EUCLIDEAN BUILDINGS

Buildings were introduced by Jacques Tits in the 1950s to give a systematic procedure for the geometric interpretation of the semi-simple Lie groups (in particular the exceptional groups) and for the construction and study of semi-simple groups over general fields. They were simplicial complexes and their apartments were euclidean spheres with a finite (Weyl) group of isometries. So these buildings were called of spherical type [Tits-74].

Later François Bruhat and Jacques Tits constructed buildings associated to semi-simple groups over fields endowed with a non archimedean valuation. When the valuation is discrete these Bruhat-Tits buildings are still simplicial (or polysimplicial) complexes, and their apartments are affine euclidean spaces tessellated by simplices (or polysimplices) with a group of affine isometries as Weyl group. So these buildings were called affine. But when the valuation is no longer discrete, the simplicial structure disappear ; so Bruhat and Tits construct (the geometric realization of) the building as a metric space, union of subspaces isometric to euclidean spaces, and they introduce facets as filters of subsets [Bruhat-Tits-72].

This is the point of view I wish to develop in these lectures, by giving a definition of euclidean buildings valid even in the non discrete case and independent of their construction. Actually such a definition has been already given by Tits [86a], but his definition emphasizes the role of sectors against that of facets. On the contrary I define here an euclidean building as a metric space with a collection of subspaces (called apartments) and a collection of filters of subsets (called facets) submitted to axioms which, in the discrete case where these filters are subsets, are the classical ones of [Tits-74]. The equivalence with Tits' definition (under some additional hypothesis) is a simple corollary of previous results of Anne Parreau [00].

So an euclidean building is defined here as a geometric object (a geometric realization of a simplicial complex in the discrete case). It is endowed with a metric with non positive curvature which makes it look like a Riemannian symmetric space. The fundamental examples are the Bruhat-Tits buildings, but the Tits buildings associated to semi-simple groups over any field [Tits-74] have also geometric realizations (called vectorial buildings) as euclidean buildings.

The building stones of a building are the apartments. They are defined as affine euclidean spaces endowed with a structure (some facets in them) deduced from a group W generated by reflections. This theory is explained in part 1, with some references to the literature for the proofs. The general theory of euclidean buildings developed in part 2 is self contained except for references to part 1 and for some final developments. Part 3 is devoted to the fundamental examples : the vectorial building associated to a reductive group and the Bruhat-Tits building of a reductive group over a local field. More details are given when the group is $\mathcal{GL}_n$.

For further developments or details, the interested reader may look at [Brown-89 and 91], [Bruhat-Tits-72, 84a and 84b], [Garrett-97], [Parreau-00], [Rémy-02], [Ronan-89 and 92], [Scharlau-95] and [Tits-74, 86a, 86b, ...].

Part I : **APARTMENTS (= thin buildings)**

The general references for this first part are to Bourbaki, Brown [89], Garrett [97 ; chap 12, 13] and Humphreys. Many proofs are omitted, specially in § 2 and § 4.

§ 1 Groups generated by reflections and apartments :

Let V be an euclidean space of finite dimension n and A an associated affine space.

A *reflection* r is an isometry of V or A (linear or affine) whose fixed point set is an hyperplane H_r. To any hyperplane H is associated the reflection r_H with respect to H.

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A *reflection group* in A is a group W of affine isometries of A which is generated by reflections and such that the subgroup W^v of $GL(V)$ consisting of the vectorial (= linear) parts of elements of W is finite.

The group W is called *irreducible* if W^v acts irreducibly on V.

A reflection group W in V is called *linear* if it fixes 0 ; it is then finite and equal to W^v.

The *walls* of W are the hyperplanes of fixed points of reflections in W. The set $\mathcal{H}$ of walls is stable under the action of W ; it completely determines W. The directions of the walls are the *vectorial walls* ; their set $\mathcal{H}^v$ is finite. We denote by $\widetilde{\mathcal{H}}$ the set of hyperplanes of direction in $\mathcal{H}^v$.

A reflection group W in A is called *affine* if the set of walls of any given direction (in $\mathcal{H}^v$) is infinite.

We denote by $\widetilde{W}$ the subgroup of $Isom(A)$ generated by all reflections whose vectorial parts are in W^v ; it is an affine reflection group with set of walls $\widetilde{\mathcal{H}}$ and isomorphic to $W^v \ltimes V'$, where $V' \subset V$ is generated by all vectors orthogonal to walls in $\widetilde{\mathcal{H}}$.

An *apartment* is a pair (A, W) (often summarized as A) where W is a reflection group in A (see also 5.4) ; it may be seen and drawn as $(A, \mathcal{H})$. Some examples of dimension 1 or 2 are in figure 1.

A wall H divides the space in 3 disjoint convex subsets : 2 open half spaces called *open-half-apartments* and the wall itself. The closure of an open-half-apartment is called an *half-apartment*.

For $Q \subset A$, the *enclosure* of Q is the intersection $cl(Q)$ of all half-apartments containing Q ; it's a closed convex set. A subset Q is called *enclosed* if $Q = cl(Q)$.

§ 2 Linear reflection groups :

2.1 If W^v is a linear reflection group, its walls are vectorial hyperplanes. One defines an equivalence relation on V by : $v_1 \sim v_2 \Leftrightarrow v_1$ and v_2 are in exactly the same open-half-apartments or walls.

The equivalence classes are the *vectorial facets* associated to W^v ; they are convex cones. The set $\mathcal{F}^v$ of these facets is finite and ordered by the following relation :

"F is a face of F' " $\Leftrightarrow$ "F' covers F " $\Leftrightarrow F \leq F' \Leftrightarrow F \subset \overline{F'}$ (topological closure of F')

The maximal facets, called *chambers* are the (open) connected components of $V \setminus \cup_{H \in \mathcal{H}^v} H$. A maximal element among non-chamber facets is a *panel*; it is an open subset of a wall (called its *support*).

More generally the *support* $supp(F^v)$ of a facet F^v is the intersection of the walls containing F^v (or its closure $\overline{F}^v$) ; its dimension is the *dimension* of F^v and F^v is the interior of $\overline{F}^v$ with respect to the topological space $supp(F^v)$.

Example 2.2 : figure 2

The only non trivial example in dimension 2 is the dihedral group W of order $2m$ in $\mathbb{R}^2$ $(m \geq 1)$. The set $\mathcal{H}^v$ consists of m vectorial walls, with angles $\frac{k\pi}{m}$ for $k \in \mathbb{Z}$. Its name is $I_2(m)$.

Proposition 2.3 : *Let C^v be a chamber and F_1^v, ..., F_p^v its (faces which are) panels. Denote by r_i the reflection with respect to the wall H_i supporting F_i^v. Then :*

a) For $i \neq j$, the angle between H_i and H_j is $\frac{\pi}{m_{i,j}}$ where $m_{i,j} \geq 2$ is an integer.

b) If $f_i \in V^*$ is such that $H_i = Ker(f_i)$, then f_1, ... ,f_p are independent in V^*.

c) The group W^v is generated by r_1, ... ,r_p . It's a Coxeter group with relations $(r_i r_j)^{m_{i,j}} = 1 = (r_i)^2$. The integer p is the rank of W^v or of the apartment.

d) The group W^v acts simply transitively on the set of chambers.

e) The fixator (pointwise) or stabilizer in W^v of a facet F^v is the group generated by the reflections with respect to walls containing F^v. It is simply transitive on the chambers covering F^v.

See [Bourbaki ; V§ 3], [Brown-89 ; chap. I], [Garrett ; 12.2, 12.3] or [Humphreys ; V§ 3].

2.4 Coxeter diagrams :

It's clear from the preceding proposition that (V, W^v) is determined up to isomorphism by the numbers $m_{i,j}$, $1 \leq i \neq j \leq p \leq n$. Except for n this is summarized in the *Coxeter diagram* :

- its vertices are indexed by $\mathbb{N} \cap [1, p] = I$ (so their number is the rank p),

- two different vertices indexed by i and j are joined by an edge if and only if $m_{i,j} \geq 3$, and this edge is labeled with the number $m_{i,j}$ if and only if $m_{i,j} \geq 4$.

(actually one uses a double edge instead of a label 4 and a triple edge instead of a label 6).

As a graph, a Coxeter diagram has connected components. If i and j are in different components, r_i and r_j commute. So W^v is the direct product of subgroups corresponding to connected components ; more precisely :

Proposition 2.5 : *There is a unique decomposition of* (V, W^v) *in orthogonal direct product :* $V = V_0 \oplus V_1 \oplus ... \oplus V_s$ *and* $W^v = W_0^v \times W_1^v \times ... \times W_s^v$ *, such that each* W_i^v *acts only on* V_i *, the group* W_o^v *is trivial and, for* $i \geq 1$ *the action of* W_i^v *on* V_i *is irreducible non trivial.*

Then one has $dim(V_i) = rank(W_i^v)$ *for* $i \geq 1$ *and the Coxeter diagrams of the groups* W_i^v *for* $i \geq 1$ *are the connected components of the Coxeter diagram of* W^v .

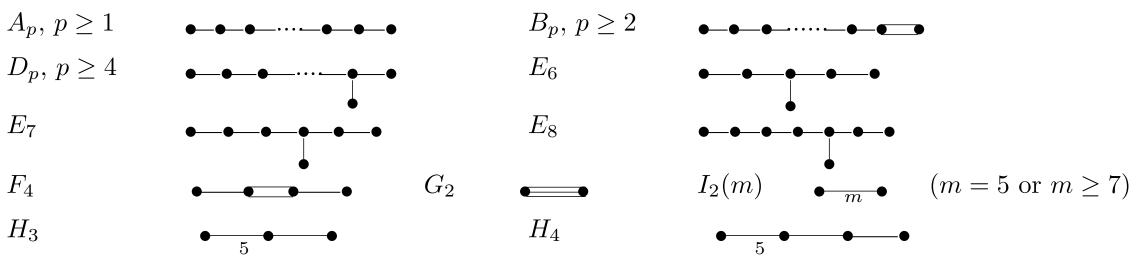
See *e.g.* [Bourbaki ; V 3.7] or [Humphreys ; 2.2].

N.B. : The Coxeter diagram determines everything except V_0 (which is the smallest facet of V).

2.6 Classification : [Bourbaki ; VI 4.1] or [Humphreys ; chap. 2].

The connected Coxeter diagrams corresponding to (irreducible) linear reflection groups are drawn in the following table :

(the index p in a name X_p is the rank of the group *i.e.* the number of vertices of the diagram).



Actually $I_2(3) = A_2$, $I_2(4) = B_2$, $I_2(6) = G_2$ and $I_2(5)$ could be named H_2.

2.7 Link with root systems : [Bourbaki ; VI], [Garrett ; 12.3] or [Humphreys ; 2.9].

For $\alpha \in V^* \setminus \{0\}$, let r_α be the reflection with respect to the hyperplane $Ker(\alpha)$ and $\alpha\check{}$ be the vector in V orthogonal to $Ker(\alpha)$ verifying $\alpha(\alpha\check{}) = 2$. For $v \in V$, one has $r_\alpha(v) = v - \alpha(v)\alpha\check{}$.

A *root system* in V^* is a finite set $\Phi \subset V^* \setminus \{0\}$ such that :

- $\forall \alpha \in \Phi$ one has $r_\alpha(\Phi) = \Phi$,
- $\forall \alpha, \beta \in \Phi$ one has $\alpha(\beta\check{}) \in \mathbb{Z}$.

This root system is called *reduced* if and only if $\forall \alpha \in \Phi$ one has $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$.

To a root system is associated its *Weyl group* W^v generated by the r_α for $\alpha \in \Phi$. It is a finite reflection group, its walls are the hyperplanes $Ker(\alpha)$ for $\alpha \in \Phi$.

Actually a finite reflection group is the Weyl group of a root system if and only if it is *crystallographic* and this means that no irreducible component is of type H_3 , H_4 or $I_2(m)$ for $m = 5$ or $m \geq 7$. Non isomorphic root systems may give rise to isomorphic Weyl groups ; the only reduced irreducible example is the following : there are reduced irreducible root systems of type B_p and C_p for $p \geq 2$, non isomorphic if $p \geq 3$ and their Weyl groups are isomorphic (of type B_p).

§ 3 General reflection groups :

3.1 Let W be a reflection group in A. Then W^v is generated by reflections with respect to hyperplanes $Ker(f_i)$ (*cf.* 2.3). But each $Ker(f_i)$ is the direction of a wall H_i in A, and the f_i are independent ; so the intersection of the H_i is non empty. If a is in this intersection the fixator W_a of a in W is isomorphic to W^v, for each vectorial wall H^v there exists a wall H of direction H^v and containing a (one says that a is *special*). Hence W is a semi-direct product $W = W_a \ltimes T$ where $W_a \simeq W^v$ and T is a group of translations generated by some vectors in V (each orthogonal to a wall).

Proposition 3.2 : *The decomposition of* (V, W^v) *in 2.5 gives a decomposition of* (A, W) *as an orthogonal direct product :* $A = A_0 \times A_1 \times ... \times A_s$ *(where* A_i *is affine under* V_i*) and* $W = W_0 \times W_1 \times ... \times W_s$ *. Each* W_i *acts only on* A_i *; the group* W_0 *is trivial and, for* $i \geq 1$, $(W_i)^v = W_i^v$ *hence* W_i *is irreducible non trivial.*

Actually each A_i is the quotient of A by the product of the V_j for $j \neq i$; in particular the decomposition is unique.

One says that W is *essential* if and only if $V_0 = \{0\}$, *i.e.* if and only if W^v has no non trivial fixed point.

Example : The apartment $I_2(m)$ of 2.2 is inessential for $m = 1$ and reducible (of type $A_1 \times A_1$) for $m = 2$.

3.3 Classification :

Let W be an essential irreducible reflection group in A. The real vector space generated by T and the greatest real vector space contained in the closure of T are stable under W^v, hence they must be $\{0\}$ or V. So there are exactly 3 cases :

- *spherical type* : if $W \simeq W^v$ is finite,
- *discrete affine type* : if $W \simeq W^v \ltimes T$ with T a lattice in V ,
- *dense affine type* : if $W \simeq W^v \ltimes T$ with T a dense subgroup of V .

3.4 A *sector* in A is a subset of A defined as $S = x + C^v$ where $x \in A$ is the *base point* of S and C^v is a vectorial chamber in V (the *direction* of S) ; one also uses for S the word "Weyl-chamber" or (in french) "quartier". Note that Ronan [89] asks moreover that the base point is special *i.e.* that S is enclosed.

One defines also a *sector-panel* or *sector-facet* as $x + F^v$ where F^v is a vectorial panel or a vectorial facet.

