The Affine Buildings of SL_n and Sp_n: A Combinatorial Perspective

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Abstract

For a local field K, we study the affine buildings Ξ_n and Δ_n naturally associated to $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively. Since $\mathrm{Sp}_n(K)$ is a subgroup of $\mathrm{SL}_{2n}(K)$, we investigate properties of a natural embedding of Δ_n in Ξ_{2n}. We use these results to derive information about Δ_n from knowledge of Ξ_{2n}. For $n \geq 3$, we relate the number of vertices of Ξ_n *close* to a given vertex of Ξ_n with the number of chambers of Ξ_n containing a given vertex, proving a conjecture of Schwartz and Shemanske. We then examine *close* vertices in Δ_n ($n \geq 2$) when the given vertex is special (all the vertices of Ξ_n are special). In particular, we use the natural embedding of Δ_n in Ξ_{2n} to prove results about *close* vertices in Δ_n analogous to those for Ξ_{2n}. Finally, we compute the spectral radius of the subgraph Y_n of Δ_n induced by the special vertices of Δ_n to show that the isoperimetric constant of Y_n is positive, from which it follows that Y_n is non-amenable.

Acknowledgements

There are times when being a graduate student feels like riding a crazy roller coaster: you feel a sense of exhilaration when things seem to be working, and you wonder what you were thinking when you cannot even understand the object with which you are working, much less the question you want to answer. But perhaps what distinguishes being a graduate student from riding a crazy roller coaster is that it is far easier to convince yourself that graduate school is not for you than it is to ask that the roller coaster you are riding be stopped before the ride is over so that you can jump off in the middle, wherever you happen to stop, even if it is not at ground level.

It is thus unclear—at least to me—whether I would have been able to finish this thesis without the help and encouragement of other people. First and foremost among these is my advisor, Tom Shemanske, whose sense of humor and perspective helped me fight through the rough spots during the thesis process. I also really appreciate that he answered the millions of questions (counted with multiplicity) I felt compelled to ask quite patiently most—if not all—of the time. The second person I need to thank is Paul Garrett, who has been like a second advisor to me. He not only answered many questions over email but he also generously took a week in June of 2005 to talk to me about buildings in person. Cristina Ballantine and Donald Cartwright also cleared up a few things over email. And there are the math professors (especially Dana Williams, Peter Winkler, and David Webb) and graduate students, both current (Chris Storm, Erik Tou, John Bourke, Allison Henrich, Dominic Klyve, Brooke Andersen, Rachel Esselstein, Geoff Goehle, Nick Scoville, Mits Kobayashi, and Jon Brown) and former (Lee Stemkoski, Nathan Ryan, and Dan Cole), here who let me ask questions and bounce ideas off them.

Finally, I am grateful for the support of my parents, my sister, Mike, Holly, and my former math professors at the University of Portland since they never seemed to doubt that I could finish even though they either did not know or did not understand the math I was trying to do.

Contents

Introduction

Buildings, which Tits first introduced in 1961 [47], were originally studied to understand certain groups—the analogues of the simple Lie groups, and particularly the exceptional groups, over an arbitrary field (see [36]). More precisely, one studies a group G by constructing a building naturally associated to G, and on which G acts nicely. Over the years, buildings have found applications to differential geometry, arithmetic groups, finite group theory, and group cohomology (see [37] and [6, Chapter VII]). Certain buildings also appear in the study of $\mathfrak{p}$-adic simple groups, where they play a role similar to that of the symmetric spaces in the theory of simple Lie groups (see [50, p. 215] and [38]). More specifically (see [21]), buildings have been used in Mostow's rigidity theorem [28], in Thorbergsson's classification of isoparametric submanifolds [45], and in Kleiner and Leeb's proof [24] of a conjecture of Margulis.

