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Blocks and gaps in the asymmetric simple exclusion process: Asymptotics

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In earlier work, the authors obtained formulas for the probability in the asymmetric simple exclusion process that at time t , a particle is at site x and is the beginning of a block of L consecutive particles. Here we consider asymptotics. Specifically, for the Kardar-Parisi-Zhang regime with step initial condition, we determine the conditional probability (asymptotically as $t \rightarrow \infty$) that a particle is the beginning of an L -block, given that it is at site x at time t . Using duality between occupied and unoccupied sites, we obtain the analogous result for a gap of G unoccupied sites between the particle at x and the next one. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5021353>

I. INTRODUCTION

The *asymmetric simple exclusion process* (ASEP) on the integer lattice \mathbb{Z} has remained an important stochastic model in nonequilibrium statistical physics and interacting particle systems since its introduction by Spitzer⁵ nearly fifty years ago. Nearly ten years ago, for the case of *step initial condition* (particles initially occupying the positive integer sites \mathbb{Z}^+), a formula for the distribution of the m th particle from the left⁸ was the starting point for the one-point probability distribution of the height function for the Kardar-Parisi-Zhang (KPZ) equation with narrow wedge initial conditions.^{1,4} For a review of these developments in stochastic growth processes, see Refs. 2 and 3.

In Ref. 9, the authors derived formulas associated with blocks of particles in ASEP analogous to those derived earlier for a single particle.^{6,7} First there was a formula for the probability that at time t the m th particle from the left is the beginning of a block of particles of length L (or “ L -block”) starting at x . Specifically it was for the probability $\mathcal{P}_{L,Y}(x, m, t)$ of the event

$$x_m(t) = x, \quad x_{m+1}(t) = x + 1, \dots, x_{m+L-1}(t) = x + L - 1,$$

given the initial configuration $Y = \{y_1, \dots, y_N\}$. Here $x_m(t)$ denotes the position of the m th particle from the left at time t . The probability was given in two forms as sums of multiple integrals. One of these permitted an extension to infinite systems unbounded on the right. In the case of step initial condition ($Y = \mathbb{Z}^+$), the sum was shown to equal an integral involving a Fredholm determinant.

Here we consider the following question for ASEP with step initial condition: what is the conditional probability (asymptotically as $t \rightarrow \infty$) that the m th particle from the left is the beginning of an L -block, given that it is at site x at time t ? Of course everything depends on how x and m depend on t . Our result is for the KPZ regime considered in Ref. 8 (Sec. 5).

Recall that in ASEP particles jump one step to the right with probability p (if the site is unoccupied) or one step to the left with probability $q = 1 - p$ (if the site is unoccupied). We assume $q > p > 0$, so there is a drift to the left. The notation of Ref. 8 was

$$m = \sigma t, \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sigma^{-1/6}(1 - \sqrt{\sigma})^{2/3}.$$

What was shown there was that, with $\gamma = q - p$,

$$\lim_{t \rightarrow \infty} \mathbb{P}(x_m(t/\gamma) \leq c_1 t + c_2 s t^{1/3}) = F_2(s),$$

uniformly for σ in a compact subset of $(0, 1)$, where F_2 is the distribution function of random matrix theory. Here we show the following, under the same assumptions. (We call this the KPZ regime.)

Theorem 1. When $m = \sigma t$ and $x = c_1 t + c_2 s t^{1/3}$, we have, as $t \rightarrow \infty$,

$$\mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) = c_2^{-1} \sigma^{(L-1)/2} F_2'(s) t^{-1/3} + o(t^{-1/3}).$$

Corollary 1. The conditional probability that the m th particle from the left is the beginning of an L -block, given that it is at x at time t/γ , has the limit $\sigma^{(L-1)/2}$. [The conditional probability that there is a block of precisely L particles, and no more, has the limit $\sigma^{(L-1)/2} - \sigma^{L/2} = \sigma^{(L-1)/2} (1 - \sqrt{\sigma})$.] The corollary follows since the conditional probability is equal to

$$\frac{\mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma)}{\mathcal{P}_{1, \mathbb{Z}^+}(x, m, t/\gamma)} = \sigma^{(L-1)/2} + o(1).$$

We also consider the probability that there is a gap of (at least) G unoccupied sites to the right of x . Specifically, we define $\overline{\mathcal{P}}_{G,Y}(x, m, t)$ to be the probability that $x_m(t) = x$ and $x_{m+1}(t) > x + G$, given the initial configuration Y . For this, we show the following.

Theorem 2. When $m = \sigma t$ and $x = c_1 t + c_2 s t^{1/3}$, we have, as $t \rightarrow \infty$,

$$\overline{\mathcal{P}}_{G, \mathbb{Z}^+}(x, m, t/\gamma) = c_2^{-1} (1 - \sqrt{\sigma})^G F_2'(s) t^{-1/3} + o(t^{-1/3}).$$

Corollary 2. The conditional probability that the m th particle from the left is followed by a gap of G unoccupied sites, given that it is at x at time t/γ , has the limit $(1 - \sqrt{\sigma})^G$. (The conditional probability that there is a gap of precisely G sites, and no more, has the limit $(1 - \sqrt{\sigma})^G \sqrt{\sigma}$. No gap is the same as a block of at least two, so this is consistent with Corollary 1 with $L = 2$.)

The starting point for the proof of Theorem 1 will be the formula for $\mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma)$ derived in Ref. 9. We set $\tau = p/q < 1$, define

$$U(\xi, \xi') = \frac{p + q\xi\xi' - \xi}{\xi' - \xi}, \quad K_x(\xi, \xi') = \frac{\xi^x e^{(p/\xi + q\xi - 1)t/\gamma}}{p + q\xi\xi' - \xi},$$

and denote by $K_{L,x}(z)$ the integral operator acting on functions on C_R with kernel

$$K_{L,x}(\xi, \xi'; z) = q^{1-L} K_{x+L-1}(\xi, \xi') \prod_{j=1}^L U(z_j, \xi).$$

(The notation is slightly different than in Ref. 9.) Here C_R is the circle with center zero and large radius R , depending on τ . The result was the formula

$$\begin{aligned} \mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) &= (-1)^{L-1} p^{L(L+1)/2} \tau^{-(m-1)(L-1)} \\ &\times \int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \frac{1}{z_1^L (qz_1 - p) z_2^{L-1} (qz_2 - p) \cdots z_L (qz_L - p)} \prod_{i < j} \frac{1}{U(z_j, z_i)} \\ &\times \left[\int \frac{\det(I - \tau^{-L} \lambda K_{L,x}(z))}{(\lambda; \tau)_m} \frac{d\lambda}{\lambda^L} \right] dz_L \cdots dz_1. \end{aligned} \tag{1}$$

The λ -integration is over a contour enclosing the singularities of the integrand at τ^{-j} for $j = 0, \dots, m - 1$.

