = \frac{1}{\pi} \int_{-1}^{1} f_F(x; \kappa)\, dx \tag{5}$$

and Leppington and Levine[10] proved rigorously that as $\kappa \to 0$,

$$Q = \frac{1}{\pi} + \frac{\kappa}{2\pi^2} \log \kappa^{-1} + \frac{\kappa}{2\pi^2}(\log \pi + 1) + o(\kappa). \tag{6}$$

The authors did not analyze the integral equation (3), but rather found an approximate solution of the related boundary value problem. In later work, Atkinson and Leppington[1] analyzed the integral equation directly by the formal method of matched asymptotic expansions (by finding "outer" and "inner" solutions). In this way they reproduced the result (6).

Gaudin used an approximate solution to (3) (see (11.105) in Ref. 4) to conjecture that

$$e_F(\gamma) = \frac{\pi^2}{12} - \frac{\gamma}{2} + o(\gamma). \tag{7}$$

We say "conjecture" since the approximate solution used is not valid near $x = \pm 1$ (as Gaudin himself pointed out); therefore, the error is not controlled.

Guan and Ma,[5] using an approximate solution of (4) due to Takahashi,[18] deduced the error bound $O(\gamma^2)$ in (7). (Although Krivnov and Ovchinnikov[9] had predicted earlier that the term $\gamma^2 \log^2 \gamma^{-1}$ appears.) Their method generalizes to the ground state energy with a weakly repulsive interaction and with polarization. Using different methods to analyze (3), Iida and Wadati[7] found

$$e_F(\gamma) = \frac{\pi^2}{12} - \frac{\gamma}{2} + \frac{\gamma^2}{6} + o(\gamma^2).$$

Their analysis was also heuristic since it involved the manipulation of divergent series. When these same methods were applied to $e_B(\gamma)$, the coefficient of the γ^2 was in error; but it is uncertain whether this is simply due to an arithmetical error or a more serious error in the method. (See the discussion in Ref. 19.)

In this note, we provide a rigorous framework for the asymptotics associated with the Fermi gas, based on the integral equation (3). Although we only verify (7) by showing that

$$e_F(\gamma) = \frac{\pi^2}{12} - \frac{\gamma}{2} + O(\gamma^2 \log^2 \gamma^{-1}), \tag{8}$$

it is in principle possible (although perhaps tedious) to use the formulas derived here to get further terms in the asymptotics.

Our analysis uses an analogue of a Wiener-Hopf method used by the second author[20] to derive asymptotics for finite Toeplitz matrices, which are the discrete analogues of finite convolution operators. The main point is that one can replace Equation (3) involving the convolution operator $\mathcal{L}_\kappa$ by one involving a pair of Hankel operators for which the Neumann series for the solution is an asymptotic expansion as $\kappa \to 0$.

This method applies to the Fermi gas because, after scaling, the operator $I + \mathcal{L}_\kappa$ (here I is the identity operator) has an invertible limit as $\kappa \to 0$. For the operator $I - \mathcal{L}_\kappa$ associated with the Bose gas, the corresponding limit is not invertible.

In Secs. II and III, we consider a general class of convolution operators and derive the equivalent equations involving a pair of Hankel operators. Then we apply this to the Gaudin operator $I + \mathcal{L}_\kappa$. In Secs. IV–VI, we derive the asymptotic results for Q and $\mathfrak{e}_F(\gamma)$, the former rigorously and the latter not. In Sec. VII, we make rigorous the asymptotics for $\mathfrak{e}_F(\gamma)$.

II. GENERALITIES

Instead of a convolution operator on a fixed interval whose kernel depends on the small parameter κ, we consider an operator with fixed kernel on a large interval of length r. The two are easily interchangeable. The convolution equation is

$$\int_{-r/2}^{r/2} k(x-y) f(y) \, dy = g(x), \quad -r/2 < x < r/2.$$

Think of k, defined on all of $\mathbb{R}$, as having a δ-summand and extend f (the unknown function) and g to be zero outside $(-r/2, r/2)$.

We use Fourier transforms. In our notation, the $x \to \xi$ Fourier transform has $e^{ix\xi}$ in the integrand; the $\xi \to x$ inverse Fourier transform has $e^{-ix\xi}$ in the integrand and the factor $1/2\pi$. At first we shall be formal, later more precise.

If σ is the Fourier transform of k and $\hat{f}$ resp. $\hat{g}$ is the Fourier transform of f resp. g, then $\sigma \hat{f} - \hat{g}$ is the Fourier transform of a function supported outside the interval $(-r/2, r/2)$. The Fourier transform of a function supported on $(r/2, \infty)$ equals $e^{ir\xi/2}$ times the Fourier transform of a function supported on $\mathbb{R}^+$, the positive reals, and the Fourier transform of a function supported on $(-\infty, -r/2)$ equals $e^{-ir\xi/2}$ times the Fourier transform of a function supported on $\mathbb{R}^-$. Therefore we can write the equation as

$$\sigma \hat{f} = \hat{g} + e^{ir\xi/2} h^+ + e^{-ir\xi/2} h^-,$$

where h^+ resp. h^- is the Fourier transform of a function supported on $\mathbb{R}^+$ resp. $\mathbb{R}^-$. We consider these the unknown functions, and we find equations for them. Once they are determined, so is $\hat{f}$.

We denote by $\psi \to \psi_\pm$ the conjugate by the Fourier transform of multiplication by $\chi_{\mathbb{R}^\pm}$, the characteristic functions of $\mathbb{R}^\pm$. These are given by

$$\psi_\pm(\xi) = \frac{1}{2}\psi(\xi) \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(\eta)}{\eta - \xi} \, d\eta,$$

where the integral is a principal value. The functions extend analytically to the upper and lower half-planes by the formulas

$$\psi_\pm(\xi) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi(\eta)}{\eta - \xi} \, d\eta,$$

where ξ is in the upper half-plane for ψ_+ and the lower half-plane for ψ_-.