§ 4 Discrete reflection groups :

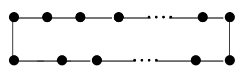
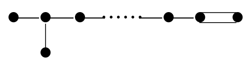
4.1 A reflection group W is called *discrete* if it has no irreducible factor of dense affine type or equivalently if T is a discrete subgroup of V. These groups are those studied in [Bourbaki ; V], [Garrett ; chap. 12], [Brown-89] or [Humphreys] where details and proofs may be found.

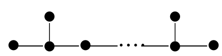
As in § 2, one can then define the facets (as equivalence classes), chambers, panels, ... They are no longer convex cones if W is infinite. When W is discrete affine, chambers are often called *alcoves.*

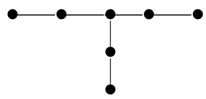
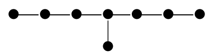
The results of proposition 2.3 are still true, except for b) and except that now $m_{i,j}$ may be infinite.

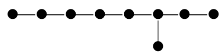
The Coxeter diagram is defined the same way. If W is essential irreducible, then the Coxeter diagram is connected and its rank is $dim(A) + 1$ if moreover it is affine.

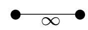
Examples are given in figures 1 or 2 and the classification of the essential irreducible ones is given in table 2.6 for spherical type and in the following table for discrete affine type.
(For a name $\tilde{X}_p$ the rank of the group *i.e.* the number of vertices of the diagram is $p + 1$).

$\tilde{A}_p$, $p \geq 2$  $\tilde{B}_p$, $p \geq 3$ 

$\tilde{C}_p$, $p \geq 2$ $\tilde{D}_p$, $p \geq 4$ 

$\tilde{E}_6$  $\tilde{E}_7$ 

$\tilde{E}_8$  $\tilde{F}_4$

$\tilde{G}_2$ $\tilde{A}_1$ 

4.2 Type : Let C be a chamber and F_1, ... ,F_p its panels.

The *type* of a face F of C is $\tau(F) = \{i \,/\, F \subset \overline{F}_i\} \subset \{1, ..., p\} = I$ and $\overline{F} = \bigcap_{i\in\tau(F)} \overline{F}_i$.
So faces of C correspond bijectively to some subsets of I ; actually when W is irreducible of spherical type (resp. of discrete affine type), any subset of I (resp. except I itself) is the type of a face.

But W is simply transitive on chambers of A and the fixator W_F of a facet F is transitive on chambers covering F. So the type of a facet F is well determined by the rule : $\tau(F) = \tau(wF) \subset I$, $\forall w \in W$.

4.3 Link with root systems :

A reduced root system Φ in V^* gives rise to a discrete affine reflection group $W = W^a(\Phi)$ in V . It is defined by its walls which are all affine hyperplanes of equation $\alpha(v) = m$, for $\alpha \in \Phi$ and $m \in \mathbb{Z}$.

$W^a(\Phi)$ is the *affine Weyl group* of Φ . Its associated vectorial group is the Weyl group W^v of Φ . Its translation group T is the $\mathbb{Z}-$module $Q\check{}$ generated by the *coroots* $\alpha\check{}$ for $\alpha \in \Phi$.

If Φ is irreducible of type X_p , then $W^a(\Phi)$ is irreducible of type $\widetilde{X}_p$ (see the above classification). As a consequence every discrete affine reflection group is the affine Weyl group of a reduced root system.

4.4 Galleries :

Two chambers C and C' are called *adjacent* if they share a common panel.

A *gallery* is a sequence of chambers $\Gamma = (C_0, ..., C_d)$ such that 2 consecutive chambers C_{i-1} and C_i are adjacent. We say that Γ is a gallery from C_0 to C_d , its *length* is d .

It's allowed that, for some i , $C_{i-1} = C_i$, we say then that the gallery *stutters.*

The *type* of the gallery Γ is $(i_1, ..., i_d) \subset I^d$, if C_{j-1} and C_j share a panel of type $\{i_j\}$. The type is well determined only if the gallery doesn't stutter.

If C and D are chambers the *(combinatorial) distance* $d(C, D)$ between C and D is the minimal length of a gallery from C to D . A gallery from C to D achieving this minimum is called *minimal.*

Actually $d(C, D)$ is the number of walls *separating* C and D (*i.e.* C and D are in opposite half-apartments defined by such a wall). And any minimal gallery from C to D is contained in the enclosure $cl(C, D)$.

Proposition 4.5 : *Let C be a chamber and F_1, ... ,F_p its panels. Denote by r_i the reflection with respect to the wall H_i support of F_i .*

a) Any non stuttering gallery Γ from C to a chamber D is defined by its type $(i_1, ..., i_d)$ by the following rule : $C_0 = C$, $C_1 = r_{i_1}C$, $C_2 = r_{i_1}r_{i_2}C$, $D = C_d = r_{i_1}r_{i_2}...r_{i_d}C$.

b) The gallery Γ is minimal if and only if $r_{i_1}r_{i_2}...r_{i_d}$ is a reduced decomposition in the Coxeter group W with set of generators $\{r_1, ..., r_p\}$.

N.B. : Actually the results c, d and e in proposition 2.3 (or their equivalents in 4.1) may be proved using these definitions (a correspondence between words in W, galleries starting from C, paths, ...) and some topological algebra : the union of facets of codimension ≥ 1 (resp. ≥ 2) is connected (resp. simply connected).

Lemma 4.6 : *Let $x \in A$, then there is $\varepsilon > 0$ (depending only on x or Wx) such that : $d(y, z) > \varepsilon$, $\forall y \neq z \in Wx$.*

Proof : Actually W is $W^v \ltimes T$ with W^v finite and T a discrete translation group. So any ball of center x contains only a finite number of elements of Wx. □

§ 5 Facets for general reflection groups :

When W is dense affine it's still possible to define facets, but they no longer are subsets of A ; they are filters. For the references see [Bruhat-Tits-72], [Parreau-00] or [Rousseau-77].

5.1 Filters :

A *filter* in a set X is a set F of subsets of X such that :

- if a subset P' of X contains a $P \in F$, then $P' \in F$,
- if $P, P' \in F$ then $P \cap P' \in F$.

Example : If $Z \subset X$, the set $F(Z)$ of subsets of X containing Z is a filter (usually identified with Z).

If $X \subset Y$, to any filter F in X is associated the filter F_Y in Y consisting of all subsets of Y containing a $P \in F$. One usually makes no difference between F and F_Y .

A filter F is said contained in another filter F' (resp. in a $Z \subset X$) if and only if any set in F' (resp. if Z) is in F.

If X is a topological space, the *closure* of a filter F in X is the filter $\overline{F}$ consisting of all subsets of X containing the closure of an element of F. If X is an apartment, the *enclosure* of F is the filter $cl(F)$ consisting of all subsets of X containing the enclosure of an element of F.

A bijection of the set X fixes (pointwise) a filter F in X if and only if it fixes pointwise a Q in F .

5.2 Facets :

A *facet* F in an apartment A is associated to a point $x \in A$ and a vectorial facet F^v in V . More precisely a subset Q of A is an element of the facet $F(x, F^v)$ if and only if it contains a finite intersection of open-half-apartments or walls containing $\Omega \cap (x + F^v)$ where Ω is an open neighborhood of x in A .

If we erase the words "a finite intersection of open-half-apartments or walls containing" in the definition of $F(x, F^v)$ above, we get the filter $\widetilde{F}(x, F^v)$ which is a facet associated to the group $\widetilde{W}$.

Actually when W is discrete, this definition is still valid : $F(x, F^v)$ is a subset Z of A (more precisely : is $F(Z)$ where Z is a subset of A) ; and this subset Z is a facet in the sense of 2.1 or 4.1 .

When W is dense, the point x is well defined by the facet $F(x, F^v)$, it is called its *center* : $\{x\}$ is the intersection of the sets in the closure $\overline{F}(x, F^v)$. When x is special (*i.e.* W_x isomorphic to W^v), the facet $F(x, F^v)$ determines F^v , but it's no longer true in general.

5.3 Chambers, panels, .. :

There is an order on facets : "F is a face of F' " $\Leftrightarrow$ "F' covers F " $\Leftrightarrow F \leq F' \Leftrightarrow F \subset \overline{F'}$

Any point $x \in A$ is contained in a unique facet $F(x, V_0)$ which is minimal ; x is a *vertex* if and only if $F(x, V_0) = \{x\}$. When W is essential, a special point is a vertex, but the converse is not true : in the (discrete, essential) examples $\widetilde{C}_2$ or $\widetilde{G}_2$ of figure 1, some vertices are not special.

The *dimension* of a facet F is the smallest dimension of an affine space generated by a $Q \in F$. The (unique) such affine space of minimal dimension is the *support* of F .

A *chamber* is a maximal facet or equivalently a facet such that all its elements contain a non empty open subset of A or a facet of dimension $n = dim(A)$.

A *panel* is a facet maximal among facets which are not chambers or equivalently a facet of dimension $n - 1$. Its support H is a wall and all its elements contain a non empty open subset of H.

So the set $\mathcal{F}$ of facets of A, completely determines the set $\mathcal{H}$ of walls. And as the affine structure of A is determined by the distance, all the structure of the apartment A may be deduced from the distance and $\mathcal{F}$.

Definition 5.4 : A metric space A endowed with a set $\mathcal{F}$ of filters in A is called an *euclidean apartment* when there is an isometry from A to an euclidean space A', exchanging the set $\mathcal{F}$ and the set $\mathcal{F}'$ of facets in A' associated to a reflection group W' . One gets then a set $\mathcal{H}_A$ of walls in A and a group $W(A)$ acting on A.

Any automorphism of $(A, \mathcal{F})$ (*e.g.* an element of W) fixing a chamber is the identity. But the action on chambers may be non transitive when W is non discrete, as we can see on the following example :

Example 5.5 : We take $A = \mathbb{R}$, $W = W^v \ltimes T_{\mathbb{Q}}$ where W^v is $\{Id, -Id\}$ and $T_{\mathbb{Q}}$ is the group of translations by rational numbers. The walls are the points in $\mathbb{Q}$, so there are 3 kinds of facets :

- panel = wall = point in $\mathbb{Q}$,
- chamber with center x in $\mathbb{R} \setminus \mathbb{Q}$; such a chamber is the filter of neighborhoods of x ,
- chamber with center x in $\mathbb{Q}$; such a chamber is determined by x and a sign :

$$Q \in F(x,+) \ \text{(resp. } F(x,-)\text{)} \ \Leftrightarrow \ \exists \eta > 0 \text{ such that } Q \supset (x, x+\eta) \ \text{ (resp. } (x-\eta, x)\text{)}$$

Part II : EUCLIDEAN BUILDINGS

I try to give a self contained exposition, at least till 9.3.

§ 6 Definitions and general properties :

Definition 6.1

An *euclidean building* is a triple $(\mathcal{I}, \mathcal{F}, \mathcal{A})$ where $\mathcal{I}$ is a set, $\mathcal{F}$ a set of filters in $\mathcal{I}$ called *facets* and $\mathcal{A}$ a set of subsets A of $\mathcal{I}$, each endowed with a distance d_A and called *apartment.* These data verify the following axioms :

(I0) Each apartment (A, d_A) endowed with the set $\mathcal{F}_A$ of facets included in A is an euclidean apartment.

(I1) For any two facets F and F' there is an apartment A containing $\overline{F}$ and $\overline{F}'$.

(I2) If A and A' are apartments, their intersection is an union of facets and, for any facets F , F' in $A \cap A'$ there is an isomorphism from A to A' fixing (pointwise) $\overline{F}$ and $\overline{F}'$.

N.B. : a) An isomorphism from A to A' is an isometry exchanging $\mathcal{F}_A$ and $\mathcal{F}_{A'}$.

b) A subset Q of A is an union of facets if and only if $\forall x \in Q$, $\exists F$ (a facet) such that $\{x\} \subset F \subset Q$ or equivalently if and only if $\forall x \in Q$, $F(x, V_0) \subset Q$.

Examples 6.2 :

a) An apartment is a *thin* building : any panel is a face of exactly 2 chambers. figure 3

b) If I is a set let $\mathcal{I} = (\mathbb{R}^+ \times I)$ with all $(0,i)$ identified (to 0) , $\mathcal{F} = \{ \{0\}, \mathbb{R}_*^+ \times \{i\} \, for \, i \in I\}$, $\mathcal{A} = \{ \{0\} \cup \mathbb{R}_*^+ \times \{i\} \cup \mathbb{R}_*^+ \times \{j\} \, for \, i \neq j \in I\}$.

c) A tree (with no endpoint) is an euclidean building ; the facets are : the vertices (= panels =walls) and the (open) edges (= chambers) ; the apartments are the doubly infinite geodesics (with group $W = \{\pm Id\} \ltimes \mathbb{Z}$). See *e.g.* figure 4.

d) A real tree is still a building. The apartments are still the doubly infinite geodesics (with group $W = \{\pm Id\} \ltimes \mathbb{R}$). As the tree branches at each point, each point is a panel or wall.

Remarks 6.3 :

a) By (I1) and (I2) an apartment contains all faces of its facets ; a facet is a panel or a chamber in an apartment if and only if it is so in any apartment containing it, so it is called a panel or a chamber in $\mathcal{I}$.

The building $\mathcal{I}$ is called *thick* if any panel in $\mathcal{I}$ is covered by at least 3 chambers.

b) By (I2) the distance $d_A(x,y)$ doesn't depend of the apartment A containing x and y ; we write $d(x,y) = d_A(x,y)$.

c) Two apartments of $\mathcal{I}$ are isomorphic (choose a third apartment containing a chamber in each of them). One often choose an apartment $\mathbb{A}$ with group W and tell that $\mathcal{I}$ is *of type* $\mathbb{A}$ or W . All qualifications given to $\mathbb{A}$ are also given to $\mathcal{I}$, *e.g.* of spherical type, of dense affine type, irreducible, essential ...

d) If $\mathcal{I}$ is a building of type W, there is a new set $\widetilde{\mathcal{F}}$ of new facets (defined under the name $\widetilde{F}$ in 5.2) such that $(\mathcal{I}, \widetilde{\mathcal{F}}, \mathcal{A})$ is a building of type $\widetilde{W}$. This new building (not very different from the preceding one) is not thick if $W \neq \widetilde{W}$. The set $\mathcal{A}$ of apartments cannot determine $\mathcal{F}$ or W when $\mathcal{I}$ is not thick.

6.4 Retractions : Let C be a chamber in an apartment A of $\mathcal{I}$.

For $x \in \mathcal{I}$, choose an apartment A' containing C and x . Then there is an isomorphism $\varphi : A' \to A$ fixing (pointwise) $\overline{C}$. If φ and φ' are 2 such isomorphisms $\varphi^{-1}\varphi'$ is an automorphism of A' fixing pointwise the chamber C, hence it's the identity : φ is unique. Moreover by (I2) $\varphi(x)$ doesn't depend of the choice of A' . So one may define : $\rho_{A,C}(x) = \varphi(x)$.

