NONINTERSECTING BROWNIAN EXCURSIONS

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Outline of Talk

1. Brownian Excursion and the Airy Distribution

2. Nonintersecting Path Models and Extended Kernels

3. Extended Kernel for $n$-Brownian Excursion

4. Distribution of Bottom Curve at One Fixed Time: $P_V$

5. Scaling of Bottom Curve to Bessel Process, $\alpha = 1/2$

6. Expected Areas under Bottom and Top Curves
Brownian Excursion

A Brownian excursion (BE) $X(\tau)$ is a Brownian path conditioned to remain positive for $0 < \tau < 1$ with boundary conditions $X(0) = X(1) = 0$. BE can be defined by scaling one-dimensional simple random walk conditioned to stay positive and conditioned to start and to end at the origin—a process known as Bernoulli excursion.

Let $B_\tau$ denote standard Brownian motion (BM), so that

$$
P_x (B_\tau \in dy) = P(x, y, \tau) \, dy = \frac{1}{\sqrt{2\pi \tau}} e^{-(x-y)^2/(2\tau)} \, dy, \tau > 0, x, y \in \mathbb{R},$$

Let

$$H_a = \inf\{s : s > 0, B_s = a\} = \text{hitting time of } a.$$

Then

$$
P_x (B_\tau \in dy, H_0 > \tau) = (P(x, y, \tau) - P(x, -y, \tau)) \, dy := P_-(x, y, \tau)dy.$$
Airy Distribution

Let $\mathcal{A}$ denote the area under a BE path.

Then

$$F_{\text{Ai}}(a) := \mathbb{P}(\mathcal{A} < a)$$

is called the Airy distribution. (Not to be confused with the Airy process!) For example,

$$\mathbb{E}(\mathcal{A}) = \sqrt{\frac{\pi}{8}}, \quad \mathbb{E}(\mathcal{A}^2) = \frac{5}{12}, \ldots$$

The Airy distribution appears as the limit law for a number of combinatorial problems: path length in trees, total displacement of a random parking sequence, ... (Flajolet & Louchard)
Extended Kernels

Given: stationary Markov process with continuous paths, transition probability density $P(x, y, \tau)$, and a family of $n$ nonintersecting paths, $\{X_i(\tau)\}_{i=1}^n$, beginning at $a_1, \ldots, a_n$ at time $\tau = 0$ and ending at $b_1, \ldots, b_n$ at time $\tau = 1$.

An extended kernel is a matrix kernel $K(x, y) = (K_{k\ell}(x, y))_{k\ell=1,\ldots,m}$ depending on $0 < \tau_1 < \cdots < \tau_m < 1$ with the following property:

Given functions $f_k$ the expected value of

$$\prod_{k=1}^m \prod_{i=1}^n (1 + f_k(X_i(\tau_k)))$$

is equal to $\det(I + Kf)$, where $f$ denotes multiplication by $\text{diag}(f_k)$. In the special case where $f_k = -\chi_{J_k}$ this is the probability that for $k = 1, \ldots, m$ no path passes through the set $J_k$ at time $\tau_k$. 
Nonintersecting Brownian Bridges and Excursions

Define *Extended Hermite kernel*:

$$K_{n}^{\text{GUE}}(x, y; \hat{\tau}) = \begin{cases} 
\sum_{j=0}^{n-1} e^{j\hat{\tau}} \varphi_{j}(x)\varphi_{j}(y) & \hat{\tau} \geq 0 \\
- \sum_{j=n}^{\infty} e^{j\hat{\tau}} \varphi_{j}(x)\varphi_{j}(y) & \hat{\tau} < 0
\end{cases}$$

- **Brownian Bridge Extended Kernel**:

  $$\frac{1}{\sqrt{2(1 - \tau_{k})\tau_{\ell}}} K_{n}^{\text{GUE}}(X_{k}, Y_{\ell}; \hat{\tau}_{k} - \hat{\tau}_{\ell}),$$

  $$X_{k} = \frac{x}{\sqrt{2\tau_{k}(1 - \tau_{k})}}, \quad Y_{\ell} = \frac{y}{\sqrt{\tau_{\ell}(1 - \tau_{\ell})}}, \quad \frac{\tau_{k}}{1 - \tau_{k}} = \hat{\tau}_{k}$$

- **Brownian Excursion Extended Kernel**:

  $$\frac{1}{\sqrt{2(1 - \tau_{k})\tau_{\ell}}} \left\{ K_{2n}^{\text{GUE}}(X_{k}, Y_{\ell}; \hat{\tau}_{k} - \hat{\tau}_{\ell}) - K_{2n}^{\text{GUE}}(X_{k}, -Y_{\ell}; \hat{\tau}_{k} - \hat{\tau}_{\ell}) \right\}$$
Case of a Single Time: $m = 1$