We explain the notation. First, $(\lambda; \tau)_m = \prod_{j=0}^{m-1} (1 - \lambda \tau^j)$. As for the iterated integral, $\Gamma_{0,\tau}$ is a contour consisting of tiny circles around the points $z = 0$ and $z = \tau$, with the circles for each z_i lying well inside the circles for z_{i-1} . Alternatively, the integral is interpreted as follows: First take the sum of the residues at $z_L = 0$ and $z_L = \tau$. In the resulting integrand, take the sum of the residues at $z_{L-1} = 0$ and $z_{L-1} = \tau$. And so on. [The order matters because of the factors $1/U(z_j, z_i)$ in the integrand. Observe that $U(\tau, z) = p$ and $U(z, \tau) = q$ for $z \neq \tau$.]

The proof of Theorem 1 will have two parts. In the first, a pair of facts on stability of Fredholm determinants allows us to replace the operators $K_{L,x}(z)$ in the λ -integral in Ref. 1 by different operators—ones for which we can do an asymptotic analysis. Since the derivation of this replacement goes along the lines of the argument in Ref. 8 for a single particle, the details will be deferred to an Appendix A. In the second part, we use the previously established formula to reduce the problem to the evaluation of an explicit L -dimensional integral (which is not completely trivial). Theorem 2 will be deduced from Theorem 1 using the duality between occupied and unoccupied sites in ASEP and some easy computations.

II. REPLACING THE OPERATORS $K_{L,x}(z)$ BY OPERATORS $J_{L,x,m}(w)$

The details of what follows will be given in Appendix A. We first make the change of variables

$$\xi = \frac{1 - \tau\eta}{1 - \eta}, \quad \xi' = \frac{1 - \tau\eta'}{1 - \eta'}$$

in the operator, which will then act on functions on a small circle about $\eta = 1$. Then, in the operator and the z_i -integrals in Ref. 1, we make the substitutions

$$z_i = \frac{w_i - \tau}{w_i - 1},$$

and we find that the integrations are over $\Gamma_{0,\tau}$ as before, with the w_i -contours well inside the w_{i-1} contours.

What comes next, the two propositions, proved in Ref. 8 (Sec. 2), depend on the stability of Fredholm determinants.

Proposition 1. Suppose $r \rightarrow \mathcal{C}_r$ is a deformation of closed curves and a kernel $H(\eta, \eta')$ is analytic in a neighborhood of $\mathcal{C}_r \times \mathcal{C}_r \subset \mathbb{C}^2$ for each r . Then the Fredholm determinant of H acting on \mathcal{C}_r is independent of r .

Proposition 2. Suppose $H_1(\eta, \eta')$ and $H_2(\eta, \eta')$ are two kernels acting on a simple closed contour \mathcal{C} , such that $H_1(\eta, \eta')$ extends analytically to η inside \mathcal{C} **or** to η' inside \mathcal{C} and that $H_2(\eta, \eta')$ extends analytically to η inside \mathcal{C} **and** to η' inside \mathcal{C} . Then the Fredholm determinants of $H_1(\eta, \eta') + H_2(\eta, \eta')$ and $H_1(\eta, \eta')$ are equal.

After using these two propositions (among other things), we arrive at an operator $J_{L,x,m}(w)$ acting on functions on a circle with center zero and radius $r \in (\tau, 1)$. It has kernel

$$J_{L,x,m}(\eta, \eta'; w) = \int \frac{\phi_{\infty,x}(\zeta)}{\phi_{\infty,x}(\eta')} \frac{\zeta^{m-L}}{(\eta')^{m-L+1}} \frac{f(\mu, \zeta/\eta')}{\zeta - \eta} \prod_{j=1}^L V(\zeta, \eta'; w_j) d\zeta, \tag{2}$$

where

$$\phi_{\infty,x}(\eta) = (1 - \eta)^{-x-L+1} e^{\frac{\eta}{1-\eta}t}, \quad f(\mu, z) = \sum_{k \in \mathbb{Z}} \frac{\tau^k}{1 - \tau^k \mu} z^k, \quad V(\zeta, \eta'; w) = \frac{w \zeta - \tau}{w \eta' - \tau}.$$

The ζ -integration is over a circle with center zero and radius in the interval $(1, r/\tau)$. The new statement, which will be derived in Appendix A, is

$$\begin{aligned} \mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) &= -\tau^{-(L^2-5L+2)/2} \int_{\Gamma_{0,\tau}} \dots \int_{\Gamma_{0,\tau}} \prod_{j=1}^L \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{i < j} \frac{w_j - w_i}{w_j - \tau w_i} \\ &\times \int \left[(\tau^L \mu; \tau)_\infty \det(I + \mu J_{L,x,m}(w)) \frac{d\mu}{\mu^L} \right] dw_L \dots dw_1. \end{aligned} \tag{3}$$

Here μ runs over a circle of radius larger than τ^{-L+1} , and the order of integration of the w_j is as indicated.

III. ASYMPTOTICS

Now we assume $m = \sigma t$ and $x = c_1 t + s c_2 t^{1/3}$. In Ref. 8 (Sec. 5), we did a saddle point analysis of the operator with kernel (2), but without the product in the integrand. We made the variable changes

$$\eta \rightarrow \xi + c_3^{-1} t^{-1/3} \eta, \quad \eta' \rightarrow \xi + c_3^{-1} t^{-1/3} \eta', \quad \zeta \rightarrow \xi + c_3^{-1} t^{-1/3} \zeta, \tag{4}$$

where $\xi = -\sqrt{\sigma}/(1 - \sqrt{\sigma})$ was the saddle point and $c_3 = \sigma^{-1/6} (1 - \sqrt{\sigma})^{5/3}$. Using Proposition 1, we found [There is a minor change here. In the expression for $\phi_{\infty,x}$, we have an exponent $-x - L + 1$ rather than the $-x$ in Ref. 8. Since changing x by $O(1)$ amounts to changing s by $O(t^{-1/3})$, this does not affect the asymptotics] that μ times the operator had the same Fredholm determinant as an operator $J^{(0)} + o(1)$, where $o(1)$ denotes a family of operators whose trace norms tend to zero as $t \rightarrow \infty$. And $\det(I + J^{(0)}) = F_2(s)$. The kernel of $J^{(0)}$ is

$$J^{(0)}(\eta, \eta') = \int_{\Gamma_\zeta} \frac{e^{-\zeta^3/3+s\zeta+(\eta')^3/3-s\eta'}}{(\zeta - \eta)(\eta' - \zeta)} d\zeta, \tag{5}$$

which is independent of μ . Here Γ_ζ is the contour consisting of the rays from $-c_3$ to $-c_3 + \infty e^{\pm 2\pi i/3}$ while the operator acts on functions on Γ_η , which consists of the rays from 0 to $\infty e^{\pm \pi i/3}$.