The *Wiener-Hopf factors* of σ, which confusingly we denote by $\sigma_\pm$, are given by

$$\sigma_\pm = e^{(\log \sigma)_\pm},$$

where the $\pm$ on the right are the projection operators defined above. For appropriate σ (defined precisely below), $\sigma_\pm$ and $1/\sigma_\pm$ equal constants plus Fourier transforms of functions supported on $\mathbb{R}^\pm$. Since the convolution of two functions supported on $\mathbb{R}^\pm$ is also supported on $\mathbb{R}^\pm$, by changing

the notation we may replace our equation by

$$\sigma_-\sigma_+\hat{f} = \hat{g} + e^{ir\xi/2}\sigma_+ h^+ + e^{-ir\xi/2}\sigma_- h^-. \tag{9}$$

The inverse Fourier transform of $\hat{f}$ is supported on $(-r/2, \infty)$, so the inverse Fourier transform of $e^{ir\xi/2}\hat{f}$ is supported on $\mathbb{R}^+$, so also the inverse Fourier transform of $e^{ir\xi/2}\sigma_+\hat{f}$ is supported on $\mathbb{R}^+$. Therefore if we multiply (9) by $e^{ir\xi/2}/\sigma_-$ and apply the minus operator we get

$$0 = (e^{ir\xi/2}\sigma_+\hat{f})_- = \left(\frac{e^{ir\xi/2}\hat{g}}{\sigma_-}\right)_- + \left(e^{ir\xi}\frac{\sigma_+}{\sigma_-}h^+\right)_- + h^-.$$

Similarly

$$0 = \left(\frac{e^{-ir\xi/2}\hat{g}}{\sigma_+}\right)_+ + h^+ + \left(e^{-ir\xi}\frac{\sigma_-}{\sigma_+}h^-\right)_+.$$

Define the operators U and V by

$$Uu^- = \left(e^{-ir\xi}\frac{\sigma_-}{\sigma_+}u^-\right)_+, \quad Vv^+ = \left(e^{ir\xi}\frac{\sigma_+}{\sigma_-}v^+\right)_-.$$

The operator U takes the Fourier transforms of functions supported on $\mathbb{R}^-$ to the Fourier transforms of functions supported on $\mathbb{R}^+$, and V does the opposite. If we define

$$G^- = -\left(\frac{e^{ir\xi/2}\hat{g}}{\sigma_-}\right)_-, \quad G^+ = -\left(\frac{e^{-ir\xi/2}\hat{g}}{\sigma_+}\right)_+, \tag{10}$$

our two relations may be written as

$$h^- + Vh^+ = G^-, \quad h^+ + Uh^- = G^+.$$

This pair of equations for the functions $h^\pm$ is equivalent to the original equation for the function f. The solution is given by

$$\begin{pmatrix} h^- \\ h^+ \end{pmatrix} = \left(I + \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} G^- \\ G^+ \end{pmatrix} \tag{11}$$

as long as the inverse on the right side makes sense.

So far everything has been formal. Now we give precise conditions and explain the exact meaning of (11).

Denote by $\widehat{L}^1$ the Fourier transforms of L^1 functions, with the operations and norm inherited from L^1, and by $\widehat{L}^1_\pm$ the Fourier transforms of L^1 functions supported on $\mathbb{R}^\pm$. (In particular convolution on L^1 becomes multiplication on $\widehat{L}^1$.) The operators $\pm$ are projections from $\widehat{L}^1$ to $\widehat{L}^1_\pm$. Thus if $\log\sigma \in \widehat{L}^1$ then also $(\log\sigma)_\pm \in \widehat{L}^1_\pm$, and it follows that the functions $\sigma_\pm$, $1/\sigma_\pm$, σ_+/σ_-, σ_-/σ_+ all belong to $\mathbb{C} + \widehat{L}^1$. If only $\log\sigma \in \mathbb{C} + \widehat{L}^1$, then we modify the definitions of $\sigma_\pm$ by multiplying by appropriate constants, so the product is σ.

Since $\hat{g} \in \widehat{L}^1$, the functions $G^\pm$ defined by (10) belong to $\widehat{L}^1_\pm$.

We abuse the notation temporarily and replace U and V by their conjugates with the Fourier transform, and keep the same notation for the conjugates. If $k_{-/+}$ resp. $k_{+/-}$ is the inverse transform of σ_-/σ_+ resp. σ_+/σ_- the operator U has kernel

$$U(x, y) = k_{-/+}(x - y + r), \quad (x \in \mathbb{R}^+, \, y \in \mathbb{R}^-)$$

and V has kernel

$$V(x, y) = k_{+/-}(x - y - r), \quad (x \in \mathbb{R}^-, \, y \in \mathbb{R}^+).$$

After the variable changes $y \to -y$ for U and $x \to -x$ for V, these become Hankel operators acting on $L^1(\mathbb{R}^+)$; their kernels are $k_{-/+}(x + y + r)$ and $k_{+/-}(x + y + r)$. Since σ_+/σ_- and σ_-/σ_+ belong to $\mathbb{C} + \widehat{L}^1$, the functions $k_{-/+}(x)$ and $k_{+/-}(x)$ both belong to L^1 away from $x = 0$. The norm of U is at most the norm of $k_{-/+}(x + r)$ in $L^1(\mathbb{R}^+)$, which is $o(1)$ as $r \to \infty$. Similarly for V. Therefore,

the inverse on the right side of (11) is well-defined and the Neumann series for the inverse gives an $\widehat{L}^1$-convergent asymptotic expansion for the left side.

In particular, we get convergent asymptotic expansions for $h^\pm(0)$, since the linear functional $h \to h(0)$ is continuous on $\widehat{L}^1$. For the Fermi gas, this will be enough to get the terms in an asymptotic expansion for $\hat{f}(0)$ and so for Q. We can also find the terms in an asymptotic expansion for $\mathfrak{e}_F(\gamma)$, but this will not be rigorous at first. The reason is that the coefficients of the higher powers of ξ in the asymptotic expansion of $h^\pm$ (and therefore the higher moments of f) are not continuous in the norm of $\widehat{L}^1$. To justify these asymptotics, we will need more properties of the operators U and V, among other things. This will be done in Section VII.