Although the axioms for buildings given in [47] were stated in terms of incidence geometries, they were later restated in terms of simplicial complexes [48]. But even the definition of a building as a simplicial complex comes in several flavors, with perhaps the most straight-forward being the one given in Chapter 1, the most abstract the one given in [35, p. 27], and those in [6, p. 76] and [20, p. 52] falling somewhere in the middle. Then someone wishing to learn about buildings has at least three different perspectives from which to choose. One is to view a building as an object attached to an algebraic group, as Tits did (see [51]), and to use the theory of root systems. Another is to view a building as a natural generalization of an incidence geometry (see [10] or [21]). And a third, which is the one we adopt, is to view a building as a combinatorial object naturally associated to a group such as $\mathrm{SL}_n(K)$ for some field K. We start with a brief history of how buildings developed.

A brief history

The history of buildings begins in 1954. At this time, it was well-known how to associate to a classical simple matrix group G a finite reflection group W called the Weyl group of G. But until the publication of Bruhat's 1954 paper [7] on the representation theory of complex Lie groups, a fundamental fact now known as the Bruhat decomposition, which says that $G = \coprod_{w \in W} BwB$ for a certain subgroup B of G, had gone unnoticed. This Bruhat discovered while studying induced representations, and his proof consisted of a separate analysis for each of the four families of classical simple groups. Soon afterwards, Chevalley used the Bruhat decomposition as a basic tool in his construction and classification of simple algebraic groups ([13], [14]), at the same time unifying Bruhat's case-by-case proof and studying the basic properties of the parabolic subgroups of certain matrix groups. Then in 1962, after analyzing Chevalley's methods, Tits extracted two axioms that imply Chevalley's results on the Bruhat decomposition and parabolic subgroups [46]. At the same time, Tits was working on a project, which he began in the mid-1950s, to construct incidence geometries associated to very general matrix groups—the semi-simple Lie groups, and in particular the exceptional groups [52]. So in the early 1960s, Tits combined the axioms for geometries and the axioms for groups with a Bruhat decomposition to arrive at the idea of a building (for more details, see [6, Section V.4]).

And just as we can classify a group as abelian, simple, or solvable (possibly several or none), so can we classify a building as "spherical," "affine," or "Moufang" (possibly several or none). "Spherical" buildings, whose "apartments" can be visualized as triangulations of a sphere, resulted from another motivation for buildings:[1] geometrically construct analogues of the exceptional groups over an arbitrary field. The irreducible spherical buildings of dimension at least 2 were classified by Tits, who also showed how such buildings give rise to large automorphism groups [49]. This connection between a spherical building and a large automorphism group meant that when Ronan and Tits later constructed buildings [39], they had an immediate existence theorem for groups such as $E_8(k)$ (one of the exceptional groups) for any field k without having recourse to the theory of algebraic groups (for more details, see [37]).

"Affine" buildings appeared in the mid-1960s as a result of Iwahori and Matsumoto's work [22] on p-adic groups. These buildings, whose apartments can be visualized as tilings of Euclidean space, arise from algebraic groups over a field with a

[1] This motiviation arose before the publication of [13].

discrete valuation. Their general theory was developed by Bruhat and Tits ([8], [9]), and they were classified by Tits [53] (more details are in [37] and [36]).

Finally, "Moufang" buildings, which include most spherical buildings and those affine buildings attached to groups over a power series field, arose after Moody and Teo introduced groups attached to Kac-Moody Lie algebras in the early 1970s [27]. These buildings come from groups attached to infinite-dimensional and non-affine Kac-Moody Lie algebras and have apartments with "negative curvature" (see [37, Section 8] for more details).

We end this section by noting that according to [50, p. 209], the "real estate" terminology of buildings first appeared in [5] and originated in the fact that the maximal simplices of a building are called "chambers" in view of their close connection with the "Weyl chamber" in the theory of root systems.