The map $\rho = \rho_{A,C} : \mathcal{I} \to A$ is a retraction of $\mathcal{I}$ onto A. It depends only on A and C and is called *the retraction of* $\mathcal{I}$ *onto* A *of center* C. Any point x in $\mathcal{I}$ such that $\{\rho(x)\} \subset \overline{C}$ is equal to $\rho(x)$.

Proposition 6.5 : a) *The function $d : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$ is a distance.*

b) *The retraction $\rho = \rho_{A,C}$ is distance decreasing, i.e. $d(\rho(x), \rho(y)) \leq d(x, y)$, $\forall x, y \in \mathcal{I}$; equality holds when x , y and C are in a same apartment (e.g. when x is in $\overline{C}$).*

c) *If x and y are in the apartment A, then the segment between x and y in A is :*

$$[x, y] = \{z \in \mathcal{I} \,/\, d(x, y) = d(x, z) + d(z, y)\} .$$

d) *For any apartments A and A' , the intersection $A \cap A'$ is closed convex in A or A' and there is an isomorphism from A to A' fixing pointwise $A \cap A'$.*

e) *Given $x, y \in \mathcal{I}$ and $t \in [0, 1]$, let $m_t = (1 - t)x + ty$ denote the point in $[x, y]$ such that $d(x, m_t) = td(x, y)$. Then the function $\mathcal{I} \times \mathcal{I} \times [0, 1] \to \mathcal{I}$, $(x, y, t) \mapsto m_t$ is a continuous map. In particular the metric space $\mathcal{I}$ is contractible.*

f) *Moreover, $\forall z \in \mathcal{I}$, $d^2(z, m_t) \leq (1 - t)d^2(z, x) + td^2(z, y) - t(1 - t)d^2(x, y)$.*

g) *When $\mathcal{I}$ is discrete, it is a complete metric space.*

Proof :

b) When x, y and C are in the same apartment, one can choose the same φ to define $\rho(x)$ and $\rho(y)$, and φ is an isometry ; so $d(\rho(x), \rho(y)) = d(x, y)$. In general, consider the segment $[x, y]$ in an apartment A' ; for all $z \in [x, y]$ such that $z \neq y$ (resp. $z \neq x$), let F^v be the vectorial facet containing the vector $y - x$ (resp. $x - y$). There is an apartment A''_z (resp. A'''_z) containing C and $F(z, F^v)$, so it contains C and $[z, z']$ for some $z' \in (z, y]$ (resp. $[z'', z]$ for some $z'' \in [x, z)$). But $(z'', z] \cup [z, z')$ is an open neighborhood of z in $[x, y]$ which is compact ; so there is a subdivision $x_0 = x, \ldots, x_n = y$ of $[x, y]$ such that, for all i, C and $[x_{i-1}, x_i]$ are in a same apartment, hence ρ is an isometry on each $[x_{i-1}, x_i]$. Then : $d(x, y) = \sum_{i=1}^n d(x_{i-1}, x_i) = \sum_{i=1}^n d(\rho x_{i-1}, \rho x_i) \geq d(\rho x, \rho y)$.

a) It remains to prove the triangle inequality : $d(x, y) \leq d(x, z) + d(z, y)$. Choose an apartment A containing x, y and any chamber C in A ; if $\rho = \rho_{A,C}$ one has $\rho x = x$, $\rho y = y$ and $d(x, y) = d(\rho x, \rho y) \leq d(\rho x, \rho z) + d(\rho z, \rho y) \leq d(x, z) + d(z, y)$.

c) Suppose $d(x, y) = d(x, z) + d(z, y)$ then $d(\rho x, \rho y) = d(\rho x, \rho z) + d(\rho z, \rho y)$ in A. So $\rho z \in [x, y]$ is the point at distance $d(x, z)$ from x. Suppose the chamber C chosen containing ρz, then (last line of 6.4) $z = \rho z \in [x, y]$.

d) By c) $A \cap A'$ is convex. As an apartment is a complete metric space, it is closed in $\mathcal{I}$, so $A \cap A'$ is closed in $\mathcal{I}$, A or A'. Let F be a facet of A contained in $A \cap A'$ and of maximal dimension ; as $A \cap A'$ is convex in A it is clear that $A \cap A'$ is in the support of F, so F contains an open subset of $A \cap A'$. Now for all facet F' in $A \cap A'$, consider the isometry $\varphi_{F'}$ from A to A' which is the identity on $\overline{F}$ and $\overline{F}'$ (axiom I2). So $\varphi_{F'}$ restricted to $A \cap A'$ is determined by its restriction to F (= identity), hence independent of F' ; this proves that $\varphi_{F'}$ restricted to $A \cap A'$ is the identity.

f) Choose an apartment $A \ni x, y$ and a chamber C in A containing m_t, then by b) and an easy calculus in the euclidean space A one has : $d(z, m_t)^2 = d(\rho z, m_t)^2 = (1-t)d(\rho z, x)^2 + td(\rho z, y)^2 - t(1-t)d(x, y)^2 \leq (1-t)d(z, x)^2 + td(z, y)^2 - t(1-t)d(x, y)^2$.

e) Apply f) to $z = (1-t')x' + t'y'$ for (x', y', t') close to (x, y, t). Then $(1-t)d^2(z, x) \approx (1-t')d^2(z, x')$, $td^2(z, y) \approx t'd^2(z, y')$ and $t(1-t)d^2(x, y) \approx t'(1-t')d^2(x', y')$. So one has $d(z, m_t)^2 \leq (1-t)d(z, x)^2 + td(z, y)^2 - t(1-t)d(x, y)^2 \approx (1-t')d(z, x')^2 + t'd(z, y')^2 - t'(1-t')d(x', y')^2 = 0$.

g) Suppose $\mathcal{I}$ discrete and fix a chamber C. There is a retraction λ of $\mathcal{I}$ onto $\overline{C}$: for $x \in \mathcal{I}$ and $\{x\} \cup C \subset A$, there is a $w \in W_A$ such that $wx \in \overline{C}$; the point $\lambda x = wx$ is independent of the choices (2.3e). For all chamber C' this retraction λ is an isometry from $\overline{C}'$ onto $\overline{C}$, so λ is distance decreasing (same proof as in a) above).

Let x_m be a Cauchy sequence in $\mathcal{I}$, then $\lambda(x_m)$ is a Cauchy sequence in $\overline{C} \subset A$. So there is a point $y \in \overline{C}$ such that $\lambda(x_m) \to y$. For each m choose a chamber C_m with $x_m \in \overline{C}_m$ and let y_m be the point in $\overline{C}_m$ such that $\lambda(y_m) = y$. Then $d(x_m, y_m) = d(\lambda x_m, \lambda y_m) \to 0$. Hence y_m is a Cauchy sequence and if y_m has a limit, x_m has the same limit. But by lemma 4.6 y_m has to be stationary, so the result follows. □

Remarks 6.6 : a) 6.5.c means that $[x, y]$ is the only geodesic between x and y and is independent from the apartment. A subset of $\mathcal{I}$ is called *convex* if it contains $[x, y]$ whenever it contains x and y.

b) The inequality in 6.5.f tells that a triangle in an euclidean building is more pinched than in an euclidean space (as we have seen that equality holds in an euclidean space). This is a typical property of negative curvature (called property CAT(0)), *cf.* [Maubon ; 4.4].

One often uses only the special case $t = \frac{1}{2}$ called property (CN) in [Bruhat-Tits-72 ; 3.2.1] :

(CN) If m is the middle of $[x, y]$ and $z \in \mathcal{I}$ then $d(z, x)^2 + d(z, y)^2 \geq 2d(m, z)^2 + \frac{1}{2}d(x, y)^2$.

6.7 Decomposition of buildings :

Let $\mathcal{I}$ be a building of type $\mathbb{A} = (A, W)$, we have explained in 3.2 the decomposition $\mathbb{A} = (A_0, \{Id\}) \times (A_1, W_1) \times ... \times (A_s, W_s)$ of $\mathbb{A}$ in a trivial factor and irreducible essential factors. A. Parreau [00] proved that there is a corresponding orthogonal decomposition of the building (as a metric space with set of facets) : $\mathcal{I} = \mathcal{I}_0 \times \mathcal{I}_1 \times ... \times \mathcal{I}_s$. Here $\mathcal{I}_0$ is trivial (a trivial apartment) and each $\mathcal{I}_j$ is an irreducible essential building of type either spherical or discrete affine or dense affine. One should pay attention to the fact that this decomposition may induce no good correspondence between the sets of apartments (except when condition (CO) of 9.3 below is verified [Parreau ; lemme 2.2]). The *essential quotient* $\mathcal{I}^e$ of $\mathcal{I}$ is always well defined (including the set of apartments) as $\mathcal{I}^e = \mathcal{I}/V_0$ and one has : $\mathcal{I}^e = \mathcal{I}_1 \times ... \times \mathcal{I}_s$.

When $\mathcal{I}$ is of spherical type (*i.e.* essential and W finite), there is a canonical facet, the unique minimal one, $\{0\}$ where 0 is in each apartment the unique point (= vertex) fixed under W ; $\mathcal{I}$ is then called a *vectorial building.* In this case one prefers to consider the space $\mathcal{I}^s = \{x \in \mathcal{I} \;/\; d(x, 0) = 1\}$, with its facets (intersections with $\mathcal{I}^s$ of facets of $\mathcal{I}$) and its apartments (= unit spheres of centre 0 of apartments of $\mathcal{I}$). One obtains this way a metrical simplicial complex which satisfy axioms analogous to axioms (I0) , (I1) and (I2) ; it is called a *spherical building.* Actually $\mathcal{I}$ is a "cone over $\mathcal{I}^s$ " , so these 2 buildings give the same information ; but $\mathcal{I}^s$ has more interesting topological properties.

When $\mathcal{I}$ is thick, irreducible, of spherical type and verify the technical condition "Moufang" (always true in rank ≥ 3), the Weyl group $W = W^v$ has to be crystallographic (*cf.* [Tits-74] or [Ronan-89 ; 6.2, App. 1, 2]). When $\mathcal{I}$ is of irreducible discrete affine type W^v is crystallographic (4.1) and this is actually still true for thick irreducible buildings of dense affine type in rank ≥ 4, as a consequence of 9.2 to 9.6 below.

In figure 5 is drawn an irreducible spherical building of rank 2 with circles cut in 6 edges (= chambers) as apartments.

6.8 Automorphisms of a building :

An *automorphism* of the building $(\mathcal{I}, \mathcal{F}, \mathcal{A})$ is an isometry φ of $\mathcal{I}$ which transforms a facet or apartment in a facet or apartment. This automorphism is said *type-preserving* (resp. *strongly type-preserving*) if and only if for each facet F contained in an apartment A and each $w \in W(A)$ such that $\varphi(F) \subset A$ and $w\varphi(F) = F$, then $w\varphi$ stabilizes each face of F (resp. fixes pointwise $\overline{F}$) (see § 8 for a justification of the terminology).

A group G of automorphisms of $\mathcal{I}$ is said *strongly transitive* if it acts transitively on $\mathcal{A}$, and $\forall A \in \mathcal{A}$ two chambers of A are conjugated by $W(A)$ if and only if they are conjugated by the stabilizer $N_G(A)$ of A in G and if and only if they are conjugated by G (as chambers in $\mathcal{I}$). When moreover G is strongly type-preserving, one has $W(A) \simeq N_G(A)/C_G(A)$, where $C_G(A)$ is the (pointwise) fixator of A in G.

When G is a strongly transitive strongly type-preserving group of automorphisms of $\mathcal{I}$, the fixators (= stabilizers) $G_F = N_G(F) = C_G(F)$ of the facets are called *parabolic subgroups* of G .

Proposition 6.9 : *In the above situation, let F and F' be 2 facets of the apartment A, then :*

a) The parabolic group G_F is transitive on the apartments containing F .

b) $G = G_F.N_G(A).G_{F'}$.

c) If F and F' are chambers, G is the disjoint union of the sets $G_F.w.G_{F'}$ for $w \in W$.

N.B. : The equality in b) is known as the *Bruhat decomposition* and c) means uniqueness in it.

Proof : a) If the apartments A and A' contain F, choose chambers $C \subset A$ and $C' \subset A'$ covering F. If A'' is an apartment containing C and C' , then there exists $n \in G_F \cap N_G(A'')$ such that $C' = nC$ (prop. 2.3e). So one may suppose $C' = C$. There exists $g \in G$ such that $A' = gA$. Now C and gC are in A', hence there exists $n' \in N_G(A')$ with $n'gC = C$. Therefore $n'g \in G_C$ (as the group is strongly type preserving) and $A' = n'gA$.

b) Let $g \in G$. There exists an apartment A' containing $g^{-1}F$ and F'. By a), there exists $g_1 \in G_{F'}$ such that $A = g_1A'$. As G is strongly transitive and type preserving, the facets F and $g_1g^{-1}F$ of A are conjugated by an $n \in N_G(A)$. But the group is strongly type preserving, so $ng_1g^{-1} \in G_F$ and $g \in (G_F)ng_1 \subset G_FN_G(A)G_{F'}$.

c) Let $n, n' \in N_G(A)$ such that $gnG_{F'} = n'G_{F'}$ with $g \in G_F$. Then $n'F'$ and nF' are chambers in A conjugated by $g \in G_F$. Using $\rho_{A,F}$ one gets $n'F' = nF'$, so n' and n have the same class in $W(A)$. □

§ 7 Metrical properties of buildings :

Most of the following properties of a building $\mathcal{I}$ are true only when $\mathcal{I}$ is complete as a metric space, *e.g.* when $\mathcal{I}$ is discrete. Actually this is still true for some dense Bruhat-Tits buildings.

Proposition 7.1 : Let G be a group of isometries of a complete metric space X with the property (CN) of 6.6. If G stabilizes a non empty bounded subset of X , then G has a fixed point.

Remark : The group G stabilizes a non empty bounded subset of X if and only if for some (any) $x \in X$, Gx is bounded ; then G is called *bounded*.

Proof : There is a proof by Bruhat and Tits [72], but we follow here Serre's proof [Brown-89 ; VI 4].

For any bounded non empty subset $Y \subset X$ and any $x \in X$, let $r(x, Y) = \mathrm{Sup}_{y \in Y}\ d(x, y)$ and $r(Y) = \mathrm{Inf}_{x \in X}\ r(x, Y)$. It's sufficient to prove the following Serre's result :

there exists a unique $z_0 \in X$ such that $r(Y) = r(z_0, Y)$ (this z_0 will be a fixed point).