III. THE FERMI GAS

To avoid factors of $\sqrt{2}$ later, we consider the integral equation

$$\frac{f(x)}{2} + \frac{1}{2\pi} \int_{-r/2}^{r/2} \frac{f(y)}{(x-y)^2 + 1} \, dy = 1,$$

so that

$$\sigma(\xi) = (1 + e^{-|\xi|})/2, \quad \hat{g}(\xi) = 2 \, \sin(r\xi/2)/\xi.$$

The κ in (1) and r are related by $r = 2/\kappa$. The solution $f_F(x; \kappa)$ of (3) and our $f(x)$ are related by $f(rx/2) = 2 f_F(x; \kappa)$. From (5), we get

$$Q = \frac{1}{r\pi} \int_{-r/2}^{r/2} f(x) \, dx = \frac{\kappa}{2\pi} \int_{-r/2}^{r/2} f(x) \, dx. \tag{12}$$

From the first part of (4), we find that

$$\gamma = \kappa \left(\frac{2}{\pi r} \int_{-r/2}^{r/2} f(x) \, dx \right)^{-1} = \frac{1}{2} \kappa Q^{-1}. \tag{13}$$

Since $\sigma_\pm(0) = 1$ and $\hat{g}(0) = r$, we have

$$\int_{-r/2}^{r/2} f(x) \, dx = \hat{f}(0) = r + h^+(0) + h^-(0), \tag{14}$$

which by (12) determines Q from $h^\pm(0)$. From both parts of (4) and a little computation, we find that

$$\mathfrak{e}_F(\gamma) = \pi^2 \frac{\displaystyle\int_{-r/2}^{r/2} x^2 f(x) \, dx}{\left(\displaystyle\int_{-r/2}^{r/2} f(x) \, dx \right)^3}. \tag{15}$$

The functions in (10) are here

$$G^- = i \left(\frac{e^{ir\xi} - 1}{\xi \sigma_-} \right)_-, \quad G^+ = i \left(\frac{1 - e^{-ir\xi}}{\xi \sigma_+} \right)_+ \tag{16}$$

and the Wiener-Hopf factors are given by

$$\log \sigma_\pm(\xi) = \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log \left((1 + e^{-|\eta|})/2 \right)}{\eta - \xi} \, d\eta.$$

For $\xi \in \mathbb{R}$, these become the limits as $\xi \to \mathbb{R}$ from above and below $\mathbb{R}$.

It is easy to see that the inverse Fourier transform of $\log(1 + e^{-|\xi|})$ belongs to L^1. It follows, as discussed earlier, that the functions $\sigma_\pm$, $1/\sigma_\pm$, σ_+/σ_-, σ_-/σ_+, restricted to $\mathbb{R}$, all belong to $\mathbb{C} + \widehat{L}^1$.

It is also easy to see that $\sigma_\pm(\xi)$ and $1/\sigma_\pm(\xi)$ extend boundedly and analytically to any substrip of $|\text{Im}\,\xi| < \pi$ cut along the imaginary axis. Explicitly, we have[1]

$$\sigma_+(\xi) = \pi^{1/2} \exp\left\{\frac{\xi}{2\pi i}\left[\log(-i\xi) - \log 2\pi - 1\right]\right\} \Gamma\left(\frac{1}{2} + \frac{\xi}{2\pi i}\right)^{-1},$$

$$\sigma_-(\xi) = \pi^{1/2} \exp\left\{-\frac{\xi}{2\pi i}\left[\log(i\xi) - \log 2\pi - 1\right]\right\} \Gamma\left(\frac{1}{2} - \frac{\xi}{2\pi i}\right)^{-1}.$$

For ξ in the upper resp. lower half-plane, $-i\xi$ resp. $i\xi$ lies in the right half-plane and the principal values of the logarithms are taken.

We first find bounds for the norms of the operators U and V better than the bound $o(1)$ which holds generally. As discussed at the end of Sec. II, the kernels of the conjugates of the operators by the Fourier transform are given in terms of the inverse Fourier transforms of σ_-/σ_+ and σ_+/σ_- for $x > 0$ and $x < 0$, respectively. The inverse Fourier transform of σ_-/σ_+ is equal to the δ-function plus $1/2\pi$ times

$$\int_{-\infty}^{\infty} e^{-ix\xi}\left(\frac{\sigma_-(\xi)}{\sigma_+(\xi)} - 1\right) d\xi \tag{17}$$

and here $x > 0$. The function $\sigma_-(\xi)$ is analytic in the lower half-plane.

We use $\sigma_-(\xi)/\sigma_+(\xi) = \sigma_-(\xi)^2/\sigma(\xi)$. The function $1/\sigma(\xi)$ extends boundedly and analytically to the lower half-plane cut along the negative imaginary axis. We swing the $\mathbb{R}^+$ part of the contour clockwise down to the right side of the imaginary axis and the $\mathbb{R}^-$ part of the contour counter-clockwise down to the left side of the imaginary axis. There we make the substitution $\xi = -iy$. The analytic continuation of $1/\sigma(\xi)$ to the right side of the imaginary axis minus its analytic continuation to the left side of the imaginary axis equals

$$\frac{2}{1 + e^{iy}} - \frac{2}{1 + e^{-iy}} = -2i\tan(y/2).$$

Therefore, the integral equals

$$-\frac{1}{\pi} \int_0^{\infty} e^{-xy} \sigma_-(-iy)^2 \tan(y/2)\, dy. \tag{18}$$

(This is a principal value integral at each odd multiple of π. The contributions of the integrals over the little semicircles on either side of the imaginary axis cancel each other.) The non-exponential part of the integrand is $O(y)$ as $y \to 0$, so the integral itself is $O(x^{-2})$ as $x \to \infty$.