Now consider the effect of the product in the integrand in (2). If we make the replacements⁴ in $V(\zeta, \eta'; w)$, a little computation shows that

$$V(\zeta, \eta'; w) \rightarrow 1 + (\zeta - \eta') \frac{w}{w \xi - \tau} c_3^{-1} t^{-1/3} [1 + O(\min(1, t^{-1/3} |\eta'|))],$$

so for the product,

$$\prod_{j=1}^L V(\zeta, \eta'; w_j) \rightarrow 1 + (\zeta - \eta') \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} + E(\zeta, \eta'; w), \tag{6}$$

where $E(\zeta, \eta'; w)$ is a polynomial in $\zeta - \eta'$ with coefficients that are functions of η' with bound $O(t^{-2/3} |\eta'|)$. By exactly the same argument as in Ref. 8, the error term $E(\zeta, \eta'; w)$ causing no difficulty, we see that $\mu J_{L,x,m}(w)$ has the same Fredholm determinant as an operator

$$J^{(0)} + o(1) + J^{(1)} \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} + o(t^{-1/3}),$$

where $J^{(1)}$ has kernel

$$J^{(1)}(\eta, \eta') = - \int \frac{e^{-\zeta^3/3+s\zeta+(\eta')^3/3-s\eta'}}{\zeta - \eta} d\zeta. \tag{7}$$

This is obtained from the kernel of $J^{(0)}$ by multiplying the integrand in (5) by the factor $\zeta - \eta'$ from (6). Both bounds are in the trace norm, and the $o(1)$ bound is independent of w . Thus the determinant in (3) is equal to the determinant of

$$I + J^{(0)} + o(1) + J^{(1)} \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} + o(t^{-1/3}),$$

which in turn equals

$$\begin{aligned} & \det(I + J^{(0)} + o(1)) \det \left(I + (I + J^{(0)})^{-1} J^{(1)} \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} + o(t^{-1/3}) \right) \\ &= (F_2(s) + o(1)) \left[1 + \text{tr} \left((I + J^{(0)})^{-1} J^{(1)} \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} \right) \right] + o(t^{-1/3}). \end{aligned}$$

Now from (5) and (7), we see that $J^{(1)} = dJ^{(0)}/ds$ and therefore by a general fact

$$\text{tr}((I + J^{(0)})^{-1}J^{(1)}) = \frac{d}{ds} \log \det(I + J^{(0)}) = \frac{F_2'(s)}{F_2(s)},$$

so the determinant in (3) is equal to

$$F_2(s) + o(1) + F_2'(s) \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} c_3^{-1} t^{-1/3} + o(t^{-1/3}). \tag{8}$$

Again the $o(1)$ term is independent of the w_j .

The limit is independent of μ , so to evaluate the μ -integral in (3) we will only need to use

$$\int (\tau^L \mu; \tau)_\infty \frac{d\mu}{\mu^L} = (-1)^{L-1} \frac{\tau^{(L-1)(3L-2)/2}}{(1-\tau) \cdots (1-\tau^{L-1})}. \tag{9}$$

Finally we have to integrate over the w_j in (3), so it remains to evaluate

$$\int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} F(w_1, \dots, w_L) dw_L \cdots dw_1 \tag{10}$$

and

$$\int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} F(w_1, \dots, w_L) \sum_{j=1}^L \frac{w_j}{w_j \xi - \tau} dw_L \cdots dw_1, \tag{11}$$

where

$$F(w_1, \dots, w_L) = \prod_{j=1}^L \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{i < j} \frac{w_j - w_i}{w_j - \tau w_i}.$$

IV. END OF THE PROOF OF THEOREM 1

Write

$$F(w_1, \dots, w_L) = \frac{(w_1 - 1)^{L-1}}{w_1(w_1 - \tau)^L} G_L(w_1, \dots, w_L),$$

where

$$G_L(w_1, \dots, w_L) = \prod_{j>1} \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{1 \leq i < j \leq L} \frac{w_i - w_j}{\tau w_i - w_j}. \tag{12}$$

The integral (10) is

$$\int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \frac{(w_1 - 1)^{L-1}}{w_1(w_1 - \tau)^L} G_L(w_1, \dots, w_L) dw_L \cdots dw_1. \tag{13}$$

It is shown in Appendix B that if $\psi(w_2, \dots, w_L)$ is analytic in the neighborhood of $\{0, \tau\}^{L-1}$, then

$$\int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} G_L(w_1, \dots, w_L) \psi(w_2, \dots, w_L) dw_L \cdots dw_2$$

is analytic for w_1 outside $\{0, \tau\}$ except for a pole of order at most $L - 1$ at $w_1 = 1$ and is $O(1)$ for large w_1 . If we use this with $\psi(w_2, \dots, w_L) = 1$, we see by expanding the w_1 -contour that the integral (13), which is the same as (10), equals zero. (The pole of order $L - 1$ is cancelled by the zero of order $L - 1$ in the first factor.) Similarly, by taking

$$\psi(w_2, \dots, w_L) = \sum_{j=2}^L \frac{w_j}{w_j \xi - \tau},$$

we see that the integral (11), but with the sum starting at $j = 2$, is also zero.

Thus the remaining integral to be evaluated is

$$\int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \frac{(w_1 - 1)^{L-1}}{w_1(w_1 - \tau)^L} G_L(w_1, \dots, w_L) \frac{w_1}{w_1 \xi - \tau} dw_L \cdots dw_1. \tag{14}$$

We do this by integrating out one variable at a time until we get to the end. Precisely, we claim that for $k = 1, \dots, L$, the integral equals

$$-\frac{\xi^{L-1}}{(1 - \xi)^L} (\tau^k / \xi - 1)^{L-k} \frac{(1 - \tau) \cdots (1 - \tau^{k-1})}{\tau^{kL}} \tag{15}$$

$$\times \int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \frac{(w_{k+1} - 1)^{L-k-1}}{w_{k+1}(w_{k+1} - \tau)^{L-k}} \prod_{j \geq k+1} \frac{\tau/\xi - w_j}{\tau^{k+1}/\xi - w_j} G_{L-k}(w_{k+1}, \dots, w_L) dw_L \cdots dw_{k+1}. \tag{16}$$

For the inductive proof, we use the easily checked fact that for $k \geq 0$,

$$G_{L-k}(w_{k+1}, \dots, w_L) = \frac{(w_{k+2} - 1)^{L-k-2}}{w_{k+2}(w_{k+2} - \tau)^{L-k-1}} \prod_{j > k+1} \frac{w_{k+1} - w_j}{\tau w_{k+1} - w_j} G_{L-k-1}(w_{k+2}, \dots, w_L). \tag{17}$$