For $x, y \in X$ and m the middle of $[x, y]$, by taking suprema one gets :

$2r^2(m, Y) \leq r^2(x, Y) + r^2(y, Y) - \frac{1}{2}d^2(x, y)$ hence $d^2(x, y) \leq 2(r^2(x, Y) + r^2(y, Y) - 2r^2(m, Y))$

and $d^2(x, y) \leq 2(r^2(x, Y) + r^2(y, Y) - 2r^2(Y))$; from this last formula one deduces the uniqueness of z_0 if it exists. Moreover if a sequence x_n is such that $r(x_n, Y) \to r(Y)$ then it's a Cauchy sequence and its limit is the wanted point z_0. □

Corollary 7.2 :*Let $\mathcal{I}$ be a complete euclidean building and G a strongly transitive strongly type-preserving group of isometries of $\mathcal{I}$. The following conditions on a subgroup H of G are equivalent :*

(i) H is bounded ,

(ii) H fixes a point in $\mathcal{I}$,

(iii) H is contained in a parabolic subgroup.

Remark : This gives a classification of maximal bounded subgroups of G .

Proof : By proposition 7.1 (i) $\Rightarrow$ (ii). For (ii) $\Rightarrow$ (iii) take the minimal facet containing the fixed point and remember that G is strongly type preserving. As the parabolic group G_F fixes at least one point $x \in \overline{F}$, (iii) $\Rightarrow$ (i). □

Proposition 7.3 : *Let X be a non empty closed convex subset of the complete euclidean building $\mathcal{I}$ and $z \in \mathcal{I}$, then there is a unique point $x \in X$ such that $d(z,x) \leq d(z,y)$, $\forall y \in X$.*

Remark : Actually we use only that $\mathcal{I}$ verify (CN) and X is convex complete, see [Maubon ; prop. 4.8].

Proof : If x and y are 2 solutions and m is the middle of $[x,y] \subset X$, property (CN) gives :

$$2d^2(z,x) = d^2(z,x) + d^2(z,y) \geq 2d^2(z,m) + \tfrac{1}{2}d^2(x,y) \geq 2d^2(z,x) + \tfrac{1}{2}d^2(x,y)$$

hence $d(x,y) = 0$ and $x = y$.

For the existence, let $\delta = \mathrm{Inf}\{\ d(z,x)\ /\ x \in X\}$. For $\varepsilon > 0$, if x and y verify $d(z,x), d(z,y) \leq \delta + \varepsilon$, one has : $2(\delta+\varepsilon)^2 \geq d^2(z,x) + d^2(z,y) \geq 2d^2(z,m) + \frac{1}{2}d^2(x,y) \geq 2\delta^2 + \frac{1}{2}d^2(x,y)$, hence $d^2(x,y) \leq 4\varepsilon^2 + 8\varepsilon\delta$. Now if $x_n \in X$ verify $d(z,x_n) \leq x + \frac{1}{n}$, it's a Cauchy sequence and its limit $x \subset X$ is the wanted point. □

7.4 Comparison with symmetric spaces : If $\mathcal{S}$ is a riemannian symmetric space, one can consider its metric and its collection $\mathcal{A}$ of *maximal flats* (= maximal totally geodesic flat subvarieties, which actually are isometric to euclidean spaces). These data verify the following weakenings of building's axioms :

(I1') For any two points x and y there is an apartment (=maximal flat) A containing x and y .

(I2') If A and A' are apartments, their intersection is closed convex and there is an isomorphism from A to A' fixing $A \cap A'$.

One can even define Weyl chambers (= sectors) in $\mathcal{S}$ [Maubon ; 5.3, 5.4].

The Riemannian symmetric space $\mathcal{S}$ verify also condition (CN) or CAT(0) [Maubon ; 4.4], and proposition 7.1 may be used to give a proof of the conjugacy of maximal compact subgroups of a semi-simple Lie group. Conversely some results on symmetric spaces (*e.g.* [Maubon ; lemma 4.3]) are also true in euclidean buildings, *cf.* [Rousseau-01].

§ 8 Discrete Euclidean buildings :

Discrete buildings are the better known buildings ; actually they were the only ones until [Bruhat-Tits-72]. Often one is essentially interested in the combinatorial structure of the ordered set $\mathcal{F}$ of facets. There are more general combinatorial discrete buildings than the euclidean ones (*e.g.* hyperbolic buildings), see *e.g.* [Rémy].

8.1 Type :

Let C be a chamber and F_1, ...,F_p be its panels. If F is a facet and A an apartment containing F and C, then the type $\tau(F) \subset \{1,...,p\}$ is defined in A (4.2), it doesn't depend on the choice of A by axiom (I2).

This type is the type "viewed from C", to see that it doesn't really depend on C we have to prove that for each apartment A', τ is invariant by $W(A')$. As 2 chambers in A' are connected by a gallery, we have to prove that, if $D \neq D'$ are adjacent chambers in A' with common panel P and r_P is the reflection in A' with respect to the wall supporting P, then $\tau(F) = \tau(r_P F)$ for all face F of D. But r_p restricted to $\overline{D}$ is the only isometry from $\overline{D}$ to $\overline{D}'$ fixing pointwise $\overline{P}$. So if D , D' and C are in a same apartment A, then the restriction of r_P to $\overline{D}$ is the same in A' and in A and the relation is clear. Otherwise, if A is an apartment containing C and D', then r_P and $\rho_{C,A}$ coincide on $\overline{D}$ and the relation is still clear.

So we have constructed a map $\tau : \mathcal{F} \to \mathcal{P}(\{1, ..., p\})$, which, in any apartment A, is invariant by $W(A)$. It is clear that an automorphism of $\mathcal{I}$ leaving τ invariant is type-preserving in the sense of 6.8 ; if moreover $\mathcal{I}$ is essential, it is even strongly type-preserving (as then any isometry of a closed facet $\overline{F}$ which stabilizes any face of F is the identity).

8.2 Galleries : They are defined in a discrete building as in a discrete apartment (*cf.* 4.4). It is clear that the image of a gallery by any retraction $\rho_{A,C}$ is a gallery of the same type.

Proposition : *Let C and D be 2 chambers in $\mathcal{I}$ and $\Gamma = (C_0 = C, ..., C_d = D)$ a minimal gallery from C to D in $\mathcal{I}$. Then, for any apartment A containing C and D, Γ is contained in A and is a minimal gallery in A from C to D ; in particular $\Gamma \subset cl(C, D)$.*

Proof : By 4.4 we only have to prove that $\Gamma \subset A$. If not let $m = \mathrm{Inf}\{ \ p \ / \ C_p \not\subset A\} \geq 1$. Then C_{p-1} and the panel $F \subset \overline{C}_p \cap \overline{C}_{p-1}$ are in A ; let $C' \neq C$ be the chamber in A covering F and $\rho = \rho_{A,C'}$. Then $\rho(C_p) = C_{p-1}$ and $\rho(\Gamma)$ is a stuttering gallery from C to D in A and of length n. So there is in A a gallery from C to D of length $n - 1$, contrary to the hypotheses. □

8.3 Strongly transitive strongly type-preserving automorphism groups :

Let G be a strongly type-preserving automorphism group of $\mathcal{I}$. The group G is strongly transitive if and only if it acts transitively on the pairs (C, A) where C is a chamber in the apartment A ; as $W(A)$ is transitive on the chambers of A, this is the same definition as in 6.8.

If (C, A) is such a pair we define $B = N_G(C) = C_G(C)$ the stabilizer or fixator of C and $N = N_G(A)$ the stabilizer of A. As C is a non empty open set in A , $B \cap N$ is the fixator $C_G(A)$ of A , $B \cap N \lhd N$ and $N/B \cap N \simeq W(A)$. By 6.9 one has the Bruhat decomposition $G = BNB = \coprod_{w \in W} \ BwB$.

Remarks 8.4 : a) The set $\mathcal{C}$ of chambers in $\mathcal{I}$ is G/B. The Bruhat decomposition allows us to define a function $\delta : \mathcal{C} \times \mathcal{C} \to W : \delta(gB, hB)$ is the $w \in W$ such that $g^{-1}h \in BwB$. This function is a "$W-$distance", its properties are explained in [Ronan-89] or [Rémy-02]. Actually this is the starting point of the new definition of (discrete) combinatorial buildings given by Tits [86b] and chosen by Ronan.

b) The Bruhat decomposition in G is better understood using the following definition :

Definition 8.5 : A *Tits system* is a triple (G, B, N) where B and N are subgroups of a group G satisfying the following axioms :

(T1) $B \cup N$ generates G and $B \cap N$ is normal in N ,

(T2) the Weyl group $W = N/B \cap N$ is generated by a system S of elements of order 2 ,

(T3) $\forall s \in S$ and $\forall w \in W$, $wBs \ \subset \ BwB \cup BwsB$,

(T4) $\forall s \in S$, $sBs \not\subset B$.

This system is said *saturated* if and only if $\quad \bigcap_{w \in W} \ wBw^{-1} \ = B \cap N$.

Proposition 8.6 : *In the situation of 8.3 suppose moreover $\mathcal{I}$ thick, then (G, B, N) is a saturated Tits system of Weyl group W .*

Proof : (T1) is a consequence of Bruhat decomposition (6.9). The group W is a discrete reflection group, so (T2) is clear with $S = \{r_1, \ldots, r_p\}$ (2.3b and 4.1). Let $g \in BsB$, then gC is a chamber in $\mathcal{I}$ adjacent to C along the panel P corresponding to s ; so wgC is a chamber adjacent to wC along a panel of type $\tau(P)$. If $\rho = \rho_{A,C}$ then $\rho(wgC)$ is a chamber of A adjacent to wC along a panel of type $\tau(P)$; this means that $\rho(wgC) = wC$ or wsC, hence $wg \in BwB \cup BwsB$ and (T3) is proved. The group sBs is the fixator of the chamber sC. But, as $\mathcal{I}$ is thick, there is a third chamber C' in $\mathcal{I}$ covering $\overline{C} \cap s\overline{C} \supset P$ and $b' \in sBs = G_{sC}$ such that $b'C = C'$ (6.9a) ; so $b' \notin B$ and $sBs \not\subset B$, now (T4) is proved. We saw in 8.3 that $B \cap N$ (resp. B) is the fixator of A (resp. C), so $B \cap N = \bigcap_{w \in W} \ wBw^{-1}$ and the system is saturated. □

8.7 Conversely let (G, B, N) be a Tits system such that its Weyl group W is a discrete reflection group (in an affine space $\mathbb{A}$) ; more precisely one asks that the set S of generators of W is the set of reflections with respect to panels of a chamber C in $\mathbb{A}$.

The type of a facet in $\mathbb{A}$ may be seen as a subset of S. If F is a face of C let W_F be the subgroup of W generated by the $s \in \tau(F)$; this is the fixator of F in W (*cf.* 2.3.e generalized in 4.1). One then defines the parabolic group $P(F) = BW_FB$; more generally for any facet in $\mathbb{A}$: $P(wF) = wP(F)w^{-1}$.

Proposition : *a) One has an equivalence relation on $G \times \mathbb{A}$ defined by :*
$(g, x) \sim (g', x') \Leftrightarrow \exists n \in N$ with $x' = nx$ and $g^{-1}g'n \in P(F(x))$ where $F(x)$ is the facet containing x .

Let $\pi : G \times \mathbb{A} \to \mathcal{I}$ be the quotient map.

b) The action of G on $G \times \mathbb{A}$ (by left action on G) gives an action of G on $\mathcal{I}$

c) The map $\varphi : \mathbb{A} \to \mathcal{I}$, $\varphi(x) = \pi(1, x)$ is an injection.

d) The stabilizer or fixator in G of $\varphi(F)$, F a facet in $\mathbb{A}$ is $P(F)$. The fixator of $\varphi(\mathbb{A})$ in G is $T = \bigcap_{w\in W} wBw^{-1} \supset B \cap N$ and the stabilizer of $\varphi(\mathbb{A})$ is TN .

e) Define an apartment in $\mathcal{I}$ as a subset $g.\varphi(\mathbb{A})$ (with its structure of euclidean space) and a facet as a subset $g.\varphi(F)$ where F is a facet in $\mathbb{A}$. Let $\mathcal{A}$ (resp. $\mathcal{F}$) be the set of apartments (resp. facets), then $(\mathcal{I}, \mathcal{F}, \mathcal{A})$ is a thick euclidean building of type W .

f) The action of G on $\mathcal{I}$ induces a strongly transitive strongly type-preserving automorphism group.

Remarks : 1) If the Tits system (G, B, N) is saturated, then it coincides with the Tits system associated to $(\mathcal{I}, G)$ in 8.5.

2) As a consequence of d) and e), the set of facets of type $\tau(F)$ may be identified with $G/P(F)$.

3) In the following proof I use freely that Bruhat decomposition $G = BNB = \coprod_{w\in W} BwB$ is valid in any Tits system and some other classical results for Tits systems proved *e.g.* in [Bourbaki ; IV n° 2.3] or [Brown-89 ;V 2].

Proof : It's proved in the above references that $P(F)$ is really a group. As $P(F(nx)) = nP(F)n^{-1}$ when $n \in N$ and $x \in \mathbb{A}$, it's easy to prove that the relation in a) is an equivalence relation. So a) and also b) are clear. If $(1, x) \sim (1, x')$ there exists $n \in N$ such that $x' = nx$ and $n \in P(F(x))$; but if F is a face of C it's a classical result that $N \cap P(F) = W_F.(B \cap N)$. So for any x, $N \cap P(F(x))$ fixes x, hence $x' = x$ and c) is proved.

By definition it's clear that $P(F)$ is the fixator of (any point in) $\varphi(F)$; but, classically, $P(F)$ is equal to its normalizer in G, so $P(F)$ is also the stabilizer of $\varphi(F)$. Clearly T is the fixator of $\varphi(\mathbb{A})$, so TN stabilizes $\varphi(\mathbb{A})$. Conversely if g is in the stabilizer of $\varphi(\mathbb{A})$, $g\varphi(C)$ is a chamber in $\mathbb{A}$, so $\exists n \in N$ such that $ng \in P(C) = B$ and ng stabilizes $\varphi(\mathbb{A})$. Hence $\forall w \in W$, $\exists w' \in W$ such that $ngw\varphi(C) = w'\varphi(C)$, so $w'^{-1}ngw \in B$ and $w \in Bw'B$ therefore $w = w'$ (uniqueness in Bruhat decomposition). Now, $\forall w \in W$, $ngw\varphi(C) = w\varphi(C)$, so $ng \in wBw^{-1}$, $ng \in T$ and d) is proved.

By construction G is strongly transitive, it is also strongly type preserving as we saw that the stabilizer of a facet is also its fixator, hence f) is proved. We note $G_F = P(F)$ the fixator of any facet F in $\mathcal{I}$.

Now for e), axiom (I0) is clear. If $g\varphi(F)$ and $g'\varphi(F')$ are facets in $\mathcal{I}$, one may suppose that F and F' are faces of C ; so the Bruhat decomposition gives $G = P(F)NP(F')$. Write $g^{-1}g' = pnp'$ with $p \in P(F)$ $n \in N$ and $p' \in P(F')$, then $g'\varphi(\overline{F'}) = gpn\varphi(\overline{F'})$ and $g\varphi(\overline{F}) = gp\varphi(\overline{F'})$ are in the apartment $gp\varphi(\mathbb{A})$; so (I1) is verified.