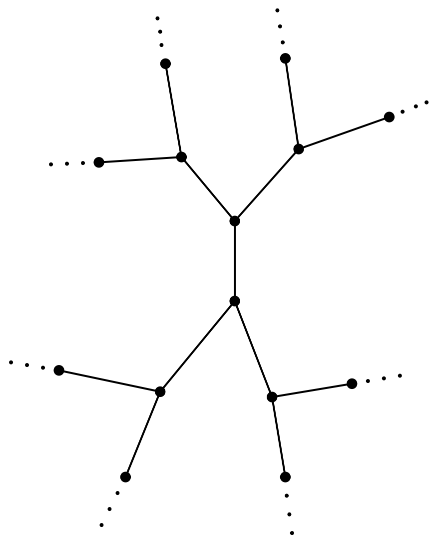
Overview

As mentioned earlier, we view a building as a combinatorial object—in fact, a highly structured simplicial complex—naturally associated to a group such as $\mathrm{SL}_n(K)$ for some field K. More concretely, suppose K is the field of p-adic numbers (p a prime) $\mathbb{Q}_p$, which can be defined as follows: for $r \in \mathbb{Q}^\times$, write $r = (a/b)p^m$, where $a, b, m \in \mathbb{Z}$ with $b \neq 0$, $\gcd(a, b) = 1$, $p \nmid a$, and $p \nmid b$. Then $\mathrm{ord}_p : \mathbb{Q}^\times \to \mathbb{Z}$ given by $\mathrm{ord}_p(r) = m$, where $r \in \mathbb{Q}^\times$ is as in the last sentence, defines a metric on $\mathbb{Q}$ by $|s-t|_p = p^{-\mathrm{ord}_p(s-t)}$ if $s \neq t$ and $|0|_p = 0$; the field $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. The affine building naturally associated to $\mathrm{SL}_2(\mathbb{Q}_p)$ is the $(p+1)$-regular tree; a partial picture in the case $p = 2$ is in Figure 1(a). Figure 1(b) shows a partial picture of an *apartment* of this building, and Figure 1(c) shows a *chamber* of this building. In the case of the affine building naturally associated to $\mathrm{SL}_3(\mathbb{Q}_p)$, a chamber is an "equilateral triangle" (together with its interior) and an apartment is a "copy" of $\mathbb{R}^2$ tiled by chambers.[2]

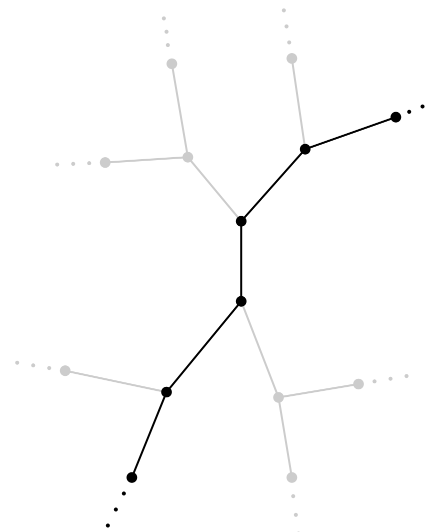
The affine building Ξ_n naturally associated to $\mathrm{SL}_n(K)$ (an $(n-1)$-dimensional simplicial complex), where K is a local field (for example, $K = \mathbb{Q}_p$ for some prime p) is perhaps the best understood and most studied affine building naturally associated to a group. This is partly because the group SL_n is well-understood and partly because all the vertices of Ξ_n are "special."[3] We consider the affine building Δ_n naturally associated to $\mathrm{Sp}_n(K)$ (an n-dimensional simplicial complex). The building Δ_n has

[2]The prime p is reflected in the fact that a codimension-one simplex (side of an equilateral triangle) is contained in exactly $p + 1$ chambers.

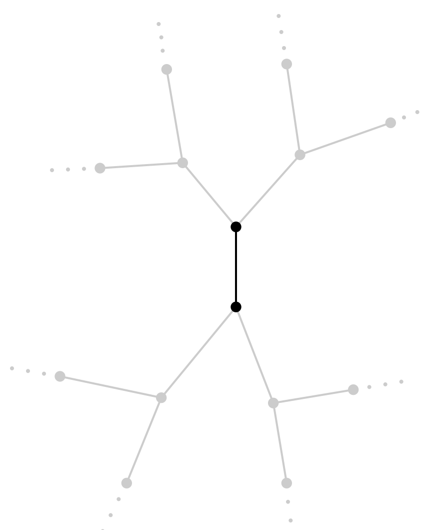
[3]Actually, more is true: all the vertices of Ξ_n are "hyperspecial."



(a) Partial picture of 3-regular tree Ξ_2.



(b) Apartment of Ξ_2.