The kernel of U, an operator from $L^1(\mathbb{R}^-)$ to $L^1(\mathbb{R}^+)$, is $k_{-/+}(x - y + r)$. We have shown that $k_{-/+}(x) = O(x^{-2})$ for $x > 0$. Therefore

$$\|U\| \le \int_0^{\infty} |k_{-/+}(x + r)|\, dx = O(r^{-1}).$$

Similarly $\|V\| = O(r^{-1})$.

IV. EXPANSION OF $G^\pm(\xi)$ NEAR $\xi = 0$

The expression for $G^+(\xi)$ for ξ in the upper half-plane is

$$G^+(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-ir\eta}}{\eta\,\sigma_+(\eta)} \frac{d\eta}{\eta - \xi}.$$

We push the contour up to $\mathbb{R} + ic$ with $c > \text{Im}\,\xi$, pass the pole at $\eta = \xi$, and obtain

$$i\frac{1 - e^{-ir\xi}}{\xi\,\sigma_+(\xi)} - \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} \frac{e^{-ir\eta}}{\eta\,\sigma_+(\eta)} \frac{d\eta}{\eta - \xi}.$$

(The reason the summand 1 does not appear in the integrand is that the contour for the integral with it can be pushed all the way up.) The integral can be expanded in powers of ξ near $\xi = 0$, and the

above equation becomes

$$i\,\frac{1-e^{-ir\xi}}{\xi\,\sigma_+(\xi)} - \frac{1}{2\pi}\sum_{k=0}^{\infty}\xi^k\int_{ic-\infty}^{ic+\infty}\frac{e^{-ir\eta}}{\sigma_+(\eta)}\,\eta^{-k-2}\,d\eta = i\,\frac{1-e^{-ir\xi}}{\xi\,\sigma_+(\xi)} + \sum_{k=0}^{\infty}g_k^+\,\xi^k. \tag{19}$$

Near zero there is an expansion, valid for $|\xi| < \pi$,

$$\frac{1}{\sigma_+(\xi)} = \sum_{0\le m\le n} a_{n,m}\,\xi^n\log^m(-i\xi),$$

with $a_{0,0} = 1$. So the first term in (19) is equal to

$$i\,(1-e^{-ir\xi})\sum_{0\le m\le n} a_{n,m}\,\xi^{n-1}\log^m(-i\xi).$$

To determine the asymptotics of the coefficients

$$g_k^+ = -\frac{1}{2\pi}\int_{ic-\infty}^{ic+\infty}\frac{e^{-ir\eta}}{\sigma_+(\eta)}\,\eta^{-k-2}\,d\eta$$

as $r \to \infty$ we do what we did for the Fourier transform of σ_-/σ_+, with some change. Now we make sure that $c \in (0,\pi)$ and deform the contour to the one from $-ic-\infty$ to $-ic-0$, up the left side of the imaginary axis, around a circle with center zero (say with radius $c/2$), then down the right side of the imaginary axis to $-ic+0$, and finally to $-ic+\infty$. The integral over the horizontal parts of the contour is exponentially small as $r \to \infty$. If we take only the terms with $n \le N$ in the series for $1/\sigma_+(\eta)$, the error is $O(\eta^N)$ near $\eta = 0$. Then we can shrink the circle around zero, getting an integral over portions of the imaginary axis of the form (after making the substitution $\eta = -iy$)

$$\int_0^c e^{-ry}\,O(y^{N-k-2})\,dy.$$

This is $O(r^{-N+k+1})$.

It follows that for an asymptotic expansion of g_k^+ as $r \to \infty$, we may replace $1/\sigma_+(\xi)$ by the sum for it and then interchange the sum over n,m with the integral. The contour goes around the circle and over portions of the imaginary axis. For each summand, we may complete the contour by adding the vertical lines from $-ic\pm 0$ to $-i\infty\pm 0$, incurring only an exponential small error. Think of the resulting contour $\mathcal{C}$ as starting from $-i\infty-0$, looping clockwise around zero, and then down to $-i\infty+0$.

We have shown that an asymptotic expansion for g_k^+ is given by

$$g_k^+ \sim -\frac{1}{2\pi}\sum_{0\le m\le n} a_{n,m}\int_{\mathcal{C}} e^{-ir\eta}\,(-i\eta)^{n-k-2}\log^m(-i\eta)\,d\eta.$$

To identify the integral, we consider first the integral

$$\int_{\mathcal{C}} e^{-ir\eta}\,(-i\eta)^{-s-1}\,d\eta = r^s\int_{\mathcal{C}} e^{-i\eta}\,(-i\eta)^{-s-1}\,d\eta,$$

with s a continuous variable and the specification $|\arg(-i\eta)| < \pi$. If we make the substitution $\eta \to -i\eta$, then this becomes

$$-i\,r^s\int_{i\mathcal{C}} e^{-\eta}\,(-\eta)^{-s-1}\,d\eta.$$

The contour $i\mathcal{C}$ loops around the positive real axis clockwise. The integral is recognized as one for the reciprocal of the gamma function (Ref. 21 [Section 12.22]), and we see that the above equals $2\pi\,r^s\,\Gamma(s+1)^{-1}$.

If we differentiate m times with respect to s, we obtain

$$\int_{\mathcal{C}} e^{-ir\eta}\,(-i\eta)^{-s-1}\log^m(-i\eta)\,d\eta = 2\pi\left(-\frac{d}{ds}\right)^m\frac{r^s}{\Gamma(s+1)}$$

and so

$$\int_{\mathcal{C}} e^{-ir\eta}\,\eta^{n-k-2}\log^m(-i\eta)\,d\eta = -2\pi\,(-i)^{k-n}\left(-\frac{d}{ds}\right)^m\frac{r^s}{\Gamma(s+1)}\Big|_{s=k-n+1}.$$

Putting things together gives the small-ξ expansion

$$G^+(\xi) = i\,(1 - e^{-ir\xi}) \sum_{0 \le m \le n} a_{n,m}\, \xi^{n-1} \log^m(-i\xi)$$

$$+ \sum_{k=0}^{\infty} (-i\xi)^k \sum_{0 \le m \le n} i^n\, a_{n,m} \left(-\frac{d}{ds}\right)^m \frac{r^s}{\Gamma(s+1)}\Big|_{s=k-n+1}, \tag{20}$$

where the sum multiplying $(-i\xi)^k$ is an asymptotic expansion as $r \to \infty$ for its coefficient.