Using [Appendix B](#) again, we see that we can expand the w_1 -contour in (14) and find now that the integral equals minus the residue at $w_1 = \tau/\xi$. Using (17) with $k = 0$, we find that (14) equals the integral over w_L, \dots, w_2 of

$$\begin{aligned} &-\frac{(\tau/\xi - 1)^{L-1}}{\xi(\tau/\xi - \tau)^L} \frac{(w_2 - 1)^{L-2}}{w_2(w_2 - \tau)^{L-1}} \prod_{j \geq 2} \frac{\tau/\xi - w_j}{\tau^2/\xi - w_j} G_{L-1}(w_2, \dots, w_L) \\ &= -\frac{\xi^{L-1}}{\tau^L(1 - \xi)^L} (\tau/\xi - 1)^{L-1} \frac{(w_2 - 1)^{L-2}}{w_2(w_2 - \tau)^{L-1}} \prod_{j \geq 2} \frac{\tau/\xi - w_j}{\tau^2/\xi - w_j} G_{L-1}(w_2, \dots, w_L). \end{aligned}$$

This verifies the claim for $k = 1$. Now assume it is true for $k < L$. From [Appendix B](#), with 1 replaced by $k + 1$ and L replaced by $L - k$, we find by expanding the contour that the integral over w_{k+1} in (16) equals minus the residue at $w_{k+1} = \tau^{k+1}/\xi$. Using (17), we see that this equals the integral over w_L, \dots, w_{k+2} of

$$\begin{aligned} &\frac{(\tau^{k+1}/\xi - 1)^{L-k-1}}{\tau^{k+1}/\xi(\tau^{k+1}/\xi - \tau)^{L-k}} (\tau/\xi - \tau^{k+1}/\xi) \prod_{j > k+1} \frac{\tau/\xi - w_j}{\tau^{k+1}/\xi - w_j} \\ &\times \frac{(w_{k+2} - 1)^{L-k-2}}{w_{k+2}(w_{k+2} - \tau)^{L-k-1}} \prod_{j > k+1} \frac{\tau^{k+1}/\xi - w_j}{\tau^{k+2}/\xi - w_j} G_{L-k-1}(w_{k+2}, \dots, w_L) \\ &= \frac{(\tau^{k+1}/\xi - 1)^{L-k-1}}{(\tau^k/\xi - \tau)^{L-k}} \frac{1 - \tau^k}{\tau^L} \\ &\times \frac{(w_{k+2} - 1)^{L-k-2}}{w_{k+2}(w_{k+2} - \tau)^{L-k-1}} \prod_{j > k+1} \frac{\tau/\xi - w_j}{\tau^{k+2}/\xi - w_j} G_{L-k-1}(w_{k+2}, \dots, w_L). \end{aligned}$$

If we multiply this by the factor (15), we obtain the statement for $k + 1$. This completes the inductive proof of the claim. When $k = L$, the integral in (16) does not appear and we obtain the result that the integral (11) equals

$$-\frac{\xi^{L-1}}{(1 - \xi)^L} \frac{(1 - \tau) \cdots (1 - \tau^{L-1})}{\tau^{L^2}}. \tag{18}$$

Putting all this together, we have shown that when we multiply $F(w_1, \dots, w_L)$ by (8) and integrate, the result is $c_3^{-1} F_2'(s) t^{-1/3} + o(t^{-1/3})$ times (18). [Recall that (10) equals zero, so the summand $F_2(s) + o(1)$ independent of the w_j drops out, while (11) gets multiplied by $c_3^{-1} F_2'(s) t^{-1/3}$ and $o(t^{-1/3})$ is added.] If we combine this with (9) and refer to (3), we obtain

$$\mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) = (-1)^{L-1} c_3^{-1} \frac{\xi^{L-1}}{(1-\xi)^L} F_2'(s) t^{-1/3} + o(t^{-1/3}).$$

Then using

$$\xi = -\sqrt{\sigma}/(1 - \sqrt{\sigma}), \quad c_3 = \sigma^{-1/6} (1 - \sqrt{\sigma})^{5/3}, \quad c_2 = \sigma^{-1/6} (1 - \sqrt{\sigma})^{2/3},$$

we see that

$$\frac{c_3^{-1}}{1 - \xi} = c_2^{-1}, \quad \frac{\xi}{1 - \xi} = -\sqrt{\sigma}.$$

This gives the statement of Theorem 1, that

$$\mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) = c_2^{-1} \sigma^{(L-1)/2} F_2'(s) t^{-1/3} + o(t^{-1/3}).$$

V. PROOF OF THEOREM 2

Let P denote ASEP with step initial condition and P' denote the process whose occupied sites are the unoccupied sites in P after the switch $x \rightarrow -x$. Then P' is an ASEP with step initial condition (Not quite since the initial configuration for P' is the set of nonnegative integers. This will make no difference and we ignore the fact.) and the same p and q .

We want to compute the probability $\bar{\mathcal{P}}_{G, \mathbb{Z}^+}(x, m, t/\gamma)$ that $x_m = x$ in P at time t/γ and sites $x + 1, \dots, x + G$ are unoccupied. In terms of P' , site $x' = -(x + G)$ is occupied, as are sites $x' + 1, \dots, x' + G - 1$, while site $x' + G = -x$ is unoccupied since x is occupied in P . Thus, if \mathcal{P}' denotes probability for blocks in P' , then

$$\bar{\mathcal{P}}_{G, \mathbb{Z}^+}(x, m, t/\gamma) = \mathcal{P}'_{G, \mathbb{Z}^+}(x', m', t/\gamma) - \mathcal{P}'_{G+1, \mathbb{Z}^+}(x', m', t/\gamma), \tag{19}$$

where m' is such that $x_{m'} = x'$ in P' . To determine m' , observe that $m - 1$ is the number of occupied sites in P to the left of x , so analogously $m' - 1$ is the number of unoccupied sites to the right of $x + G$ in P . The number of unoccupied sites to the right of x equals $m - x$, so the number of unoccupied sites to the right of $x + G$ equals $m - x + O(1)$. Thus $m' = m - x + O(1)$.