For (I2) one may suppose the apartments are $A' = \varphi(\mathbb{A})$ and $A'' = g\varphi(\mathbb{A})$. By definition, if $\varphi(x) \in \varphi(\mathbb{A})$ is in $A'' = g\varphi(\mathbb{A})$, then $gn \in P(F(x))$ for an $n \in N$, so $A' \cap A'' \supset \varphi(\overline{F(x)})$ and $A' \cap A''$ is an union of closed facets. Let F' and F'' be facets in $A' \cap A''$, choose chambers C' in A' covering F' and C'' in A'' covering F'' ; one may suppose $C' = \varphi(C)$. Let A_1 be an apartment containing $\overline{C}'$ and $\overline{C}''$; by strong transitivity $\exists g' \in G_{C'} = B$ and $g'' \in G_{C''}$ such that $A_1 = g'A' = g''A''$. We are reduced to prove that g' is the identity on $\overline{F}''$ (and the corresponding assertion for g''). One has $g'^{-1}F'' \subset \varphi(\mathbb{A})$ and F'' or $g'^{-1}F''$ have the same type ; so $\exists n_1, n_2 \in N$ such that $\varphi(F_1) = n_1F'' = n_2g'^{-1}F'' \subset \varphi(\overline{C})$, hence $n_2g'^{-1}n_1^{-1} \in P(F_1)$ and $g'^{-1}n_1^{-1} \in n_2^{-1}BW_{F_1}B$. But this last group is included in $Bn_2^{-1}W_{F_1}B$ [Bourbaki ; IV 2.5 prop.2], hence

$Bn_1^{-1}B \subset Bn_2^{-1}W_{F_1}B$. By uniqueness in Bruhat decomposition, the class of $n_2n_1^{-1}$ in W is in W_{F_1} ; so $F'' = n_1^{-1}\varphi(F_1) = n_2^{-1}\varphi(F_1) = g'^{-1}F''$, hence $g' \in P(F'')$ is the identity on $\overline{F}''$.

Any panel in $\mathcal{I}$ is conjugated to the panel P separating $\varphi(C)$ and $\varphi(sC)$ (with $s \in S$). If $g \in sBs \setminus B$, then $g\varphi(C) \neq \varphi(C)$ and $g\varphi(C) \neq \varphi(sC) \neq \varphi(C)$; but these 3 chambers share the same panel P, so $\mathcal{I}$ is thick. □

§ 9 Apartments and points at infinity : $\mathcal{I}$ is an euclidean building (discrete or not).

Proposition 9.1 : *Let A and A' be apartments, then $A \cap A'$ is enclosed in A (or A'), more precisely it is the intersection of a finite number of half-apartments.*

Proof : If $A \cap A' = \{x\}$, then by axiom (I2) $\{x\}$ is a facet and a finite intersection of walls.

1) Suppose now $A \cap A' \supset [x, y]$ with $x \neq y$ (6.5d). Let F^v be the vectorial facet containing $y - x$ and $F = F(x, F^v)$, then $A \cap A' \supset \overline{F} \ni x$:

Let F' be the facet defined as F but in A'. There is an apartment A'' containing $\overline{F}$ and $\overline{F}'$, so it contains $[x, z]$ for some $z \in (x, y]$ and there is an isomorphism from A'' to A (or A') fixing $[x, z]$. This isomorphism exchanges the facets ; so the definition of F (or F') in A'' is the same as in A (or A'), as it involves only $[x, z]$ we have $F = F'$ and $\overline{F} = \overline{F}'$, so $\overline{F} \subset A \cap A'$.

2) Let $x \in A \cap A'$. As $A \cap A'$ is convex the union of the facets $\overline{F^v}$ defined as in 1) is a convex cone ; so $A \cap A'$ contains the intersection of a neighborhood of x with a finite number of half-apartments, each containing $A \cap A'$ and with boundary a wall containing x. In particular any point in $A \cap A'$ is interior to $A \cap A'$ or in a wall boundary of an half-apartment containing $A \cap A'$. As $A \cap A'$ is closed (6.5d) and there is only a finite number of directions of walls, this proves the result. □

9.2 The complete system of apartments :

Any bounded subset Q of an apartment is contained in the enclosure of 2 points x and y : If $z \in Q$, choose x and y such that $y - z = z - x$ is in no vectorial wall and sufficiently far from these walls.

So if $\mathcal{A}' \supset \mathcal{A}$ is another system of apartments for $\mathcal{I}$ (*i.e.* axioms I0 , I1 and I2 are verified for $(\mathcal{I}, \mathcal{F}, \mathcal{A}')$) and $A' \in \mathcal{A}'$, then every bounded subset Q' of A' is contained in the enclosure of $x, y \in A'$ and there is an apartment $A \in \mathcal{A}$ containing x and y. Hence $Q' \subset cl(x, y) \subset A \cap A' \subset A$. Therefore A' is an increasing union of bounded subsets of apartments in $\mathcal{A}$.

Conversely let A' be a subset of $\mathcal{I}$ which is isometric to $\mathbb{A}$ (6.3c) and a union of bounded subsets of apartments in $\mathcal{A}$; more precisely one requires an isometry $\varphi : A' \to \mathbb{A}$ such that $\forall Q$ bounded in $\mathbb{A}$, $\varphi^{-1}(Q)$ is in an apartment of $\mathcal{A}$. Then it is clear that $\mathcal{A} \cup \{A'\}$ is a system of apartments (as axiom (I2) involves only bounded subsets of apartments).

Hence there is a maximal system of apartments $\mathcal{A}^m$ consisting of all A' verifying the above condition.

N.B. : This system $\mathcal{A}^m$ is also called the *complete system of apartments.*

Actually, at least when $\mathcal{I}$ is discrete, any subset A' of $\mathcal{I}$ isometric to an euclidean space is in an apartment of $\mathcal{A}^m$ [Brown-89 ; VI 7]. Moreover if $\mathcal{I}$ is of spherical type $\mathcal{A}$ is already complete : there is a unique system of apartments [Brown-89 ; IV 5].

Proposition 9.3 : *Suppose the apartment system complete, then it verifies the following condition :*

(CO) Two sectors of $\mathcal{I}$ sharing the same base point and "opposite" are in the same apartment.

N.B. : Suppose that sectors S^1 and S^2 share the same base point x. The intersection of S^i with the filter of neighborhoods of x is contained in a chamber C^i ; hence there is an apartment A'' containing a neighborhood U^i of x in S^i , $\forall i = 1, 2$. One says that S^1 and S^2 are *opposite* if and only if $\exists y^i \in U^i$ different from x such that x is the middle of $[y^1, y^2]$.

Proof : Using the above notations, let A^i be an apartment containing the sector S^i. Define $y^i_n \in S_i \subset A^i$ by $y^i \in [x, y^i_n]$ and $d(x, y^i_n) = nd(x, y^i)$. It's clear that S^i is in the increasing union of the enclosures $cl(x, y^i_n)$.

Consider now the retraction $\rho = \rho_{A^1,C^1}$. There is an apartment A''_n containing $\overline{C}^1$ and y^2_n, but $\overline{C}^1 \ni x$ so A''_n contains $[x, y^2_n]$ and $y^1_\varepsilon = (1-\varepsilon)x + \varepsilon y^1$ for some $\varepsilon > 0$. Moreover x is the middle of y^1_ε and $y^2_\varepsilon = (1-\varepsilon)x+\varepsilon y^2$ (calculus in A''), so $[y^1_\varepsilon, y^2_n] = [y^1_\varepsilon, x] \cup [x, y^2_n]$ and $\rho([x, y^2_n])$ is, in A^1, the segment opposite to $[y^1_n, x]$. In particular $d(\rho y^1_n, \rho y^2_n) = d(y^1_n, x) + d(x, y^2_n) = 2nd(x, y^1)$. But $d(y^1_n, y^2_n) \le d(y^1_n, x) + d(x, y^2_n)$, and, as ρ is distance decreasing, we have $d(y^1_n, y^2_n) = d(y^1_n, x) + d(x, y^2_n)$ and $x \in [y^1_n, y^2_n]$.

If A'_n is an apartment containing y^1_n and y^2_n, then A'_n contains x and the enclosure (in A'_n) $cl(y^1_n, y^2_n)$ contains $cl(y^1_n, x) \supset C^1$ and $cl(y^2_n, x)$. So S^1 and S^2 are contained in $A' = \bigcup_{n\in\mathbb{N}} cl(y^1_n, y^2_n)$ which is clearly in $\mathcal{A}^m$ (the isomorphism of A' with A^1 (or $\mathbb{A}$) is given by ρ). □

Proposition 9.4 : a) Condition (CO) is equivalent to the following condition :

(A4) Given 2 sectors S_1 and S_2, there exists sectors $S'_1 \subset S_1$ and $S'_2 \subset S_2$ and an apartment containing S'_1 and S'_2 .

b) If $\mathcal{I}$ verify (CO), then, given a facet F and a sector S, there exists a sector $S' \subset S$ and an apartment containing F and S' .

Consequence : Under condition (CO), for S a sector in an apartment A, there exists a retraction $\rho_{A,S}$ associated to A and (the subsectors of) S. It is defined as in 6.4 using property b) above and is distance-decreasing.

N.B. : 1) For Bruhat-Tits buildings (which always verify (CO)) the results of this proposition are still true for sector-facets instead of sectors : see [Rousseau-77 or 01] where the notion of "cheminée" (chimney) is introduced. Till now I saw no proof of this under the more general hypothesis (CO).

2) The system of apartments $\mathcal{A} \cup \{A'\}$ constructed in 9.2 may not verify (CO) even if $\mathcal{A}$ verify it.

Remark 9.5 : The results of 9.4 are proved by Anne Parreau [00]. Actually she proves the equivalence of several definitions of euclidean buildings, in particular the 2 following ones :

1) euclidean building (for the definition given here) verifying (CO) but with no emphasis on $\mathcal{F}$ and a particular W (*i.e.* replacing $(\mathcal{F}, W)$ by $(\widetilde{\mathcal{F}}, \widetilde{W})$ defined in 6.3 d).

2) a pair $(\mathcal{I}, \mathcal{A})$ verifying (to be short) (I0) , (I1') , (I2') (as in 7.4) , condition (A4) above and a condition (A5') asking the existence of a retraction of $\mathcal{I}$ on an apartment A with center a point $x \in A$.

This last definition is that given by Tits [86a] with later corrections, see [Ronan-89]. It is particularly suited for the classification purposes of [Tits-86a] and was the basis of almost all abstract definitions of (eventually non discrete) euclidean buildings.

The only exception (as far as I know) was a definition by Kleiner and Leeb [97], which Parreau proved to be equivalent to "euclidean building with the complete apartment system". In their definition an euclidean building is a metric space $\mathcal{I}$ with a collection of sub-metric-spaces (which are apartments in the sense of § 1) and the axioms are entirely geometric, *e.g.* : $\mathcal{I}$ is CAT(0), there exists geodesics in $\mathcal{I}$ and any geodesic or geodesic ray is contained in an apartment, ...

9.6 The spherical building at infinity : *cf.* [Parreau-00 ; 2.1.9 and 1.5.1]

Suppose that the building $\mathcal{I}$ verifies the condition (CO) (or equivalently A4).

If Q and Q' are subsets of $\mathcal{I}$, one says that Q' is asymptotically greater than Q (resp. asymptotic to Q) if and only if $sup_{x\in Q}\, d(x, Q') < \infty$ (resp. and $sup_{x\in Q'}\, d(x, Q) < \infty$). Asymptoticity is an equivalence relation.

The *spherical building* $\mathcal{I}^\infty$ is the set of asymptotic classes of rays in apartments of $\mathcal{I}$. Its facets are the asymptotic classes of sector-facets, they are ordered by the relation "asymptotatically greater". Its apartments are in bijection with apartments of $\mathcal{I}$; the apartment A^∞ corresponding to $A \in \mathcal{A}$ is the set of asymptotic classes of rays contained in A, it is thus in bijection with the unit sphere of the vector space V of A and this gives him its distance.

This building $\mathcal{I}^\infty$ is spherical in the sense of 6.7 ; it depends strongly on the set $\mathcal{A}$ of apartments. It was introduced by Tits [86a] (see also [Ronan-89]) in a more combinatorial manner and used to classify the affine buildings (irreducible of rank ≥ 4) as one already knew the classification of spherical buildings (irreducible of rank ≥ 3) [Tits-74].

When $\mathcal{I}$ is discrete, locally finite (*i.e.* each facet is covered by a finite number of chambers) and the apartment system is complete, the disjoint union $\mathcal{I} \coprod \mathcal{I}^\infty$ may be endowed with a compact topology. The subset $\mathcal{I}$ is open and with its usual topology, but the topology on $\mathcal{I}^\infty$ is new. This is the Borel-Serre compactification, see [Brown-89 ; VII 2.A].

A similar construction can be made for symmetric spaces, *cf.* [Maubon ; th. 5.10].

9.7 Satake compactification : Suppose still that $\mathcal{I}$ verifies condition (CO).

We used in 9.6 the compactification of an affine space by adding its unit sphere at infinity. There is another compactification of an apartment A involving the group W, at least when W^v is crystallographic. In this case W^v is associated to a root system Φ as in 2.7.

The set A^∞ is the quotient of the set of rays in A by the following equivalence relation :
$\delta \sim \delta' \Leftrightarrow \forall \alpha \in \Phi$ either $\alpha(\delta) = \alpha(\delta')$ is reduced to a point or $\alpha(\delta)$ and $\alpha(\delta')$ are asymptotic rays in $\mathbb{R}$.

The compactification $\overline{A} = A \coprod A^\infty$ is constructed inside the compact set $[-\infty, +\infty]^\Phi$ and it is the closure of the image of the map : $A \to [-\infty, +\infty]^\Phi$, $x \mapsto (\alpha(x))_{\alpha \in \Phi}$,
(actually, as above, one has to choose an origin in A, but it doesn't depend on it).

This compactification is the starting point of the construction of the Satake compactification of an affine building (discrete and locally finite). For the details see [Landvogt-96] or the forthcoming article by Y. Guivarc'h and B. Rémy.

PART III : BRUHAT-TITS BUILDINGS

§ 10 Reductive groups : The references for many results are to [Borel-Tits] and [Bruhat-Tits-72].

10.1 Relative roots : Let $\mathcal{G}$ be a connected reductive algebraic group over a field K. We consider a *maximal split torus* $\mathcal{S}$ in $\mathcal{G}$ (*i.e.* $\mathcal{S}$ is isomorphic to $(K^*)^n$ as an algebraic group and is maximal for this property). We note $X(\mathcal{S})$ (resp. $Y(\mathcal{S})$) the group of characters (resp. cocharacters) of $\mathcal{S}$: $X(\mathcal{S}) = Hom(\mathcal{S}, K^*)$ and $Y(\mathcal{S}) = Hom(K^*, \mathcal{S})$; they are dual free $\mathbb{Z}-$modules of rank $n = dim(\mathcal{S})$, the *relative rank* of $\mathcal{G}$ over K.