Order paths: $X_n(\tau) > \cdots > X_1(\tau) > 0$

$$\mathbb{P}(X_1(\tau) \geq x) = \det (I - K\chi_{J_1}), \quad J_1 = (0, s)$$

$$\mathbb{P}(X_n(\tau) < x) = \det (I - K\chi_{J_2}), \quad J_2 = (s, \infty)$$

where

$$K = K_{2n}^{\text{GUE}}(x, y) - K_{2n}^{\text{GUE}}(x, -y), \quad s = \frac{x}{\sqrt{\tau(1 - \tau)}}$$

- Operator $K$ is finite rank $\implies$ distributions are $n \times n$ dets.
- Operator $K$ has kernel of the form

$$\frac{\varphi(x)\psi(y) - \psi(x)\varphi(y)}{x - y}$$

where

$$\varphi(x) = n^{1/4}x^{1/4}\varphi_{2n}(\sqrt{x}), \quad \psi(x) = n^{1/4}x^{-1/4}\varphi_{2n-1}(\sqrt{x})$$
Painlevé V Representation

\[ P(X_1(\tau) \geq x) = \exp \left( - \int_0^{s^2 \tau} \frac{r(t)}{t} dt \right) \]

where \( r \) satisfies the Jimbo-Miwa \( \sigma \) form of Painlevé V,\(^a\)

\[ s = \frac{x}{\sqrt{\tau (1 - \tau)}} \text{ with boundary condition as } s \to 0^+ \]

\[ r(s) = r_0 s^{3/2} + O(s^{5/2}), \quad r_0 = \frac{1}{\sqrt{\pi}} \frac{1}{2^{2n}} \frac{(2n)}{n} \frac{4n(2n + 1)}{3} \]

- The number of BE curves, \( n \), appears only as a parameter in the Painlevé V DE.
- Similar Painlevé V representation for \( P(X_n(\tau) < x) \).

\(^a\) \( r = -\sigma, \nu_0 = n, \nu_1 = n + 1/2, \nu_2 = \nu_3 = 0.\)
Limit Theorems as $n \to \infty$

• Look at scaling of bottom curve as $n \to \infty$:

\[
x, y \to \sqrt{\frac{2n(1-\tau)}{\tau}} \cdot x, y, \quad J_k \to \sqrt{\frac{2n(1-\tau)}{\tau}} J_k
\]

\[
\tau_k \to \tau + \frac{\tau(1-\tau)}{2n} \cdot \tau_k
\]

Then extended BE kernel approaches (with trace norm convergence) the extended Bessel kernel with $\alpha = 1/2$:

\[
\frac{2}{\pi} \int_0^1 e^{(\tau_k - \tau_\ell)t^2/2} \sin xt \sin yt \, dt, \quad k \geq \ell
\]

\[
-\frac{2}{\pi} \int_1^\infty e^{(\tau_k - \tau_\ell)t^2/2} \sin xt \sin yt \, dt, \quad k < \ell.
\]

• Expect top curve to scale to Airy process. Top curve does not “feel presence of the wall” as $n \to \infty$. Checked that diagonal matrix elements approach (with trace norm convergence) the Airy kernel.
Area under Bottom and Top Curves

Define

\[ A_{n,L} = \int_0^1 X_1(\tau) \, d\tau, \quad A_{n,H} = \int_0^1 X_n(\tau) \, d\tau. \]

• Expected area

\[ \mathbb{E}(A_{n,L}) = \int_0^1 \mathbb{E}(X_1(\tau)) \, d\tau \]

\[ = \int_0^1 \int_0^\infty \mathbb{P}(X_1(\tau) \geq x) \, dx \, d\tau \]

\[ = \int_0^1 \sqrt{2\tau(1-\tau)} \, d\tau \cdot \int_0^\infty \det (I - K\chi(0,s)) \, ds \]

\[ = \frac{\pi}{4\sqrt{2}} \int_0^\infty \det (\delta_{j,k} - (\Psi_j, \Psi_k))_{j,k=0}^{n-1} \, ds \]

• Last expression good for numerical evaluation for small \( n \).

• Higher moments are expressible in terms of integrals over Fredholm dets with extended kernel.
**Expected Area Asymptotics**

Using convergence of BE kernel to Bessel kernel we derive (with some additional estimates to get convergence of the moments)

\[
\mathbb{E}(A_{n,L}) \sim \frac{c_L}{\sqrt{n}}, \quad n \to \infty \quad \text{where}
\]

\[
c_L = \frac{\pi}{8\sqrt{2}} \int_{0}^{\infty} \det \left( I - K_{\text{Bessel}} \chi_{(0,s^2)} \right) ds \simeq 0.682808.
\]

Compare with \( \sqrt{n} \mathbb{E}(A_{n,L}) \) for \( n = 5, \ldots, 11 \)

0.667334, 0.669708, 0.671449, 0.672784, 0.673838, 0.674691, 0.675396

- Top curve

\[
\mathbb{E}(A_{n,H}) = \frac{\pi}{23/2} \sqrt{n} + \frac{c_H}{n^{1/6}} + o(n^{-1/6})
\]

\[
c_H = \frac{\pi}{8 \cdot 2^{1/6}} \mu_2 \simeq -0.619623767170
\]

Here \( \mu_2 \) is first moment of \( F_{\text{GUE}} \) largest eigenvalue distribution.