(c) Chamber of Ξ_2.

both special and non-special vertices. Though there are other affine buildings with both special and non-special vertices, Δ_n seems the "closest" to Ξ_n in the sense that it has only one "kind" of special vertex. Note that $\mathrm{Sp}_n(K)$ is a proper subgroup of $\mathrm{SL}_{2n}(K)$.[4] In general, the fact that not all the vertices of Δ_n are special—exactly two vertices in any chamber (maximal simplex) of Δ_n are special—means that studying Δ_n requires more care than is needed to study Ξ_n. However, we discovered that by thinking of Δ_n as embedded in Ξ_{2n} and by using the fact that $\mathrm{Sp}_n(K)$ is a subgroup of $\mathrm{SL}_{2n}(K)$, we can use facts about Ξ_{2n} to derive information about Δ_n. In particular, if we focus on the *special* vertices of Δ_n, we can usually adapt proofs of results for Ξ_{2n} to obtain analogous results for Δ_n.

For the rest of this section, K is a local field with valuation ring $\mathcal{O}$, uniformizing parameter π, and residue field isomorphic to $\mathbb{F}_q$ (if $K = \mathbb{Q}_p$, then $\mathcal{O} = \mathbb{Z}_p$, the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$, $\pi = p$, and $\mathbb{F}_q = \mathbb{F}_p$). In Chapter 1, we give the definitions related to simplicial complexes and buildings that we need. We also describe the buildings Ξ_n and Δ_n in terms of lattices of a vector space. Then we prove some results about how Δ_n embeds in Ξ_{2n}. In particular, we show how a given chamber C of Δ_n embeds in Ξ_{2n} (Proposition 1.3.2), which allows us to count the number of chambers of Ξ_{2n} that contain C (Corollary 1.3.1). In addition, we derive information about Δ_n from knowledge about Ξ_{2n}. For example, we show that "labelling" the vertices of Ξ_{2n} "labels" the vertices of Δ_n. This provides a way to tell when certain vertices of Ξ_{2n} are *not* vertices of Δ_n and allows us to characterize the *special* vertices of Δ_n in terms of "types":

Proposition 1.5.1. *If $t \in \Delta_n$ is a vertex, then t has type i for some $i \equiv n, \ldots, 2n$ mod $2n$.*

Proposition 1.5.3. *A vertex of Δ_n is special if and only if it has type 0 or n.*

We end Chapter 1 by considering whether the group $\mathrm{GSp}_n(K)$ of symplectic similitudes—the analogue of $\mathrm{GL}_{2n}(K)$ for $\mathrm{Sp}_n(K)$—acts on (the vertices of) Δ_n. It turns out that we can use the action of $\mathrm{GSp}_n(K)$ on Ξ_{2n} to show that $\mathrm{GSp}_n(K)$ does *not* act on (the vertices of) Δ_n (see Corollary 1.6.2) but that $\mathrm{GSp}_n(K)$ *does* act on the *special* vertices of Δ_n. More precisely, we prove:

Proposition 1.6.3. *The group $\mathrm{GSp}_n(K)$ takes adjacent special vertices of Δ_n to adjacent special vertices of Δ_n.*

[4] More precisely, we can think of $\mathrm{Sp}_n(K)$ as preserving a non-degenerate, alternating bilinear form defined on a $2n$-dimensional K-vector space.

Corollary 1.6.3. *The group* $\mathrm{GSp}_n(K)$ *acts transitively on the special vertices of* Δ_n.

To prove Proposition 1.6.3, we show that $\mathrm{GSp}_n(K)$ acts on a certain set of chambers of Ξ_{2n} containing a chamber of Δ_n (see Lemmas 1.6.1 and 1.6.2). With only a few exceptions, the results contained in Chapter 1 are either used in Chapters 2 and 3 or their proofs are the basis of proofs in Chapters 2 and 3.