The assumption of Theorem 1 is that we are in the KPZ regime in P so that if $m = \sigma t$, then σ is in a compact subset of $(0, 1)$ and

$$x = c_1 t + c_2 s t^{1/3} \quad \text{where} \quad c_1 = -1 + 2\sqrt{\sigma}, \quad c_2 = \sqrt{\sigma}^{-1/3} (1 - \sqrt{\sigma})^{2/3}. \tag{20}$$

To apply Theorem 1 to the right side of (19), we have to show that we are in the KPZ regime in P' . This means that if we define

$$\sigma' = m'/t, \quad c'_1 = -1 + 2\sqrt{\sigma'}, \quad c'_2 = \sqrt{\sigma'}^{-1/3} (1 - \sqrt{\sigma'})^{2/3}, \tag{21}$$

then σ' is in a compact subset of $(0, 1)$ and

$$x' = c'_1 t + c'_2 s t^{1/3} + o(t^{1/3}). \tag{22}$$

[It will turn out to be the same s . The $o(t^{1/3})$ does not matter since it can be removed by replacing s by $s + o(1)$, which will give the same asymptotics.] Since $m = \sigma t$ and $m' = m - x + O(1)$, it follows from (20) that

$$\begin{aligned} \sigma' &= \sigma - c_1 - c_2 s t^{-2/3} + O(t^{-1}) = (1 - \sqrt{\sigma})^2 - c_2 s t^{-2/3} + O(t^{-1}), \\ \sqrt{\sigma'} &= 1 - \sqrt{\sigma} - \frac{1}{2}(1 - \sqrt{\sigma})^{-1} c_2 s t^{-2/3} + O(t^{-1}). \end{aligned} \tag{23}$$

Clearly σ' is in a compact subset of $(0, 1)$, so it remains to verify (22). First, from (21) and (23), it follows that

$$\begin{aligned} c'_1 &= -1 + 2\left[1 - \sqrt{\sigma} - \frac{1}{2}(1 - \sqrt{\sigma})^{-1}c_2 s t^{-2/3} + O(t^{-1})\right] \\ &= -c_1 - (1 - \sqrt{\sigma})^{-1}c_2 s t^{-2/3} + O(t^{-1}), \end{aligned}$$

whence

$$c'_1 t = -c_1 t - (1 - \sqrt{\sigma})^{-1}c_2 s t^{1/3} + O(1).$$

Then, from (21) and (23) again, we see that

$$c'_2 = (1 - \sqrt{\sigma})^{-1/3} \sqrt{\sigma}^{2/3} + O(t^{-2/3}),$$

so

$$c'_2 s t^{1/3} = (1 - \sqrt{\sigma})^{-1/3} \sqrt{\sigma}^{2/3} s t^{1/3} + O(t^{-1/3}).$$

Thus,

$$c'_1 t + c'_2 s t^{1/3} = -c_1 t - \left[(1 - \sqrt{\sigma})^{-1}c_2 - (1 - \sqrt{\sigma})^{-1/3} \sqrt{\sigma}^{2/3}\right] s t^{1/3} + O(1).$$

Miraculously, the expression in brackets above equals c_2 , as is easily checked. Therefore

$$c'_1 t + c'_2 s t^{1/3} = -c_1 t - c_2 s t^{1/3} + O(1).$$

Since $x' = -(x + G) = -c_1 t - c_2 s t^{1/3} - G$, relation (22) follows.

Hence we are in the KPZ regime in P' and may apply Theorem 1 to the right side of (19). We obtain

$$\begin{aligned} \overline{\mathcal{P}}_{G, \mathbb{Z}^+}(x, m, t/\gamma) &= c_2'^{-1} (\sigma'^{(G-1)/2} - \sigma'^{G/2}) F_2'(s) t^{-1/3} + o(t^{-1/3}) \\ &= c_2'^{-1} (1 - \sqrt{\sigma})^{G-1} \sqrt{\sigma} F_2'(s) t^{-1/3} + o(t^{-1/3}). \end{aligned}$$

Finally, one sees that $c_2'^{-1} \sqrt{\sigma} = c_2^{-1} (1 - \sqrt{\sigma}) + O(t^{-2/3})$, so the above equals

$$c_2^{-1} (1 - \sqrt{\sigma})^G F_2'(s) t^{-1/3} + o(t^{-1/3}).$$

This completes the proof of Theorem 2.

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APPENDIX A: DEFORMATION TO THE OPERATORS $J_{L,x,m}(w)$

In order to simplify notation, in this section, we do not display the subscripts L and x , since they will have the same meaning throughout. In particular, the operator we denoted by $K_{L,x}(z)$ before we now denote by $K(z)$.

We follow the steps in the argument in Ref. 8. We first make the changes of variable

$$\xi = \frac{1 - \tau\eta}{1 - \eta}, \quad \xi' = \frac{1 - \tau\eta'}{1 - \eta'}, \quad z_i = \frac{w_i - \tau}{w_i - 1}$$

in the kernel of $K(z)$. The result is the operator $K_2(w)$ with kernel

$$K_2(\eta, \eta'; w) = \frac{\phi(\eta')}{\eta' - \tau\eta} \prod_{j=1}^L \frac{w_j \eta' - \tau}{w_j \eta' - 1},$$

where

$$\phi(\eta) = \left(\frac{1 - \tau\eta}{1 - \eta}\right)^{x+L-1} e^{[\frac{1}{1-\eta} - \frac{1}{1-\tau\eta}]t}.$$

This operator, which has the same Fredholm determinant as $K(z)$, acts on functions on a small (clockwise) circle γ about 1. Using

$$\prod_{j=1}^L \frac{1}{z_j^{L-j+1}(qz_j - p)} \prod_{i < j} \frac{1}{U(z_j, z_i)} \prod_j \frac{dz_j}{dw_j} = (-1)^L q^{-L(L+1)/2} \prod_{j=1}^L \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{i < j} \frac{w_j - w_i}{w_j - \tau w_i},$$

we see that (1) becomes

$$\begin{aligned} \mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) &= -\tau^{L(L+1)/2 - (m+L-1)(L-1)} \int_{\Gamma_{0,\tau}} \dots \int_{\Gamma_{0,\tau}} \prod_{j=1}^L \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{i < j} \frac{w_j - \tau w_i}{w_j - w_i} \\ &\times \left[\int \frac{\det(I - \lambda K_2(w))}{(\tau^L \lambda; \tau)_m} \frac{d\lambda}{\lambda^L} \right] dw_L \dots dw_1, \end{aligned} \tag{A1}$$

where we made the substitution $\lambda \rightarrow \tau^L \lambda$ in the λ -integral. The λ -contour is large enough so that all the zeros of $(\tau^L \lambda; \tau)_m$ are inside the contour.