The adjoint action of $\mathcal{S}$ on $\mathfrak{g} = Lie(\mathcal{G})$ is diagonalizable : $\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\alpha \in \Phi}\, \mathfrak{g}_\alpha)$, where $\mathfrak{g}_0$ is the fixed point set of $\mathcal{S}$ and $\Phi \subset X(\mathcal{S}) \setminus \{0\}$ is the set on non-zero weights of $\mathcal{S}$ in $\mathfrak{g}$.

Let $V = V(\mathcal{S}) = Y(\mathcal{S}) \otimes \mathbb{R}$; it's a real space of dimension n . It may be endowed with a scalar product and then Φ is a root system in V^* in the sense of 2.6 : it's the *relative root system.*

10.2 Associated subgroups over K : Let $G = \mathcal{G}(K)$.

If $\mathcal{N}(\mathcal{S})$ (resp. $\mathcal{Z}(\mathcal{S})$) is the normalizer (resp. centralizer) of $\mathcal{S}$ in $\mathcal{G}$ and if $N = \mathcal{N}(\mathcal{S})(K)$ and $Z = \mathcal{Z}(\mathcal{S})(K)$ are the corresponding groups of rational points, then N/Z is an automorphism group of $\mathcal{S}$, *i.e.* of $Y(\mathcal{S})$ *i.e.* of V. Actually it's the Weyl group W^v of the root system Φ. Let $\nu^v : N \to W^v$ be the quotient map.

If $\alpha \in \Phi$, then $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ (or $\mathfrak{g}_\alpha$ if $2\alpha \notin \Phi$) is a Lie subalgebra of $\mathfrak{g}$. There is a unique connected algebraic subgroup $\mathcal{U}_\alpha$ of $\mathcal{G}$ associated to this Lie subalgebra ; it is normalized by $\mathcal{S}$. We define $U_\alpha = \mathcal{U}_\alpha(K)$.

A great part of Borel-Tits theory may be summarized (in the Bruhat-Tits manner) by telling that $(G, (U_\alpha)_{\alpha \in \Phi}, Z)$ is a generating root datum, as defined below :

Definition 10.3 : Let Φ be a root system in the dual V^* of a vector space. A *root datum of type* Φ is a triple $(G, (U_\alpha)_{\alpha \in \Phi}, Z)$ satisfying the following axioms :

(RD1) The group Z is a subgroup of G and, for all $\alpha \in \Phi$, U_α is a non trivial subgroup of G, normalized by Z .

(RD2) For each pair of roots $\{\alpha, \beta\}$, the commutator group $[U_\alpha, U_\beta]$ is contained in the group generated by the U_γ for $\gamma = p\alpha + q\beta \in \Phi$ with p and q strictly positive integers.

(RD3) If α and 2α are in Φ , then : $U_{2\alpha} \subset U_\alpha$ and $U_{2\alpha} \neq U_\alpha$.

(RD4) For all $\alpha \in \Phi$ and all $u \in U_\alpha$, $u \neq 1$, there exists u' and u" $\in U_{-\alpha}$ such that $m(u) := u'uu$" conjugates U_β into $U_{r_\alpha(\beta)}$ for all $\beta \in \Phi$.

Moreover, for all $u, v \in U_\alpha \setminus \{1\}$ one asks that $m(u)Z = m(v)Z$.

(RD5) If Φ^+ is some (any) positive root system in Φ and if U^+ (resp. U^-) is the subgroup of G generated by the U_α for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$), then : $ZU^+ \cap U^- = \{1\}$.

This root datum is called *generating* when :

(GRD) The group G is generated by the subgroups Z and U_α for $\alpha \in \Phi$.

N.B. : a) ZU^+ is a minimal parabolic subgroup in G (not necessarily a Borel subgroup).

b) $\forall \alpha \in \Phi$ and $\forall u \in U_\alpha \setminus \{1\}$, $m(u) \in N$ and $\nu^v(m(u))$ is the reflection r_α.

Examples 10.4 : a) When $\mathcal{G}$ is split over K, then $\mathcal{S}$ is a maximal torus and $\mathcal{Z} = \mathcal{S}$. The root system Φ is reduced, and for each $\alpha \in \Phi$, $\mathcal{U}_\alpha$ is isomorphic to the additive group. More precisely a choice of a "Chevalley basis" in $\mathfrak{g}$ gives isomorphisms $x_\alpha : K \to U_\alpha$ and there is a more precise result than (RD2) : $[x_\alpha(u), x_\beta(v)] = \prod_\gamma x_\gamma(c^\gamma_{\alpha,\beta}.u^p.v^q)$, where the product on the right is on the $\gamma = p\alpha + q\beta$ as in (RD2) in a chosen order and the constants $c^\gamma_{\alpha,\beta}$ are integers. For $s \in S$, $sx_\alpha(u)s^{-1} = x_\alpha(\alpha(s).u)$. Axiom (RD4) is nothing else than a classical calculus in SL_2 .

b) If $\mathcal{G} = \mathcal{GL}_n$, one chooses for $\mathcal{S}$ the group of diagonal matrices. Then $\mathcal{Z} = \mathcal{S}$ ($\mathcal{G}$ is split) and $\mathcal{N}(\mathcal{S})$ is the group of monomial matrices (exactly one coefficient $\neq 0$ in each column or line), so $W^v = \mathcal{N}/\mathcal{Z}$ is the symmetric group on n letters and its canonical system of generators is $\Sigma = \{ r_i = (i, i+1) \ / \ i \in I = \{1, \dots, n-1\} \}$.

If $\varepsilon_i(M)$ is the coefficient $m_{i,i}$ of a matrix M, then $\varepsilon_1, ..., \varepsilon_n$ is a $\mathbb{Z}-$basis of $X(\mathcal{S})$. The root system is $\Phi = \{\alpha_{i,j} = \varepsilon_i - \varepsilon_j \ / \ 1 \leq i \neq j \leq n\}$; the reflection corresponding to $\alpha_{i,j}$ is the transposition (i, j). The group $U_{\alpha_{i,j}}$ consists of the matrices $x_{i,j}(u)$ (for $u \in K$) with 1's on the diagonal, u as coefficient (i, j) and 0 elsewhere. For axiom (RD4) $x_{i,j}(u)' = x_{i,j}(u)'' = x_{j,i}(-u^{-1})$ and $m(x_{i,j}(u))$ is the matrix with 1's on the diagonal except for coefficients (i, i) and (j, j) , u as coefficient (i, j), $-u^{-1}$ as coefficient (j, i) and 0 elsewhere.

The system of positive roots associated to Σ is $\Phi^+ = \{\alpha_{i,j} \ / \ i \leq j\}$ and ZU^+ (resp. U^+ or U^-) is the group of matrices which are upper triangular (resp. upper or lower triangular with 1's on the diagonal).

Proposition 10.5 : *Let $(G, (U_\alpha)_{\alpha\in\Phi}, Z)$ be a generating root datum, Φ^+ a positive root system in Φ, $B = ZU^+$ and N the subgroup of G generated by Z and the $m(u)$ for $\alpha \in \Phi$, $u \in U_\alpha$, then (G, B, N) is a saturated Tits system.*

Proof : The first assertion of (T1) is clear, by (RD4) and (GRD), as $r_\alpha(\alpha) = -\alpha$. There is an action ν^v of N on Φ given by (RD4) with image $\nu^v(N) = W^v$ and with kernel containing Z. Actually $Ker(\nu^v) = Z = B \cap N$; see [Bruhat-Tits-72 ; 6.1.11], I don't prove it as it's clear in the above case of a reductive group. Now (T2) and the second assertion of (T1) are clear.

If $s = r_\alpha$, $\alpha > 0$, then $sBs \supset sU_\alpha s = U_{-\alpha} \subset U^-$ and (T4) is a consequence of (RD5). The saturation is also a consequence of (RD5).

Suppose α is a simple root, then by (RD2), $U^+ = U^\alpha U_\alpha$, where U^α is generated by the U_β for $\beta > 0$ such that $\frac{\beta}{2} \notin \Phi$ and $\beta \neq \alpha$. This group U^α is normalized by Z, U_α and r_α , hence :

$$BwBr_\alpha B = BwU_\alpha U^\alpha r_\alpha B = Bwr_\alpha U_{-\alpha} U^\alpha B = Bwr_\alpha U_{-\alpha} B$$

If $(wr_\alpha)(\alpha) < 0$, then $wr_\alpha U_{-\alpha} \subset Bwr_\alpha$ hence $BwBr_\alpha B \subset Bwr_\alpha B$.

If $(wr_\alpha)(\alpha) > 0$, then $w(\alpha) < 0$ and (by (RD4))

$$Bwr_\alpha(U_{-\alpha} \setminus \{1\})B \subset Bwr_\alpha U_\alpha r_\alpha U_\alpha B = Bwr_\alpha U_\alpha r_\alpha B = BwU_{-\alpha}B = BU_{-w\alpha}wB = BwB.$$

(T3) is now proved in both cases. □

Corollary 10.6 : *There is a (vectorial) building of type (V, W^v) associated to the reductive group $\mathcal{G}$ over K ; it is called $\mathcal{I}^v(\mathcal{G}, K)$. Its apartments are in bijection with maximal split tori of $\mathcal{G}$ over K. The apartment $A^v(\mathcal{S})$ associated to $\mathcal{S}$ is (V, W^v) with the action of N via ν^v. The chamber associated to $(\mathcal{S}, \Phi^+)$ is the set $\{v \in V \ / \ \alpha(v) > 0, \forall \alpha \in \Phi^+\} \subset A^v(\mathcal{S})$.*

This is a simple consequence of 8.6 and 10.5 as it is proved in 8.6 that the stabilizer in G of $A(\mathcal{S})$ is $N = N_G(\mathcal{S})$ and as one knows that all maximal split tori are conjugated under G. Actually $\forall \alpha \in \Phi$ the fixator of $\{v \in V \ / \ \alpha(v) \geq 0\} \subset A^v(\mathcal{S})$ is ZU_α.

The spherical building $\mathcal{I}^s(\mathcal{G}, K)$ associated to the essential quotient of $\mathcal{I}^v(\mathcal{G}, K)$ is called the *Tits building* of $\mathcal{G}$ over K.

10.7 Example of GL_n :

With the notations of 10.4b, if $\tau \subset I$ and $I \setminus \tau = \{i_1, \ldots, i_k\}$ with $i_1 < \ldots < i_k$, then $P_\tau = BW_\tau B$ is the group of matrices triangular by blocks of successive sizes $i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k$ *i.e.* P_τ is the stabilizer of the flag $\{ \bigoplus_{i=1}^{i_j} Ke_i \}_{1 \leq j \leq k}$ (where $(e_1, \ldots, e_n)$ is the canonical basis of K^n). Moreover we saw in 8.7.2 that the set of facets of type τ is G/P_τ. So the facets of $\mathcal{I}^v(\mathcal{GL}_n, K)$ are the flags of K^n (*i.e.* the sets F of non trivial subspaces of K^n totally ordered by inclusion) ; the type of a flag F is the set $\tau(F) \subset I$ of the dimensions of the subspaces in F. Actually the 2 maps facet $F \mapsto$ group $P(F) \mapsto$ flag reverse the natural orders, so the order on facets correspond to the inclusion of flags (seen as sets of subspaces of K^n).

The apartments correspond to maximal split tori *i.e.* to sets $\{L_1, \ldots, L_n\}$ of n lines in K^n generating K^n. The facets in the apartment corresponding to such a set are the flags made of vector spaces which are generated by the L_i they contain.

For $n = 3$ and $|K| = 2$, there are 7 lines (numbered 1 to 7) (resp. 7 planes (numbered I to VII)) in K^3 ; they correspond to the vertices of type 1 (resp. 2) in the spherical building $\mathcal{I}^s(\mathcal{GL}_3, K) = \mathcal{I}^s(\mathcal{PGL}_3, K)$. The chambers correspond to pairs (*line* $\subset$ *plane*) their number is 21. There are 28 apartments each made of 6 vertices and 6 chambers. This building is drawn in figure 5.

§ 11 Reductive groups over local fields :

The references are to [Bruhat-Tits-72 or 84a] or [Tits-79].

11.1 We suppose the existence of a *real valuation* ω on K *i.e.* a map $\omega : K \to \mathbb{R} \cup \{+\infty\}$ such that :

$$\omega(0) = +\infty \quad , \quad \Lambda = \omega(K^*) \subset \mathbb{R} \text{ is not reduced to a point,}$$

$$\omega(xy) = \omega(x) + \omega(y) \quad \text{and} \quad \omega(x+y) \geq inf(\omega(x), \omega(y)) \quad (\forall x, y \in K).$$

Then $\mathcal{O} = \omega^{-1}(\mathbb{R}_+ \cup \{+\infty\})$ is the *valuation ring*, it's a local ring with unique maximal ideal $\mathfrak{m} = \omega^{-1}(\mathbb{R}_+^* \cup \{+\infty\})$. The *residue field* is $\kappa = \mathcal{O}/\mathfrak{m}$.

With the hypotheses of § 10, the group G is now endowed with a topology (induced by that of K^{m^2} for any embedding of $\mathcal{G}$ in $\mathcal{GL}_m$).

Proposition 11.2 : *There is an homomorphism $\nu : Z \to V(\mathcal{S})$ such that : $\forall z \in S$, $\chi(\nu(z)) = -\omega(\chi(z))$, $\forall \chi \in X(\mathcal{Z}) \subset V^*$.*

N.B. : We define $\widehat{T} = \nu(Z)$ and $Z_0 = Ker(\nu)$ then $Z_0 \supset \mathcal{S}(\mathcal{O}) \simeq (\mathcal{O}^*)^r$ with equality when $\mathcal{G}$ is split (*i.e.* $\mathcal{S} = \mathcal{Z}$). This group Z_0 is open in Z and actually it is compact when K is locally compact.

Proof : The group $\mathcal{Z}$ is reductive with center containing $\mathcal{S}$, hence the restriction map $X(\mathcal{Z}) \to X(\mathcal{S})$ has an image of finite index. So there exists $m > 0$ such that $\forall \chi \in X(\mathcal{S})$, $m\chi \in X(\mathcal{Z})$ and ν is defined by $\chi(\nu(z)) = -\frac{1}{m}\omega(m\chi(z))$, $\forall \chi \in X(\mathcal{S})$ and $\forall z \in Z$. □

Proposition 11.3 : *There is an homomorphism ν from N to the group of affine automorphisms of $V(\mathcal{S})$ such that, $\forall z \in Z$, $\nu(z)$ is the translation by the vector $\nu(z)$ defined in 11.2 and $\forall n \in N$ the linear part of $\nu(n)$ is $\nu^v(n) \in W^v$ defined in 10.2.*

This action is unique up to an affine automorphism (which may be non unique).