The motivation behind Chapter 2 was Section 3 of a paper by Schwartz and Shemanske [41], who consider the building Ξ_n. In particular, they point out how the *combinatorial distance* between two chambers of a building can be generalized to define the distance between two vertices of a building. They prove that the vertices of Ξ_n ($n \geq 3$) distance one away or *close* to a given vertex of Ξ_n are given by a certain Hecke operator [41, Theorem 3.3], so that the Hecke operator acts as a generalized adjacency operator on Ξ_n. From this, they conjecture a relationship between the number ω_n of vertices of Ξ_n close to a given vertex and the number r_n of chambers of Ξ_n containing a given vertex, with $r_1 := 1$ (see the Remark following Proposition 3.4 of [41]). In an attempt to better understand the computation underlying Schwartz and Shemanske's conjecture, we use an observation of Paul Garrett to reformulate the question in terms of modules and vector spaces, thus allowing us to prove the conjecture:

Theorem 2.1.1. *For all* $n \geq 3$, $q \cdot r_n = r_{n-2}\, \omega_n$.

Although this proof uses little more than the description of the chambers of Ξ_n in terms of lattices of an n-dimensional K-vector space, it does not really give any insight into why the conjecture is true. On the other hand, the proof of Theorem 2.1.2, which takes a more combinatorial approach, provides a structural explanation—an explanation intrinsic to the building—for the relationship between ω_n and r_n. We devote the rest of Chapter 2 to considering the same question in the building Δ_n when the given vertex is *special* (recall that all the vertices of Ξ_n are special). We prove the following analogue of Theorem 3.3 of [41]:

Theorem 2.2.1. *It* $t \in \Delta_n$ *is a special vertex, then the number of vertices of* Δ_n *close to* t *is the number of left cosets of* $\mathrm{Sp}_n(\mathcal{O})$ *in*

$$\mathrm{Sp}_n(\mathcal{O})\mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\mathrm{Sp}_n(\mathcal{O}).$$

Remark. In Appendix A, we prove the analogue of the last theorem if $\mathrm{Sp}_n(\mathcal{O})$ is replaced by $\mathrm{GSp}_n(\mathcal{O}) = \mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$.

Let $r(\Delta_n)$ be the number of chambers of Δ_n containing a given special vertex (of Δ_n), with $r(\Delta_1) := q + 1$, and $\omega(\Delta_n)$ the number of vertices of Δ_n close to a given special vertex (of Δ_n). The following is an analogue of Theorem 2.1.1 for Δ_n:

Theorem 2.2.2. *For all $n \geq 2$, $q \cdot r(\Delta_n) = r(\Delta_{n-1})\ \omega(\Delta_n)$.*

We also prove a partial analogue (Proposition 2.2.11) of Theorem 2.1.2 for Δ_n using combinatorial arguments when the given vertex has a certain "type." The overlap of Chapter 2 with Chapter 3 consists of Lemma 2.2.1 and Proposition 2.2.1.

We change gears a bit in Chapter 3, which was motivated by an example of Woess, who considers Ξ_n [55, Example (12.20)]. Woess' example implies that the *one-complex* X_n of Ξ_n (i.e., the graph with vertices the vertices of Ξ_n and edges the 1-simplices of Ξ_n) is *non-amenable*. More precisely, it tells us the *spectral radius* of X_n, which implies that the *isoperimetric constant* of X_n—a measure of expansion in X_n by [3, p. 116]—is positive (see [3, Theorem 3.1]); hence, X_n is an analogue of a *family of expanders* (an example of a family of expanders is a family of *Ramanujan graphs*, see [16, pp. 1 – 4]). We adapt this example to the case of the subgraph Y_n of Δ_n induced by the *special* vertices of Δ_n to compute the spectral radius of Y_n, from which we deduce

Theorem 3.2.1. *The isoperimetric constant of Y_n satisfies $i(Y_n) > 0$.*

Consequently, Y_n is both non-amenable and an analogue of a *family of expanders*. The computation of the spectral radius of Y_n involves finding groups (some solvable) that act transitively on the vertices of Y_n and characterizing the vertices of Y_n adjacent to certain ones in terms of orbits.