Now we shall use the two propositions on stability of Fredholm determinants stated in Sec. II. We introduce the operator $K_1(w)$ with kernel

$$K_1(\eta, \eta'; w) = \frac{\phi(\tau\eta)}{\eta' - \tau\eta} \prod_{j=1}^L \frac{\tau w_j \eta - \tau}{\tau w_j \eta - 1}.$$

Observe that $K_2(\eta, \eta'; w)$ is analytic for η inside γ while $K_1(\eta, \eta'; w)$ is analytic for both η and η' inside γ . Therefore, by Proposition 2, the Fredholm determinant of $K_2(\eta, \eta'; w)$ acting on γ clockwise is equal to the Fredholm determinant of

$$K_1(\eta, \eta'; w) - K_2(\eta, \eta'; w)$$

acting on γ counterclockwise. Furthermore, if \mathcal{C} denotes a (counterclockwise) circle with center zero and radius $R \in (1, \tau^{-1})$, then the above difference is analytic in η and η' in the deformation from γ to \mathcal{C} , the singularities at $\eta' = \tau\eta$ of the two kernels canceling. Using Proposition 1, we conclude that in the integral (A1), the operator $K_2(w)$ may be replaced by $K_1(w) - K_2(w)$ acting on \mathcal{C} .

Consider the kernel $K_1(\eta, \eta'; w)$ acting on $r\mathcal{C}$, where $0 \leq r < 1$. It is analytic for $\eta, \eta' \in r\mathcal{C}$ and so by Proposition 1, the Fredholm determinants are independent of r . On the other hand, their Fredholm determinants are the same as that for $r K_1(r\eta, r\eta'; w)$ acting on \mathcal{C} . As $r \rightarrow 0$, these operators converge in the trace norm to $\tau^L K_0$, where K_0 has kernel

$$K_0(\eta, \eta') = \frac{1}{\eta' - \tau\eta}.$$

(The operator with kernel $r K_1(r\eta, r\eta'; w)$ equals the trace class operator K_0 left-multiplied by a function tending uniformly on \mathcal{C} to the constant function τ^L .) We conclude that

$$\det(I - \lambda K_1(w)) = \det(I - \tau^L \lambda K_0) = (\tau^L \lambda; \tau)_\infty.$$

For the last identity, see the end of the proof of Proposition 4 of Ref. 8.

Next, define $R(\lambda; w) = \lambda K_1(w)(I - \lambda K_1(w))^{-1}$, the resolvent operator of $K_1(w)$. From the preceding equation, we conclude that $\det(I - \lambda K_2(w))$ appearing in (A1) is equal to

$$\begin{aligned} \det[I - \lambda K_1(w) + \lambda K_2(w)] &= \det(I - \lambda K_1(w)) \det[I + \lambda K_2(w)(I + R(\lambda; w))] \\ &= (\tau^L \lambda; \tau)_\infty \det[I + \lambda K_2(w)(I + R(\lambda; w))], \end{aligned}$$

the operators acting on functions on \mathcal{C} . Substituting this into (A1), we obtain

$$\begin{aligned} \mathcal{P}_{L, \mathbb{Z}^+}(x, m, t/\gamma) &= -\tau^{L(L+1)/2 - (m+L-1)(L-1)} \int_{\Gamma_{0,\tau}} \cdots \int_{\Gamma_{0,\tau}} \prod_{j=1}^L \frac{(w_j - 1)^{L-j}}{w_j(w_j - \tau)^{L-j+1}} \prod_{i < j} \frac{w_j - \tau w_i}{w_j - w_i} \\ &\times \left[\int (\lambda \tau^{m+L}; \tau)_\infty \det[I + \lambda K_2(w)(I + R(\lambda; w))] \frac{d\lambda}{\lambda^L} \right] dw_L \cdots dw_1. \end{aligned} \tag{A2}$$

For the computation of $R(\eta, \eta'; \lambda; w)$, the kernel of $R(\lambda; w)$, we can see by induction on n that the kernel of $K_1(w)^n$ is

$$\tau^{nL} \frac{\phi_n(\tau\eta)}{\eta' - \tau^n\eta} \prod_{j=1}^L \frac{w_j\eta - 1}{\tau^n w_j\eta - 1},$$

where

$$\phi_n(\eta) = \phi(\eta)\phi(\tau\eta) \cdots \phi(\tau^{n-1}\eta) = \left(\frac{1 - \tau^n\eta}{1 - \eta} \right)^{x+L-1} e^{[\frac{1}{1-\eta} - \frac{1}{1-\tau^n\eta}]t}.$$

Therefore

$$R(\eta, \eta'; \lambda; w) = \sum_{n=1}^\infty (\lambda \tau^L)^n \frac{\phi_n(\tau\eta)}{\eta' - \tau^n\eta} \prod_{j=1}^L \frac{w_j\eta - 1}{\tau^n w_j\eta - 1}.$$

Since

$$\phi_n(\eta) = \frac{\phi_\infty(\eta)}{\phi_\infty(\tau^n\eta)},$$

where

$$\phi_\infty(\eta) = \lim_{n \rightarrow \infty} \phi_n(\eta) = (1 - \eta)^{-x-L+1} e^{\frac{t}{1-\eta}},$$

we can write the kernel of $K_2(w)R(\lambda; w)$ as

$$\sum_{n=1}^\infty (\tau^L \lambda)^n \int \frac{\phi_\infty(\zeta)}{\phi_\infty(\tau^{n+1}\zeta)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{\tau^n w_j\zeta - 1} \frac{d\zeta}{(\zeta - \tau\eta)(\eta' - \tau^n\zeta)}.$$

Here $|\zeta| = R$, but by analyticity we may take any radius such that $1 < |\zeta| < R$. This is equal to

$$\sum_{n=1}^\infty (\tau^L \lambda)^n \int \frac{\phi_\infty(\zeta)}{\zeta - \tau\eta} \prod_{j=1}^L (w_j\zeta - \tau) \left[\int \frac{1}{\phi_\infty(u\zeta)(\eta' - u\zeta/\tau) \prod_j (uw_j\zeta/\tau - 1)} \frac{du}{u - \tau^{n+1}} \right] d\zeta,$$

where on the u -contour we have $\tau^2 < |u| < \tau R/|\zeta|$. (The u -integral equals the residue at τ^{n+1} .) Using

$$\frac{1}{u - \tau^{n+1}} = \sum_{k=0}^\infty \frac{\tau^{(n+1)k}}{u^{k+1}}$$

and summing first on n gives

$$\sum_{k=0}^\infty \frac{\lambda \tau^{2k+L}}{1 - \lambda \tau^{k+L}} \int \frac{\phi_\infty(\zeta)}{\zeta - \tau\eta} \prod_{j=1}^L (w_j\zeta - \tau) \left[\int \frac{1}{\phi_\infty(u\zeta)(\eta' - u\zeta/\tau) \prod_j (uw_j\zeta/\tau - 1)} \frac{du}{u^{k+1}} \right] d\zeta.$$