Remark. An easy check shows that the sum of terms with fixed n and $m = 0$ when $n = 0$ and $i\,\xi^{n-1}\, a_{n,0}$ when $n > 0$. In particular, only terms with $n > 0$ contribute to the sum.

The coefficient of ξ^k is of order r^k. Any term with $n > 1$ will contribute to this coefficient at most $r^{k-1} \log^2 r$. So for a first approximation, we may take $n = 1$ only. When $m = 0$, we use the remark. We have (here γ is the Euler $\gamma = -\Gamma'(1)$)

$$a_{1,0} = \frac{i}{2\pi}(\gamma - \log(\pi/2) - 1), \;\; a_{1,1} = \frac{i}{2\pi}$$

and using this and (20) we find the following coefficients of ξ^k up to $k = 2$, with error $r^{k-1} \log^2 r$:

$$k = 0: \;\; i\,a_{1,0} - i(\log r + \gamma)\,a_{1,1} = \frac{1}{2\pi}(\log r + \log(\pi/2) + 1),$$

$$k = 1: \;\; -a_{1,1}\,r\,(\log r + \gamma - 1) = -\frac{i}{2\pi}r\,(\log r + \gamma - 1),$$

$$k = 2: \;\; i\,a_{1,1}\frac{r^2}{2}\,(\log r + \gamma - 3/2) = -\frac{1}{4\pi}r^2\,(\log r + \gamma - 3/2).$$

In particular,

$$G^+(0) = \frac{1}{2\pi}(\log r + \log(\pi/2) + 1) + O(r^{-1}\log^2 r). \tag{21}$$

Remark. The coefficients above come from the last sum in (20). The $n = m = 1$ term from the other sum does not contribute to the constant term, and its eventual contribution to the ξ^2 term on the right side of (9) will be of lower order and so may be ignored.

V. CALCULATION OF Q

It follows from the fact that the norms of U and V are $O(r^{-1})$ that

$$|h^+(0) - G^+(0)| \le \|h^+ - G^+\| = O(r^{-1}\|G^+\|), \tag{22}$$

the last norm being that in $\widehat{L^1}$. This is the same as the L^1-norm of the inverse Fourier transform of G^+. This inverse Fourier transform is the convolution of the inverse Fourier transform of $i(1 - e^{-ir\xi})/\xi$, which is minus $\chi_{(-r,0)}$, and the inverse Fourier transform of $1/\sigma_+(\xi)$. If we denote the latter by $s(x)$, then the convolution at x equals

$$\int_x^{x+r} s(y)\, dy.$$

The norm of this in $L^1(\mathbb{R}^+)$ is bounded by

$$\int_0^{\infty} \min(r, y)\, |s(y)|\, dy.$$

Since $1/\sigma_+ \in \mathbb{C} + \widehat{L^1}$, we know that $s(y)$ is a linear combination of a δ-function, which drops out, and an L^1 function. Therefore, the integral over $(0, 1)$ is bounded. For the rest, an argument analogous to the one in which we bounded the Fourier transform of σ_-/σ_+ shows that $s(y) = O(y^{-2})$.

One sees from this that the integral over $(1, \infty)$ is $O(\log r)$. Hence (22) is $O(r^{-1} \log r)$, and with this and (21) we have shown

$$h^+(0) = \frac{1}{2\pi}(\log r + \log(\pi/2) + 1) + O(r^{-1}\log^2 r).$$

Similarly for $h^-(0)$. Therefore, from (14) and (12) we obtain

$$\int_{-r/2}^{r/2} f(x)\,dx = \hat{f}(0) = r + \frac{1}{\pi}(\log r + \log(\pi/2) + 1) + O(r^{-1} \log^2 r), \tag{23}$$

$$Q = \frac{1}{\pi} + \frac{\kappa}{2\pi^2}\log \kappa^{-1} + \frac{\kappa}{2\pi^2}(\log \pi + 1) + O(\kappa^2 \log^2 \kappa^{-1}). \tag{}$$

This agrees with (6).

VI. CALCULATION OF $\mathfrak{e}_F(\gamma)$

It follows from (23) that

$$\left(\int_{-r/2}^{r/2} f(x)\,dx\right)^{-3} = r^{-3}\left(1 - 3\pi^{-1}(\log r + \log(\pi/2) + 1)\,r^{-1} + O(r^{-2} \log^2 r)\right). \tag{24}$$

This gives the denominator in (15). For the numerator is easy to see that

$$\int_{-r/2}^{r/2} x^2 f(x)\,dx = -\hat{f}''(0)$$

$$= \hat{f}(0)/2 - 2 \times \text{the coefficient of } \xi^2 \text{ in the expansion of } \sigma(\xi)\hat{f}(\xi). \tag{25}$$

The first term is $O(r)$, as we know, and will be of lower order than the rest. The $\hat{g}(\xi)$-term in (9) gives to the coefficient the contribution $-r^3/24 + O(r)$, so we have to consider the $h^\pm$ terms. *Now we assume that the coefficients in the expansion of $G^\pm$ are first-order approximations to the corresponding coefficients for $h^\pm$.*