N.B. : We write $A(\mathcal{S})$ for the affine space underlying $V(\mathcal{S})$, then $\widehat{W} = \nu(N)$ is a subgroup of $Aut_{af}(A(\mathcal{S}))$ and there exists points x in $A(\mathcal{S})$ such that : $\widehat{W} = \widehat{W}_x \ltimes \widehat{T}$, with $\widehat{W}_x \simeq W^v$ (see the proof below).

Proof : a) For all z in Z, n in N and χ in $X(\mathcal{Z})$, $\nu(nzn^{-1})\chi = -\omega(\chi(nzn^{-1})) = -\omega((\nu^v(n^{-1}).\chi)(z)) = \nu(z)(\nu^v(n^{-1}).\chi) = (\nu^v(n).\nu(z)).\chi$, hence $\nu(nzn^{-1}) = \nu^v(n).\nu(z)$.

b) It remains only to find a subgroup of N/Z_0 isomorphic to W^v by ν^v. As $Z/Z_0 = \widehat{T} \subset V(\mathcal{S})$ has no torsion, it is sufficient to find a finite subgroup N_1 of N such that $\nu^v(N_1) = W^v$. This group is well known when $\mathcal{G}$ is split : one chooses a Chevalley basis as in 10.4a, to each simple root α is associated an element $s_\alpha = m(x_\alpha(1))$ such that $\nu^v(s_\alpha) = r_\alpha$ and s_α is of order 2 or 4 ; then N_1 is generated by these s_α. In the general case Borel and Tits have shown that there is a split reductive subgroup of $\mathcal{G}$ sharing the same $\mathcal{S}$ and the same W^v, hence we are done. □

11.4 The affine Weyl group and the apartment :

For all root α and all $u \in U_\alpha \setminus \{1\}$, $\nu^v(m(u)) = r_\alpha$ is the linear part of $\nu(m(u))$. And actually $\nu(m(u))$ is the orthogonal reflection with respect to an hyperplane $H(u)$ (it's easy to prove in the split case as one verifies that $m(u)^4 = 1$). This hyperplane has $Ker(\alpha)$ for direction and it's the *wall* of u or $m(u)$. The *half-apartment defined by* u is $D(u) = \{x \in A(\mathcal{S}) \ / \ \alpha(x) \geq \alpha(M(u)) \}$.

The *affine Weyl group* is the subgroup W of $\widehat{W}$ generated by these $\nu(m(u))$; it is a reflection group with set of walls $\mathcal{H} = \{H(u) \ / \ \alpha \in \Phi, u \in U_\alpha \setminus \{1\} \}$. Thus $(A(\mathcal{S}), W)$ is an apartment.

For a special point $x \in A(\mathcal{S})$, $W \simeq W_x \ltimes T$ with $W_x = \widehat{W}_x \simeq W^v$ and $T \subset \widehat{T}$. But N normalizes all the situation, hence $W \triangleleft \widehat{W}$ and N stabilizes $\mathcal{H}$. By construction $Y(\mathcal{S}) \otimes \Lambda \subset \widehat{T} \subset (1/m)Y(\mathcal{S}) \otimes \Lambda$ (*cf.* proof of 11.2). So W is discrete if and only if $\mathcal{H}$ is discrete *i.e.* if and only if $\Lambda = \omega(K^*)$ is discrete in $\mathbb{R}$.

Examples 11.5 : a) When $\mathcal{G}$ is split, $\forall \alpha \in \Phi$, $\forall u \in K^*$, $m(x_\alpha(u)) = \alpha\check{}(u)m(x_\alpha(1))$, where $\alpha\check{} \in Y(\mathcal{S})$ is the coroot. The special point may be chosen fixed by all $m(x_\alpha(1))$ ($\forall \alpha \in \Phi$) hence the translation group T is generated by the $\nu(\alpha\check{}(u))$. By definition of ν in 11.2, it is clear that $T = Q\check{} \otimes \Lambda$ where $Q\check{} \subset Y(\mathcal{S})$ is the "coroot-lattice" generated by the $\alpha\check{}$ for $\alpha \in \Phi$. But $\widehat{T} = Y(\mathcal{S}) \otimes \Lambda$, hence $W = \widehat{W}$ if $Q\check{} = Y(\mathcal{S})$ *i.e.* if $\mathcal{G}$ is semi-simple and simply connected.

b) Suppose now $\mathcal{G} = \mathcal{GL}_n$. The $\mathbb{Z}-$basis $\varepsilon_1, ..., \varepsilon_n$ of $X(\mathcal{S})$ gives a dual $\mathbb{Z}-$basis $\mu_1, ..., \mu_n$ of $Y(\mathcal{S})$ and hence an $\mathbb{R}-$basis of $V(\mathcal{S}) = Y(\mathcal{S}) \otimes \mathbb{R}$. The element $m(x_{i,j}(1))$ acts linearly on $V(\mathcal{S})$ by the reflection with respect to the hyperplane " $\alpha_{i,j}(x) = 0$ " *i.e.* by permutation of μ_i and μ_j. The coroot $\alpha_{i,j}\check{}$ is $\mu_i - \mu_j$ and $\nu(\alpha_{i,j}\check{}(u))$ is the translation by $-\omega(u)\alpha_{i,j}\check{}$. So $m(x_{i,j}(u))$ is the reflection with respect to the wall $H(x_{i,j}(u))$ of equation $\alpha_{i,j}(x = \sum x_k\mu_k) = x_i - x_j = -\omega(u)$ $(\in \Lambda = \omega(K^*))$.

This gives the definition of all walls. In particular a special point is a point $x = \sum x_k\mu_k$ such that $x_i - x_j \in \Lambda$, $\forall i \neq j$.

The group $\widehat{T}$ of translations in $\widehat{W}$ is $\Lambda^n \subset \mathbb{R}^n$, whereas T is the subgroup of elements $(\lambda_1, ..., \lambda_n) \in \Lambda^n$ such that $\sum \lambda_k = 0$ (as $Q\check{}$ is defined by the same equation in $Y(\mathcal{S})$.

11.6 Valuation of the root datum :

Let's choose an origin in the apartment $A(\mathcal{S})$. Then, $\forall \alpha \in \Phi$, $\forall u \in U_\alpha \setminus \{1\}$, the wall $H(u)$ has for equation : $\alpha(x) + k = 0$ for some $k \in \mathbb{R}$. We define $\varphi_\alpha(u) = k$, (and $\varphi_\alpha(1) = +\infty$).

One would like that these φ_α verify the following definition of [Bruhat-Tits-72] :

Definition 11.7 : A *real valuation* of the root datum $(G, (U_\alpha)_{\alpha\in\Phi}, Z)$ is a family $(\varphi_\alpha)_{\alpha\in\Phi}$ satisfying the following axioms :

(V0) For all $\alpha \in \Phi$, φ_α is a map from U_α to $\mathbb{R} \cup \{+\infty\}$ and the image of φ_α contains at least 3 elements.

(V1) For all $\alpha \in \Phi$ and all $\lambda \in \mathbb{R} \cup \{+\infty\}$, the set $U_{\alpha,\lambda} = \varphi_\alpha^{-1}([\lambda, +\infty])$ is a subgroup of U_α and $U_{\alpha,+\infty} = \{1\}$.

(V2) a) For all $\alpha \in \Phi$ and all $u \in U_\alpha \setminus \{1\}$, the function $v \mapsto \varphi_{-\alpha}(v) - \varphi_\alpha(m(u)vm(u)^{-1})$ is a constant on $U_{-\alpha} \setminus \{1\}$.

b) For all $\alpha \in \Phi$ and all $t \in Z$, the function $v \mapsto \varphi_\alpha(v) - \varphi_\alpha(tvt^{-1})$ is a constant on $U_\alpha \setminus \{1\}$.

(V3) For any pair $\{\alpha, \beta\}$ of roots verifying $-\alpha \notin \mathbb{R}^+\beta$, and all $\lambda, \mu \in \mathbb{R}$ the commutator group $[U_{\alpha,\lambda}, U_{\beta,\mu}]$ is contained in the subgroup generated by the $U_{p\alpha+q\beta,p\lambda+q\mu}$ for $p, q \in \mathbb{N}^*$ and $p\alpha + q\beta \in \Phi$.

(V4) If α and 2α are in Φ, then $\varphi_{2\alpha}$ is the restriction of $2\varphi_\alpha$ to $U_{2\alpha}$.

(V5) For all $\alpha \in \Phi$, if u, u' and u'' are as in the axiom (RD4) , then $\varphi_{-\alpha}(u') = \varphi_{-\alpha}(u'') = -\varphi_\alpha(u)$.

11.8 Results about these axioms : a) If $\mathcal{G}$ is split, one chooses as origin the special point of 11.5.a ; then $\varphi_\alpha(x_\alpha(u)) = \omega(u)$ and the verification of the axioms in 11.7 is easy (particularly for $\mathcal{GL}_n$).

b) Axioms V0 , V2 , V4 and V5 are always verified in our situation. J. Tits conjectures that V1 and V3 are always true when K is complete. This was proven when ω is discrete with a perfect residue field [Bruhat-Tits-84a], more generally when $\mathcal{G}$ quasi-splits over a tamely ramified extension [Rousseau-77] but also when the residual characteristic is not 2 (unpublished results of Tits, *cf.* [Rousseau-77]).

In the following we assume these axioms verified.

11.9 Parahoric subgroups : If $x \in A(\mathcal{S})$, $\widehat{P}_x$ is defined as the group generated by the fixator N_x of x in N (for the action ν) and the groups $U_{\alpha,-\alpha(x)}$; one has $\widehat{P}_x \cap N = N_x$.

If F is a facet in $A(\mathcal{S})$, the *parahoric subgroup* $P(F)$ is the group generated by Z_0 and the groups $U_{\alpha,k}$ where $\alpha \in \Phi$ and $\alpha(F) + k \subset [0, +\infty)$. If N_F is the (pointwise) fixator of F in N, it normalizes $P(F)$ and one defines $\widehat{P}(F) = N_F P(F)$. If F is a chamber, $P(F)$ is an *Iwahori subgroup*.

11.10 The Bruhat-Tits building : We want an euclidean building with a strongly transitive action of G, with an apartment isomorphic to $A(\mathcal{S})$ stabilized by N (acting through ν) and such that $\forall s \in A(\mathcal{S})$ the fixator of x in G is $\widehat{P}_x$. Note that this implies that, $\forall \alpha \in \Phi$, an $u \in U_\alpha \setminus \{1\}$ fixes (pointwise) the half-apartment $D(u)$.

The Bruhat-Tits definition of this building is as the quotient $\mathcal{I}^a(\mathcal{G}, K)$ of $G \times A(\mathcal{S})$ by the equivalence relation : $\quad (g, x) \sim (h, y) \quad \Leftrightarrow \quad \exists n \in N$ such that $y = \nu(n)x$ and $g^{-1}hn \in \widehat{P}_x$.

As in 8.7, Bruhat and Tits prove that $\mathcal{I}^a(\mathcal{G}, K)$ is a solution to our problem. Actually proposition 8.7 is still true except for assertions d) and f). The fixator of a facet F in G is $\widehat{P}(F)$. The stabilizer (resp. fixator) of $A(\mathcal{S})$ is N (resp. Z_0) ; so one sees as in 10.6 that the apartments are in bijection with maximal split tori of $\mathcal{G}$ over K. The action of G is strongly transitive but non type-preserving in general. The subgroup G' of G consisting of the strongly type preserving automorphisms is generated by $N' = \nu^{-1}(W)$ and the groups U_α, it is also strongly transitive on $\mathcal{I}^a(\mathcal{G}, K)$.

N.B. : a) This Bruhat-Tits building verify the conditions A4 or CO of § 9 [Bruhat-Tits-72 ; 7.4.18]. Actually, when K is complete, the system of apartments is the complete one (9.2) [Rousseau-77 ; 2.3.7]. The spherical building at infinity is the spherical building $\mathcal{I}^s(\mathcal{G}, K)$ defined in 10.6.

b) When $\mathcal{G}$ is reductive and its center contains a split torus, the building $\mathcal{I}^a(\mathcal{G}, K)$ is not essential. In the literature it is often called the *extended Bruhat-Tits building* and the Bruhat-Tits building is then its essential quotient $\mathcal{I}^{ae}(\mathcal{G}, K)$ (quotient by $V_0 = Y(\mathcal{S}_1) \otimes \mathbb{R}$ where $\mathcal{S}_1$ is the maximal split torus of the center). This building is drawn in figure 4 when $\mathcal{G} = \mathcal{GL}_2$ and the residue field κ has 2 elements.

11.11 Discrete case :

When the valuation ω is discrete, we saw that the reflection group W and hence the building are discrete. If moreover the relative rank of $\mathcal{G}$ over K is 1, then it's easy to see that $\mathcal{I}^a(\mathcal{G}, K)$ is a tree with no endpoint [Tits-79 ; 2.7].

If C is a chamber in $A(\mathcal{S})$, $B = P(C)$ and G', N' are as in 11.10, then (G', B, N') is a saturated Tits system and $\mathcal{I}^a(\mathcal{G}, K)$ is its building (defined as in 8.6).

When K is locally compact (*i.e.* ω discrete and the residue field finite) then the group G is locally compact and each group $\widehat{P}_x$ or $\widehat{P}(F)$ is compact open. So the orbits of a compact subgroup of G are all finite : a compact subgroup of G is bounded and hence is contained in a group $\widehat{P}_x$ (*cf.* 7.1). This gives a classification of maximal compact subgroups : each conjugacy class (under G) contains a group $\widehat{P}_x$ where x is the barycenter of a face of a given chamber. In particular there is only a finite number of conjugacy classes of maximal compact subgroups.

§ 12 The Bruhat-Tits building of GL_n :

I shall give a more concrete description of this building following [Goldman-Iwahori-63] (where no building language is used) , [Bruhat-Tits-84b] and [Parreau-00].

Definition 12.1 : An *additive norm* on the $K-$vector space E of dimension n is a map $\gamma : E \to \mathbb{R} \cup \{+\infty\}$ such that : 1) $\gamma^{-1}(+\infty) = \{0\}$; $\gamma(\lambda y) = \omega(\lambda) + \gamma(y)$; $\gamma(x+y) \geq inf(\gamma(x), \gamma(y))$; $\forall \lambda \in K$, $\forall x, y \in E$,

2) there exists a basis (e_i) of E and reals (c_i) such that $\gamma(\sum \lambda_i e_i) = inf\{\omega(\lambda_i) + c_i \;/\; i = 1, n\}$.

One tells then that this basis and γ are *adapted* and note $\gamma = \gamma_{(e_i)}^{(c_i)}$.