If we take $|u| > \tau$ (as we may since $|\zeta| < R$), then when we write

$$\frac{\lambda \tau^{2k+L}}{1 - \lambda \tau^{k+L}} = \frac{\tau^k}{1 - \lambda \tau^{k+L}} - \tau^k,$$

the above sum may be written as the sum of two, in the obvious way, since both series will converge. Summing the second gives

$$- \int \frac{\phi_\infty(\zeta)}{\zeta - \tau\eta} \prod_{j=1}^L (w_j\zeta - \tau) \left[\int \frac{1}{\phi_\infty(u\zeta)(\eta' - u\zeta/\tau)(u - \tau) \prod_j (uw_j\zeta/\tau - 1)} du \right] d\zeta.$$

It is easy to check that the only singularity inside the u -contour is at $u = \tau$, so evaluating the u -integral in the above equation gives

$$-\int \frac{\phi_\infty(\zeta)}{\phi_\infty(\tau\zeta)} \frac{1}{(\zeta - \tau\eta)(\eta' - \zeta)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{w_j\zeta - 1} d\zeta = -\int \frac{\phi(\zeta)}{(\zeta - \tau\eta)(\eta' - \zeta)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{w_j\zeta - 1} d\zeta.$$

If we expand the contour so that $R < |\zeta| < \tau^{-1}$, then we pass a pole at $\zeta = \eta'$ and get

$$-\frac{\phi(\eta')}{\eta' - \tau\eta} \prod_{j=1}^L \frac{w_j\eta' - \tau}{w_j\eta' - 1} - \int \frac{\phi(\zeta)}{(\zeta - \tau\eta)(\eta' - \zeta)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{w_j\zeta - 1} d\zeta.$$

The first term in the above is exactly $-K_2(\eta, \eta'; w)$, so we have shown that

$$\begin{aligned} (K_2(w)(I + R(\lambda; w))(\eta, \eta')) &= -\int \frac{\phi(\zeta)}{(\zeta - \tau\eta)(\eta' - \zeta)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{w_j\zeta - 1} d\zeta \\ &+ \sum_{k=0}^{\infty} \frac{\tau^k}{1 - \lambda\tau^{k+L}} \int \frac{\phi_\infty(\zeta)}{\zeta - \tau\eta} \prod_{j=1}^L (w_j\zeta - \tau) \left[\int \frac{1}{\phi_\infty(u\zeta)(\eta' - u\zeta/\tau) \prod_j (uw_j\zeta/\tau - 1)} \frac{du}{u^{k+1}} \right] d\zeta. \end{aligned}$$

In the first integral on the right, $|\zeta| > R$, while in the second $1 < |\zeta| < R$ and $\tau < |u| < \tau R/|\zeta|$.

The first term on the right extends analytically for η, η' inside \mathcal{C} (the circle with radius R) and the sum extends analytically for η inside \mathcal{C} . It follows by Proposition 2 that for the Fredholm determinant, we may replace $K_2(w)(I + R(\lambda; w))$ by the operator whose kernel is the sum on the right.

If in the sum the index k were negative, then the u -integration would give zero since the integrand would be analytic inside the u -contour. Therefore we may take the sum over $k \in \mathbb{Z}$. In addition, we make the variable change $u \rightarrow u/\zeta$ and the sum becomes

$$\sum_{k \in \mathbb{Z}} \frac{\tau^k}{1 - \lambda\tau^{k+L}} \int \frac{\phi_\infty(\zeta)}{\zeta - \tau\eta} \zeta^k \prod_{j=1}^L (w_j\zeta - \tau) \left[\int \frac{1}{\phi_\infty(u)(\eta' - u/\tau) \prod_j (uw_j/\tau - 1)} \frac{du}{u^{k+1}} \right] d\zeta,$$

where now $\tau|\zeta| < |u| < \tau R$.

Recall that

$$f(\mu, z) = \sum_{k \in \mathbb{Z}} \frac{\tau^k}{1 - \tau^k \mu} z^k,$$

which is analytic for $1 < |z| < \tau^{-1}$. When we sum those terms involving k , we get

$$\sum_{k \in \mathbb{Z}} \frac{\tau^k}{1 - \lambda\tau^{k+L}} (\zeta/u)^k = f(\lambda\tau^L, \zeta/u)$$

so that the last sum equals

$$\tau^L \int \int \frac{\phi_\infty(\zeta)}{\phi_\infty(u)} \frac{f(\lambda\tau^L, \zeta/u)}{(\zeta - \tau\eta)(\eta' - u/\tau)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{uw_j - \tau} d\zeta \frac{du}{u}.$$

With the substitutions $\eta, \eta' \rightarrow \eta/\tau, \eta'/\tau$, we see that the above equation has the same Fredholm determinant as

$$\tau^L \int \int \frac{\phi_\infty(\zeta)}{\phi_\infty(u)} \frac{f(\lambda\tau^L, \zeta/u)}{(\zeta - \eta)(\eta' - u)} \prod_{j=1}^L \frac{w_j\zeta - \tau}{uw_j - \tau} d\zeta \frac{du}{u},$$

where now the operator acts on a circle with radius $r \in (\tau, 1)$ and in the integrals $1 < |\zeta| < r/\tau, \tau|\zeta| < |u| < r$.

We shall make the substitution $\lambda = \tau^{-m}\mu$ in (A2). When we do that here, we use the easy fact $f(\tau^{-n}\mu, z) = (\tau z)^n f(\mu, z)$ and see that the above equation becomes

$$\tau^m \int \int \frac{\phi_\infty(\zeta)}{\phi_\infty(u)} \left(\frac{\zeta}{u}\right)^{m-L} \frac{f(\mu, \zeta/u)}{(\zeta - \eta)(\eta' - u)} \prod_{j=1}^L \frac{w_j \zeta - \tau}{u w_j - \tau} d\zeta \frac{du}{u}.$$

If we expand the u -contour, so that $r < |u| < 1$ on the new contour, then we pass the pole at $u = \eta'$ with minus the residue equal to

$$\tau^m \int \frac{\phi_\infty(\zeta)}{\phi_\infty(\eta')} \frac{\zeta^{m-L}}{(\eta')^{m-L+1}} \frac{f(\mu, \zeta/\eta')}{\zeta - \eta} \prod_{j=1}^L \frac{w_j \zeta - \tau}{w_j \eta' - \tau} d\zeta = \tau^m J_{L,x,m}(\eta, \eta'; w),$$

by definition (2). [The function $f(\mu, \zeta/\eta')$ remains analytic during the deformation.] The new double integral is analytic for $|\eta|, |\eta'| \leq r$ and $J_{L,x,m}(\eta, \eta'; w)$ is analytic for $|\eta| \leq r$, so by Proposition 2, we may replace the last double integral by $J_{L,x,m}(\eta, \eta'; w)$. Substituting $\lambda = \tau^{-m}\mu$ in (A2) and replacing $K_2(w) (I + R(\lambda; w))$ by $\tau^m J_{L,x,m}(w)$, we arrive at (3).