From (9), we must multiply $G^+(\xi)$ by $e^{ir\xi/2}$ and take the coefficient of ξ^2. (We may ignore the factor $\sigma_+(\xi)$ because it would only contribute extra powers of ξ without accompanying powers of r.) From the coefficients of $G^+(\xi)$ we computed, we find that the coefficient of ξ^2 in $e^{ir\xi/2}\,G^+(\xi)$ is

$$-\frac{r^2}{16\pi}(\log r + \log(\pi/2) - 1) + O(r \log^2 r).$$

We double this (there is an equal term from the h^- summand), add the term $-r^3/24$ from the $\hat{g}$-term in (9), and multiply by (24). The result is

$$-\frac{1}{24}\left(1 - \frac{6}{\pi r}\right) + O(r^{-2} \log^2 r) = -\frac{1}{24}\left(1 - \frac{6\gamma}{\pi^2}\right) + O(\gamma^2 \log^2 \gamma^{-1}),$$

since $\gamma = \pi/r + O(\log^2 r/r^2)$. (See (13) and the asymptotics of Q.) Then multiplying by -2 from (25) and π^2 from (15) gives

$$\mathfrak{e}_F(\gamma) = \frac{\pi^2}{12} - \frac{\gamma}{2} + O(\gamma^2 \log^2 \gamma^{-1}).$$

To make this rigorous, we have to show that the coefficients in the asymptotic expansions of $h^\pm$ are, to a first order, equal to the corresponding coefficients in the asymptotic expansions of $G^\pm$. This, and more, will be done in Sec. VII.

VII. MAKING THE ASYMPTOTICS RIGOROUS

In the following lemmas, when we consider the result of the operators $\psi \to \psi_\pm$, the function $\psi_\pm$ will mean its analytic extension to the corresponding half-plane.

We use the following notation: A bound $O(r^{n+})$ as $r \to \infty$ means $O(r^n \log^m r)$ for some $m \geq 0$. Analogously, $O(\xi^{n+})$ as $\xi \to 0$ means $O(|\xi|^n \log^m |\xi|^{-1})$ for some $m \geq 0$. This notation makes it unnecessary to keep the track of logarithm factors in bounds.

We define $\mathcal{L}^+$ to be those families of functions $\varphi \in \widehat{L}^1_+$ (Fourier transforms of L^1 functions supported on $\mathbb{R}^+$), depending on the parameter r, satisfying the following:

(i) $\|\varphi\| = O(r^{0+})$ as $r \to \infty$;
(ii) There is an asymptotic expansion as $\xi \to 0$,

$$\varphi(\xi) \sim \sum_{0 \leq m \leq n} c_{n,m,r} \, \xi^n \log^m(-i\xi),$$

where each $c_{n,m,r} = O(r^{n+})$ as $r \to \infty$; stopping the series with terms $n < N$ leads to an error $O((r\xi)^{N+})$.

Similarly, we define $\mathcal{L}^-$. It is easy that $\mathcal{L}^+$ is closed under multiplication by $e^{\pm ir\xi/2}$ or $\sigma_+(\xi)^{\pm 1}$. This will be relevant when we go back to (9). In the following lemma, $r\mathcal{L}^+$ denotes r times the functions in $\mathcal{L}^+$. Analogous notation will be used later.

Lemma 1. We have $\hat{g}_+, (\sigma \hat{f})_+ \in r\mathcal{L}^+$.

Proof. The statement concerning $\hat{g}_+$ is almost immediate. As for $\hat{f}$, if K is the operator with convolution kernel then $f = (I + K)^{-1}1 = (I - K^2)^{-1}(I - K)1 > 0$ since $(K1)(x) < 1$. Since $f + Kf = 1$, it follows that $0 < f < 1$, so $\|\hat{f}\| = \|f\| < r$. Therefore $\|\sigma \hat{f}\| = O(r)$, and then $\|(\sigma \hat{f})_+\| = O(r)$. Thus (i), with the extra factor r, is satisfied.

For (ii), we write $\hat{f} = \hat{f}_+ + \hat{f}_-$ and first consider $(\sigma \hat{f}_-)_+$. For ξ in the upper half-plane

$$(\sigma f_-)_+(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \sigma(\eta) \, f_-(\eta) \, \frac{d\eta}{\eta - \xi}.$$

We do something similar to what we did before, namely swing the $\mathbb{R}^+$ part of the contour down to the right part of the imaginary and the $\mathbb{R}^-$ part of the contour down to the left part of the imaginary axis. Then we set $\eta = -iy$. The continuation of $\sigma(\eta)$ to the right part of the imaginary axis minus the continuation of $\sigma(\eta)$ to the left part of the imaginary axis equals $i \sin y$, so the integral becomes

$$\frac{1}{2\pi} \int_0^{\infty} \hat{f}_-(-iy) \, \sin y \, \frac{dy}{y - i\xi}$$

and $-i\xi$ is in the right half-plane. We rewrite this as

$$\frac{1}{2\pi} \int_0^{r/2} f(-x) \, dx \int_0^{\infty} e^{-xy} \, \sin y \, \frac{dy}{y - i\xi}. \tag{26}$$

For the inner integral, we use the general formula (29) below in the limit $s \to 0$, with ξ replaced by $-\xi$ and r by $x' := x \pm i$, to obtain

$$\int_0^{\infty} e^{-x'y} \, \frac{dy}{y - i\xi} = -e^{-ix'\xi} \log(-ix'\xi) + \sum_{n=0}^{\infty} \frac{(-ix'\xi)^n}{n!} \, \psi(n+1),$$

where $\psi = \Gamma'/\Gamma$. Then we replace x' by $x \pm i$, multiply by $f(-x)$, integrate, subtract the two results, and divide by $2i$. We find a series of the form in (ii) with bounds on the coefficients that are required.

For what comes below, we note that the $\log(-i\xi)$ terms in the expansion of the inner integral in (26) combine as

$$-\frac{e^{-ix\xi}}{2i} (e^{-\xi} - e^{\xi}) \log(-i\xi). \tag{27}$$

The error after stopping the power series above at ξ^{N-1} is $O((x\xi)^N)$ and stopping the series for the logarithm term there is $O((x\xi)^N \log x)$. After multiplying by $f(-x)$ and integrating over $(0, r/2)$, we get the bound $O(\xi^N r^{N+1} \log r)$ for the error. This will give the rest of (ii), with the extra factor r.