Let $\mathcal{N}$ (resp. $\mathcal{N}((e_i))$) be the set of these norms (resp. adapted to (e_i)). The group $GL(E)$ acts on $\mathcal{N}$ by $(g\gamma)(x) = \gamma(g^{-1}x)$.

Remarks : a) Actually when K is locally compact the second condition is a consequence of the first [Goldman-Iwahori-63]. One can even prove that the flag $Ke_1 \subset Ke_1 \oplus Ke_2 \subset ...$ may be chosen in advance ; this is more or less a proof of 9.4.b for GL_n.

Proof : A map γ verifying 1) is adapted to $(e_1, \ldots, e_n)$ if and only if $\gamma(\sum \lambda_i e_i) = inf\{\gamma(\lambda_i e_i)\}$. By induction it's sufficient to prove that, if F is an hyperplane in E, $\exists e_1 \in E \setminus E_1$ such that $\gamma(\lambda e_1 + f) = inf\{\gamma(\lambda e_1), \gamma(f)\}$, $\forall \lambda \in K$, $\forall f \in F$. By hypothesis one knows that $\gamma(\lambda e_1 + f) \geq inf\{\gamma(\lambda e_1), \gamma(f)\}$ with equality when $\gamma(\lambda e_1) \neq \gamma(f)$; so it's sufficient to prove that $\gamma(\lambda e_1 + f) \leq \gamma(\lambda e_1)$.

Let $\varphi \in E^*$ be such that $F = Ker(\varphi)$ and choose e_1 minimizing $\omega(\varphi(e)) - \gamma(e)$ (this function is defined and continuous on the compact projective space $\mathbb{P}(E)$ with values in $\mathbb{R} \cup \{\infty\}$). One may suppose $\varphi(e_1) = 1$. Then $\gamma(e = \lambda e_1 + f) \leq \omega(\varphi(e)) - \omega(\varphi(e_1)) + \gamma(e_1) = \omega(\lambda) + \gamma(e_1) = \gamma(\lambda e_1)$. □

b) When ω is discrete, one can also easily prove that 2 norms are adapted to a same basis [Bruhat-Tits-84b ; 1.2.6], this is axiom (I1') (see 12.2). Many other properties of buildings can be proven directly.

Proof : If $\gamma \in \mathcal{N}$, then $\gamma(E \setminus \{0\})$ is finite modulo $\omega(K^*)$ (which is discrete). Hence if $\gamma, \delta \in \mathcal{N}$ the function $\gamma - \delta$ on $E \setminus \{0\}$ is bounded and contained in a finite number of classes modulo $\omega(K^*)$. So there is $e_1 \in E \setminus \{0\}$ achieving the minimum of $\gamma - \delta$: $\gamma(e) - \delta(e) \geq \gamma(e_1) - \delta(e_1)$, $\forall e \in E \setminus \{0\}$.

One may define a norm γ^* on the dual E^* of E by $\gamma^*(\varphi) = \mathrm{Inf}_{0 \neq e \in E}\, (\omega(\varphi(e)) - \gamma(e))$, $\forall \varphi \in E^*$. It is adapted to the basis of E^* dual to a basis adapted to γ, and a simple calculus in these bases proves that $\gamma^{**} = \gamma$. Hence $\gamma(e_1) = \mathrm{Inf}_{0 \neq \varphi \in E^*}\, (\omega(\varphi(e_1)) - \gamma^*(\varphi)) = \omega(\varphi_1(e_1)) - \gamma^*(\varphi_1)$ for some $\varphi_1 \in E^*$ (as this function is bounded and finite modulo $\omega(K^*)$). One may suppose $\varphi_1(e_1) = 1$ and then $(\forall e \in E)$:

$$(*) \qquad \omega(\varphi_1(e_1)) - \gamma(e_1) = \gamma^*(\varphi_1) \leq \omega(\varphi_1(e)) - \gamma(e)$$

and $\delta(e) - \delta(e_1) \leq \gamma(e) - \gamma(e_1) \leq \omega(\varphi_1(e)) - \omega(\varphi_1(e_1))$,

hence $$(**) \qquad \omega(\varphi_1(e_1)) - \delta(e_1) \leq \omega(\varphi_1(e)) - \delta(e)$$

Using (*) and (**) one proves as in a) above that e_1 is the first vector of a basis adapted to γ and δ (and $Ker(\varphi_1)$ is the vector space generated by the other vectors of this basis, by induction on n). □

12.2 Apartments : 1) A basis (e_i) of E enables us to identify $GL(E)$ with $GL_n(K)$ and so to use 10.4.b, 10.7 and 11.5.b. So $Z = S$ is the diagonal torus and N the group of monomial matrices.

There is a bijection of $V(S)$ onto $\mathcal{N}((e_i))$ given by : $\sum c_i e_i \mapsto \gamma_{(e_i)}^{(c_i)}$. The normalizer Z of S stabilizes $\mathcal{N}((e_i))$, hence $\mathcal{N}((e_i))$ depends only on S and is denoted by $\mathcal{N}_S$. More precisely the induced action of N on $V(S)$ is given by :

- the finite group N_1 consisting of matrices in N with all coefficients equal 0 or ± 1 verifies $\nu^v(N_1) = W^v$ and acts linearly on $V(S)$ by permutation of the basis (μ_i) *i.e.* through ν^v,

- the group $Z = S$ acts via translations : if $s \in S$ has for diagonal coefficients $s_1, ..., s_n$, then $(s.\gamma_{(e_i)}^{(c_i)})(\sum \lambda_i e_i) = \gamma_{(e_i)}^{(c_i)}(\sum s_i^{-1}\lambda_i e_i) = inf_i\{\omega(s_i^{-1}\lambda_i) + c_i\} = \gamma_{(e_i)}^{(c_i - \omega(s_i))}(\sum \lambda_i e_i)$. Hence s acts through the translation $\nu(s)$ of vector $\nu(s) = -\sum_i\ \omega(s_i)\mu_i \in V(S)$ which verify $\chi(\nu(s)) = -\omega(\chi(s))$, $\forall \chi \in X(S)$.

2) So $\mathcal{N}_S$ is isomorphic to $V(S) = A(S)$ with its affine action of N as defined in 11.3. This is the *apartment* of S inside $\mathcal{N}$. More generally the apartments of $\mathcal{N}$ correspond (bijectively) to the bases of E (up to permutation of the vectors and multiplication of vectors by constants) *i.e.* to sets $\{L_1, \ldots, L_n\}$ as in 10.7.

The walls in $\mathcal{N}_S$ are defined by $H(x_{i,j}(u)) = \{\gamma \in \mathcal{N}_S\ /\ \gamma(e_i) - \gamma(e_j) + \omega(u) = 0\} = \{\gamma_{(e_i)}^{(c_i)}\ /\ c_i - c_j + \omega(u) = 0\}$ for $i \neq j$ and $u \in K^*$. The group $\widehat{W} = \nu(N)$ is $W^v \ltimes (Y(S) \otimes \omega(K^*))$; its reflection subgroup is $W = W^v \ltimes (Q\check{}\otimes \omega(K^*))$. The subgroup $N' = \nu^{-1}(W)$ of N is defined by the condition " determinant $\in \mathcal{O}^*$ " (the same condition defines the subgroup G' of strongly type-preserving automorphisms, *cf.* 11.11).

Proposition 12.3 : *The fixator* G_γ *of* $\gamma = \gamma_{(e_i)}^{(c_i)}$ *in* G *identified with* $GL_n(K)$ *via the basis* (e_i) *is defined by the following conditions on the coefficients of a matrix* g *:*

$$det(g) \in \mathcal{O}^* \quad \text{and,} \quad \forall i,j\ ,\ \omega(g_{i,j}) \geq c_j - c_i \quad (= -\alpha_{i,j}(\gamma) \text{ for the identification } \mathcal{N}_S \simeq V(S)\ ).$$

It is the subgroup of $G = GL_n(K)$ *generated by* N_γ *and the groups* $U_{\alpha_{i,j}, -\alpha_{i,j}(\gamma)}$ *for* $i \neq j$ *.*

Proof : Let L be an $\mathcal{O}-$submodule of E of the form $L = \bigoplus_{i=1}^n\ \omega^{-1}([-c'_i, +\infty])e_i$. It is more or less clear that an element $g \in GL_n(K)$ stabilizes L if $\omega(g_{i,j}) \geq c'_j - c'_i$, $\forall i,j$ and this condition is necessary if $c'_i \in \omega(K^*)$. So, using the considerations in 12.5c, it's easy to prove that g fixes the norm γ if and only if the coefficients of g and g^{-1} verify $\omega(g_{i,j}) \geq c_j - c_i$, $\forall i,j$. The formulas for the coefficients of g^{-1} enable us to prove that this is also equivalent to the conditions of the proposition. Details for this sketch of proof may be found in [Parreau ; cor. 3.4].

The group P generated by N_γ and the groups $U_{\alpha_{i,j}, -\alpha_{i,j}(\gamma)}$ is clearly in G_γ and one knows (Bruhat decomposition) that $G = PNP$, hence $G_\gamma = PN_\gamma P = P$. □

12.4 Consequences : The fixator G_γ is the group $\widehat{P}_\gamma$ defined in 11.9. The action of N on $\mathcal{N}_S \simeq A(S)$ is that of 11.3. The space $\mathcal{N}$ is the union of all $\mathcal{N}_{S'}$ which are of the form $\mathcal{N}_{S'} = g\mathcal{N}_S$ for some $g \in G$, as any two maximal split tori are conjugated by G.

So the obvious ($G-$equivariant) map $G \times A(S) \to \mathcal{N}$ is onto and factors through the quotient $\mathcal{I}^a(\mathcal{G}, K)$ defined in 11.10. But any two points in $\mathcal{I}^a(\mathcal{G}, K)$ are, up to conjugacy, both in $A(S)$ and the map $A(S) \to \mathcal{N}$ is injective ; so we get a $G-$equivariant bijection from $\mathcal{I}^a(\mathcal{G}, K)$ to $\mathcal{N}$.

Hence $\mathcal{N}$ (with its apartments) is the (extended) Bruhat-Tits building of $GL(E)$.

12.5 Lattices and vertices : a) Let L be a *lattice* in E, *i.e.* an $\mathcal{O}-$submodule which is free and generates E as a vector space. If $e_1, ..., e_n$ is an $\mathcal{O}-$basis of L, the norm $\gamma_L = \gamma_{(e_i)}^{(0)}$ is entirely defined by L : $\gamma_L(x) = sup\{\omega(\lambda)\ /\ \lambda \in K\ ,\ x \in \lambda L\}$ and L is defined by γ_L : $L = \gamma_L^{-1}([0, +\infty])$.

Such a norm is a special point and conversely any special point in $\mathcal{N}((e_i))$ is $\gamma_{(e_i)}^{(c_i)}$ with $c_i - c_j \in \omega(K^*)$, $\forall i \neq j$ and this special point may be written $\gamma = \gamma_L + c_1$ for the lattice $L = \gamma^{-1}([c_1, +\infty])$.

b) The essential quotient $\mathcal{I}^{ae}(GL_n, K)$ is the quotient by $V_0 = Y(S_1) \otimes \mathbb{R}$ where S_1 is the group of scalar matrices. On $\mathcal{N}$ this means that : $\gamma \sim \gamma' \iff \gamma - \gamma'$ is a constant . Hence the vertices (= special points) of $\mathcal{I}^{ae}(GL_n, K)$ are the classes (up to constants) of the norms γ_L and they may be identified to the classes (up to homotheties) of lattices in E (see [Serre-77] for the case n=2).

c) More generally any norm γ determines a filtration of E by the $\mathcal{O}-$submodules $L_c = \gamma^{-1}([c, +\infty])$ and $L_c^+ = \gamma^{-1}((c, +\infty])$ for $c \in \mathbb{R}$. These $\mathcal{O}-$submodules generate E but are not always lattices (*i.e.* free) when ω is not discrete. This filtration is stable by homothety.

Conversely the norm γ is entirely determined by the family of these $\mathcal{O}-$submodules L_c (verifying $L_c \subset L_{c'}$ if $c > c'$, $\lambda L_c = L_{c+\omega(\lambda)}$, $KL_c = E$ and $\cap_c\ L_c = \{0\}$). But the set of these submodules determines γ up to a constant only when ω is dense.

12.6 Facets in the discrete case :As ω is now discrete, we normalize it such that $\Lambda = \omega(K^*) = \mathbb{Z}$ and we choose a generator π of the maximal ideal $\mathfrak{m}$ ($\omega(\pi) = 1$).

a) As $\gamma^{(c_i)}_{(\lambda_i e_i)} = \gamma^{(c_i+\omega(\lambda_i))}_{(e_i)}$, a change of basis enables us to write any norm as $\gamma = \gamma^{(c_i)}_{(e_i)}$ with $0 \leq c_1 \leq c_2 \leq ... \leq c_n < 1$. Then, for $0 \leq c < 1$ and $m \in \mathbb{N}$, $L_{c+m} = \pi^m((\oplus_{c_i \geq c}\mathcal{O}e_i) \oplus (\oplus_{c_i<c}\pi\mathcal{O}e_i))$ is a lattice. The closure of the facet $F(\gamma)$ containing γ is the enclosure of $\{\gamma\}$, it is the set of $\gamma = \gamma^{(c'_i)}_{(e_i)}$ such that $\exists a \in \mathbb{R}$ with $a \leq c'_1 \leq c'_2 \leq ... \leq c'_n \leq a+1$ and $c'_i = c'_j$ when $c_i = c_j$. In particular this enclosure contains the norms γ_{L_c} for any $c \in \mathbb{R}$.

b) Let's look now to the essential quotient $\mathcal{I}^{ae}(GL_n, K) = \mathcal{N}/constants$. There is a finite number of classes (up to constants) of norms γ_{L_c} : $\gamma_{L_{c_1}}, ..., \gamma_{L_{c_n}}$. The closed facet $\overline{F}(\gamma)$ is a simplex whose vertices are these classes. We get so the classical definition of the (essential) Bruhat-Tits building $\mathcal{I}^{ae}(GL_n, K)$: its vertices are the classes (up to homotheties) of lattices in E and a facet is the simplex built on p vertices associated to lattices $L_1, ..., L_p$ verifying $L_1 \supset\neq L_2 \supset\neq ... \supset\neq L_n \supset\neq \pi L_1$ (up to modification of the L_i by homotheties and permutation) ; see [Serre-77] for the tree case ($n = 2$), it is drawn in figure 4 when $|\kappa| = 2$.

The subgroup G'' of G acting by type preserving automorphisms on $\mathcal{I}^{ae}(GL_n, K)$ or $\mathcal{I}^{a}(GL_n, K)$ is defined by the equation " $\omega(det(g)) \in n\mathbb{Z}$ ". The group G induces a cyclic group of permutations of types.

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