APPENDIX B: SINGULARITY OF THE G_L -INTEGRAL

Here we show that if $\psi(w_2, \dots, w_L)$ is analytic in the neighborhood of $\{0, \tau\}^{L-1}$, then

$$\int_{\Gamma_{0,\tau}} \dots \int_{\Gamma_{0,\tau}} G_L(w_1, \dots, w_L) \psi(w_2, \dots, w_L) dw_L \dots dw_2 \tag{B1}$$

is analytic for w_1 outside $\{0, \tau\}$ except for a pole of order at most $L - 1$ at $w_1 = 1$ and is $O(1)$ for large w_1 . This is straightforward except for the nature of the singularity $w_1 = 1$, which occurs when some w_j is integrated over Γ_τ . We show that it is a pole of order at most $L - 1$.

The integral (B1) is a sum of integrals in each of which some w_j are taken over Γ_0 and some over Γ_τ . Let Z be the set of former indices and integrate first with respect to the w_j with $j \in Z$. There are simple poles at $w_j = 0$. When evaluating the integrals, we recall the convention, which is relevant only for the integrals over Γ_0 , that we integrate first with respect to the w_j with the largest j . If in the double product in (12) some $j \in Z$, then the product over i equals τ^{j-1} , while if $i \in Z$ but $j \notin Z$, then that factor becomes 1. Thus after these integrations, we are left with a constant times (B1) except that all indices run over Z^c , and all integrals are over Γ_τ . In the function $\psi(w_2, \dots, w_L)$, the variables w_j with $j \in Z$ are set equal to zero.

To be more explicit, set $n = |Z|$ and let the indices in Z^c be $\ell_2 < \ell_3 < \dots < \ell_{L-n}$. (Note that $\ell_2 \geq 2$.) Then the integrand becomes a function analytic near all $w_{\ell_j} = \tau$ and $w_1 = 1$ times

$$\frac{\prod_{1 < i < j} (w_{\ell_i} - w_{\ell_j})}{\prod_{j > 1} [(w_{\ell_j} - \tau)^{L-\ell_j+1} (\tau w_1 - w_{\ell_j})]}.$$

We make the variable changes $w_{\ell_j} \rightarrow w_j + \tau$, so the integrations are over Γ_0 and the integrand equals a function analytic near all $w_j = 0$ ($j \geq 2$) and $w_1 = 1$ times

$$\frac{\prod_{1 < i < j} (w_i - w_j)}{\prod_{j > 1} [w_j^{L-\ell_j+1} (\tau(w_1 - 1) - w_j)]}. \tag{B2}$$

It is convenient to set

$$v = \tau(w_1 - 1)$$

so that $\tau(w_1 - 1) - w_j = v - w_j$. The analytic multiplying function has a series expansion

$$\sum_{\alpha_2, \dots, \alpha_{L-n} \geq 0} c(\alpha_2, \dots, \alpha_{L-n}) \prod_{j=2}^{L-n} w_j^{\alpha_j}, \tag{B3}$$

where the coefficients are polynomials in v .

The numerator in (B2) is equal to a Vandermonde determinant $\det(w_j^i)$. (Rows i run from 0 to $L - n - 2$ and columns j from 2 to $L - n$.) Multiplying by the rest of (B2) and the product in (B3) has the effect of multiplying column j by $w_j^{-L+\ell_j-1+\alpha_j}/(v-w_j)$. So we obtain a linear combination over $\alpha_2, \dots, \alpha_{L-n}$ of determinants with the i, j entry

$$\frac{w_j^{i-L+\ell_j-1+\alpha_j}}{v-w_j}. \tag{B4}$$

We are to integrate this over all $w_j \in \Gamma_0$. Since w_j appears only in column j , the multiple integral is obtained by integrating each entry. The i, j -entry integrates to $v^{i-L+\ell_j-1+\alpha_j}$ if $i-L+\ell_j+\alpha_j \leq 0$, when the exponent is negative, and zero otherwise. (If any $\alpha_j > L-\ell_j$, then the j th column is zero and therefore so is the determinant. Thus the linear combination is a finite one.)

We make the substitution $i \rightarrow L-n-2-i$ (the new i also runs from 0 to $L-n-2$) and set $V=v^{-1}$. Then the i, j entry is $V^{i+n+3-\ell_j-\alpha_j}$ if the exponent is positive and zero otherwise. We factor out V from each entry and get V^{L-n-1} times the determinant whose i, j entry is $V^{i+n+2-\ell_j-\alpha_j}$ if the exponent is non-negative and zero otherwise. Since for what follows we want j , as well as i , to start from 0, we replace the nonzero entries by

$$V^{i+n+2-\ell_{j+2}-\alpha_{j+2}}. \tag{B5}$$

Lemma. Suppose we have a determinant with the i, j entry $V^{a_i-b_j}$ if the exponent is non-negative and zero otherwise. (The indices begin at 0.) If the determinant is nonzero, then the a_i and b_j , after reordering, satisfy

$$b_0 \leq a_0 < b_1 \leq a_1 < b_2 \leq a_2 < \dots,$$

in which case the determinant equals $\pm V^{\sum_i(a_i-b_i)}$.

Proof. We may assume the sequences $\{a_i\}$ and $\{b_i\}$ are nondecreasing. If the determinant is nonzero, then some exponent $a_0 - b_j$ in the top row is non-negative. Then this would hold for the smallest b_j , which is b_0 . If also $a_0 - b_j$ were non-negative for some $j > 0$, then columns 0 and j would be linearly dependent since there would be no zero entries in these columns. (Because the remaining $a_i \geq a_0$.) Hence $b_j > a_0$ for $j > 0$ and the 0, 0-entry is the only nonzero one in the top row. Therefore the determinant equals $V^{a_0-b_0}$ times the 0, 0 cofactor, which is of the same form as the original determinant. The result follows by induction.

It follows from the lemma that when $a_i = i$ and the b_j are integers, the determinant is nonzero only if the b_j when reordered satisfy $b_0 \leq 0$ and $b_j = j$ when $j > 0$, in which case the determinant is $\pm V^{-b_0}$. For the determinant with entries (B5), we have $a_i = i$ and

$$b_j = -n - 2 + \ell_{j+2} + \alpha_{j+2} \geq -n,$$

for all j since $\alpha_{j+2} \geq 0$ and $\ell_{j+2} \geq 2$. Thus the determinant is $O(V^n)$. Recalling the factor V^{L-n-1} we had for this determinant, we have shown that our original determinant is $O(V^{L-1}) = O((w_1 - 1)^{-L+1})$. Therefore the singularity at $w_1 = 1$ is a pole of order at most $L - 1$.

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