We are not finished because we want $(\sigma \hat{f})_+$ and what we considered was $(\sigma \hat{f}_-)_+$. That leaves

$$(\sigma \hat{f}_+)_+(\xi) = \frac{1}{2\pi i} \int_0^{r/2} f(x) \, dx \int_{-\infty}^{\infty} e^{ix\eta} \, \sigma(\eta) \frac{d\eta}{\eta - \xi}.$$

For the inner integral, we deform to the imaginary axis, and when we swing the $\mathbb{R}^+$ part of the contour we use $\sigma(\eta) = (e^{-\eta} + 1)/2$ and when we swing the $\mathbb{R}^-$ part of the contour we use $\sigma(\eta) = (e^{\eta} + 1)/2$. Suppose for definiteness that ξ, which is in the upper half-plane, is also in the right half-plane. Then we pass the pole at $\eta = \xi$ when we swing the $\mathbb{R}^+$ part of the contour and find that the inner integral becomes (after the substitution $\eta = iy$ on the positive imaginary axis)

$$\pi i \, e^{ix\xi} (e^{-\xi} + 1) - i \int_0^\infty e^{-xy} \, \sin y \, \frac{dy}{y + i\xi}.$$

The integral here is the inner integral in (26) with ξ replaced by $-\xi$. Thus the integral with its factor equals a power series in ξ (with the bound after stopping at powers of ξ less than N) plus the logarithm term. This is obtained from (27) after the substitution $\xi \to -\xi$ and multiplying by the factor $-i$, with the result

$$\frac{e^{ix\xi}}{2} (e^{\xi} - e^{-\xi}) \log(i\xi). \tag{28}$$

How does $\log(i\xi)$ relate to $\log(-i\xi)$, the logarithm analytic for all ξ in the upper half-plane? All our logarithms are principal value. Thus for ξ in the upper half-plane, $\log(-i\xi)$ is determined by $|\arg(-i\xi)| < \pi/2$. With ξ in the first quadrant, as we have been taking it, we have $\pi/2 < \arg(i\xi) < \pi$ whereas $-\pi/2 < \arg(-i\xi) < 0$. Thus $\log(i\xi) = \log(-i\xi) + i\pi$. It follows that the term before the integral plus (28) equals

$$i\pi \, e^{ix\xi}(e^{-\xi} + 1) + i\frac{\pi}{2} \, e^{ix\xi} (e^{\xi} - e^{-\xi}) + \frac{e^{ix\xi}}{2} (e^{\xi} - e^{-\xi}) \log(-i\xi)$$

$$= \frac{i\pi}{2} \, e^{ix\xi} (e^{\xi} + e^{-\xi} + 2) + \frac{e^{ix\xi}}{2} (e^{\xi} - e^{-\xi}) \log(-i\xi).$$

Although we assumed that ξ is in the right half-plane, we now have a function that extends analytically to the entire upper half-plane, because $\log(-i\xi)$ does, so the same identity holds throughout this half-plane. We then multiply by $f(x)$ and integrate over $(0, r/2)$. The result satisfies condition (ii), with the extra factor r coming from the first summand. We then continue as before with the power series. □

Lemma 2. If $\varphi \in \mathcal{L}^+$ then $V\varphi \in r^{-1} \mathcal{L}^-$, and if $\varphi \in \mathcal{L}^-$ then $U\varphi \in r^{-1} \mathcal{L}^+$.

Proof. We derive the first statement. For (i), we use that $\|V\| = O(r^{-1})$. For (ii), an argument analogous to the one that leads from (17) to (18) gives

$$V\varphi(\xi) = -\frac{1}{\pi} \int_0^\infty e^{-ry} \, \sigma_+(iy)^2 \, \varphi(iy) \, \tan(y/2) \frac{dy}{y + i\xi}.$$

Here ξ is in the lower half-plane, so $i\xi$ is in the right half-plane. We have $\varphi(iy) = O(r)$ uniformly in y by (i). The integral over $(1, \infty)$ equals a power series in ξ each of whose coefficients is exponentially small in r. The error after stopping the series at ξ^{N-1} is bounded by ξ^N times an exponentially small factor. So we may replace the interval of integration by $(0, 1)$.

If we truncate the series for $\sigma_+(iy)^2$ at all powers of y less than N the error is $O(y^{N+})$. For $\varphi(iy)$, the error is $O((ry)^{N+})$, by assumption (ii). Therefore, because of the factor $\tan(y/2)$, if we truncate the series for $\sigma_+(iy)^2 \varphi(iy)$ at all powers of y less than N the error is $O(r^N y^{N+1+})$. With the error only, we use

$$\frac{1}{y + i\xi} = \sum_{n=0}^N (-i\xi)^n \, y^{-n-1} + O(\xi^N \, y^{-N-1}).$$

For $n < N$, the nth term of the series, after multiplying by the error and integrating, gives ξ^n times $O(r^{n-1+})$. So the integral with the error term instead gives an expansion with powers of ξ up to $N-1$ satisfying the condition (ii) with an extra factor r^{-1}. It follows that to establish (ii), with this extra factor, for the full integrand it suffices to establish it for each summand of the integrand. And for this we may go back to the interval of integration $(0, \infty)$.

So consider the integral when $\sigma_+(iy)^2 \varphi(iy) \tan(y/2)$ is replaced by $y^{n+1} \log^m y$ with $m \leq n$, which may come from one term of its expansion. (The extra power of y comes from the factor $\tan(y/2)$.) The coefficient is $O(r^{n+})$. We use

$$\int_0^\infty e^{-ry} \, y^s \, \frac{dy}{y + i\xi} = \frac{\pi}{\sin \pi s} \int_{iC} e^{-ry} \, (-y)^s \, \frac{dy}{y + i\xi},$$

where as in Section IV, the contour iC loops around the positive real axis clockwise. The pole at $-i\xi$ is outside the contour. If we expand the contour to pass the pole, we get a contribution from the residue at the pole, and the integrand in the resulting integral can be expanded in powers of ξ. The coefficients are, as in Section IV, integrals for the reciprocal of the gamma function. The result is

$$\int_0^\infty e^{-ry} \, y^s \, \frac{dy}{y + i\xi} = \frac{\pi}{\sin \pi s} \left[-e^{ir\xi} (i\xi)^s + r^{-s} \sum_{k=0}^\infty \frac{(ir\xi)^k}{\Gamma(k - s + 1)} \right]. \tag{29}$$

For $s = n + 1$, the expression in brackets vanishes and the integral equals $(-1)^{n+1}$ times the derivative with respect to s of the expression in brackets, evaluated at $s = n + 1$. The error after stopping at powers of ξ less than N is $O(r^{-n-1+}(r\xi)^N) = O(r^{-1+}(r\xi)^N)$, which is the part of (ii) with the extra factor r^{-1}. To get

$$\int_0^\infty e^{-ry} \, y^s \log^m y \, \frac{dy}{y + i\xi},$$

with $m \leq n$ we first differentiate m times with respect to s.

Consider the coefficient of ξ^k, with $k \neq n + 1$, coming from the series. After differentiating m times with respect to s and setting $s = n + 1$, the coefficient is at most $O(r^{k-n-1} \log^m r)$. Multiplying by the original coefficient $O(r^{n+})$ gives $O(r^{k-1+})$.

For the term involving ξ^{n+1}, we get a factor of at most $\log^{m+1}(i\xi)$, so at most $\log^{n+1}(i\xi)$. For the coefficient, we get at most $O(\log^{m+1} r)$ which, when multiplied by the original coefficient $O(r^{n+})$, gives $O(r^{n+}) = (r^{-1} r^{n+1+})$.

This verifies the rest of (ii) with the extra factor r^{-1}. □

Lemma 3. We have $G^\pm \in \mathcal{L}^\pm$.

Proof. We already know that $\|G^+\| = O(\log r)$. For (ii), we write

$$2\pi \, G^+(\xi) = \int_{-\infty}^\infty \frac{1}{\eta \, \sigma_+(\eta)} \frac{d\eta}{\eta - \xi} - \int_{-\infty}^\infty \frac{e^{-ire}}{\eta \, \sigma_+(\eta)} \frac{d\eta}{\eta - \xi}$$

$$= \frac{2\pi i}{\xi \, \sigma_+(\xi)} - \int_{-\infty}^\infty \frac{e^{-ir\eta}}{\eta \, \sigma_+(\eta)} \frac{d\eta}{\eta - \xi}.$$

We deform the contour in the integral to one going up the left side of the imaginary axis, then in a little circle around zero, then down the right side of the imaginary axis. The integral around the little circle becomes in the limit $2\pi i/\xi$, and combined with the first term gives

$$\frac{2\pi i}{\xi} \left(\frac{1}{\sigma_+(\xi)} - 1 \right) = \log(-i\xi) + \text{const.} + o(1)$$

near zero.

After using $1/\sigma_+ = \sigma_-/\sigma$, the integral over the negative imaginary axis becomes, as in the proof of Lemma 2,

$$\int_0^\infty \frac{e^{-ry}}{y} \, \sigma_-(iy) \, \tan(y/2) \frac{dy}{y - i\xi}$$

and this is handled as in the proof of that lemma. The only difference is the denominator y which gives a $\log(-i\xi)$ summand, which cancels the $\log(-i\xi)$ we had above, and because of which the extra factor r^{-1} does not occur. We omit the details. □

Lemma 4. For each $k \geq 0$,

$$\begin{pmatrix} h^- \\ h^+ \end{pmatrix} - \sum_{j=0}^{k-1} (-1)^j \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}^j \begin{pmatrix} G^- \\ G^+ \end{pmatrix} \in r^{-k}(\mathcal{L}^- \oplus \mathcal{L}^+). \tag{30}$$

Proof. It follows from Lemma 1, by applying the plus operator to (9), that the function $e^{ir\xi/2}\sigma_+(\xi)\,h^+(\xi)$ belongs to $r\mathcal{L}^+$. Therefore the same is true for $h^+(\xi)$, and similarly $h^-(\xi)$. From this, the representation (11) and Lemma 2, we deduce that for each k

$$\begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}^{k+1} \left(I + \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} G^- \\ G^+ \end{pmatrix} \in r^{-k}(\mathcal{L}^- \oplus \mathcal{L}^+).$$

By the general fact $L^{k+1}(I-L)^{-1} = (I-L)^{-1} - \sum_{j=0}^{k} L^j$, this is the same as

$$\begin{pmatrix} h^- \\ h^+ \end{pmatrix} - \sum_{j=0}^{k} (-1)^j \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}^j \begin{pmatrix} G^- \\ G^+ \end{pmatrix} \in r^{-k}(\mathcal{L}^- \oplus \mathcal{L}^+).$$

But by Lemmas 2 and 3, we know that

$$\begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}^k \begin{pmatrix} G^- \\ G^+ \end{pmatrix} \in r^{-k}(\mathcal{L}^- \oplus \mathcal{L}^+).$$

Multiplying this by $(-1)^k$ and adding gives the statement of the lemma.   □

Formula (20) gives the asymptotic expansion of G^+, and therefore also for G^- since functions are complex conjugates of each other. It follows from Lemma 4 that the coefficient of ξ^n in $h^\pm$ is obtained with error $O(r^{n-k+})$ by taking the corresponding coefficient in the sum in (30) up to $k-1$. With the formulas in the proof of Lemmas 2 and 3 further coefficients can be computed from the asymptotics of some integrals. And then we would determine the coefficients in the expansions of their products with $e^{\pm ir\xi/2}\sigma_\pm(\xi)$ appearing in (9). Thus with the result for general k one could in principle find as may terms as desired in the asymptotic expansions of Q and $\mathfrak{e}_F(\gamma)$.

The special case $n=2$, $k=1$ justifies the asymptotics we found for $\mathfrak{e}_F(\gamma)$. Because we did not keep the track of powers of logarithms, we cannot guarantee that the power two of the logarithm in the error term in (8) is correct, only that it is correct with some power.

ACKNOWLEDGMENTS

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