We end this section by noting that about six months after we derived the formulas in Propositions 2.1.1 and 2.2.8, we learned that both are special cases of a result of Parkinson [32, Theorem 5.15] and that the formula in Proposition 2.1.1 also follows from a result of Cartwright [11, Lemma 2.2]. In contrast to our method, Parkinson uses deep properties of buildings (root systems and Poincaré polynomials of Weyl groups), and Cartwright takes a more combinatorial approach. In Parkinson's general approach, our numbers ω_n and $\omega(\Delta_n)$ are special cases of his number N_λ. Parkinson uses the number N_λ in his definition of vertex set averaging operators on certain affine buildings, and he uses the formula for N_λ to prove results about these operators. At about the same time, we also discovered (about seven months after we had obtained the formula independently) that the formula in Proposition 3.2.3 is also derivable from results of Parkinson. Again, Parkinson's approach is quite general, as it takes a building-theoretic perspective rather than the group-theoretic one we use. As in [55, Example (12.20)], Parkinson uses the idea of a random walk on Y_n, as follows: in [33, Theorem 2.16], Parkinson gives a local limit theorem for isotropic random walks on certain affine buildings. Then [34, Theorem 6.3] and general facts of $C^\star$-algebras imply that the spectral radius of the isotropic random walk is $\widehat{A}(1)$, where A is the transition operator of the random walk and $\widehat{A}$ its Gelfand transform. To express $\widehat{A}(1)$ in terms of the order q of the residue field of K, we use the underlying root system of the building together with results about the Macdonald spherical functions defined in [34, p. 9]. Parkinson's machinery thus gives a means of determining the spectral radius of an isotropic random walk on certain affine buildings.

A final word

We end this chapter by commenting on some of the phrases and notation we use. In general, we try to avoid using phrases such as "It is well-known," but when we do use such phrases (only in Chapter 3), we do so when stating facts that we learned in a paper that uses the same phrase, and we reference the appropriate paper. We learned the information contained in the paragraph following Lemma 3.2.1 on page 84 from Paul Garrett, and we learned about the relationship between $\rho(P)$ and $\rho(X)$ given in Appendix B from Donald Cartwright.

The letters $a, b, c, d, i, j, \ell, m, n$, and r, along with any variations, always denote integers unless specifically stated otherwise. Since the use of $\langle\ \rangle$ arises quite a bit in mathematics, we sometimes use it to denote a subgroup generated by certain elements of a given group and sometimes (more often) to denote a certain bilinear form; it should be "clear" from context which meaning we intend. Other than that, when we "overload" notation (for example, μ denotes an integer in Chapter 1, while $\mu(\cdot)$ is a left Haar measure in Chapter 3 and a measure in Appendix B), it should be "clear" from context which meaning we intend. Finally, we try to use relatively standard notation, but for clarity's sake, we now give the notation we use that we do not define elsewhere. The other notation we use frequently can be found in the Index of Notation on page 106.

$\mathbb{Z}$	set of integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{Z}^{\geq i}$	set of integers greater than or equal to i
$\mathbb{R}$	set of real numbers
$\mathbb{R}^{\geq 0}$	set of non-negative real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{F}_q$	finite field with q elements
$\mathbb{Q}_p$ (p a prime)	set of p-adic numbers
$H \leq G$ (H and G groups)	H is a subgroup of G
$H \trianglelefteq G$ (H and G groups)	H is a normal subgroup of G
$H \cong G$ (H and G groups)	H and G are isomorphic groups
$L + L'$ (L and L' R-modules, R a ring)	$\{\ell + \ell' : \ell \in L,\ \ell' \in L'\}$
R^m (R a ring)	set of m-tuples with entries from R
A^t or g^t (A and g matrices)	transpose of A or g
$M_n(R)$ (R a ring)	set of $n \times n$ matrices with entries in R
$\mathrm{GL}_n(R)$ (R a ring)	$\{g \in M_n(R) : \det g \in R^\times\}$
$\mathrm{SL}_n(R)$ (R a ring)	$\{g \in M_n(R) : \det g = 1\}$
δ_{ij} (Kronecker delta)	has value 1 if $i = j$ and 0 otherwise
$\mathrm{Card}(S)$ (S a set)	the number of elements in S
$\coprod_i S_i$ (S_i sets)	the disjoint union of the S_i
$\Delta \cong \Delta'$ (Δ and Δ' simplicial complexes)	Δ and Δ' are isomorphic as posets

Chapter 1

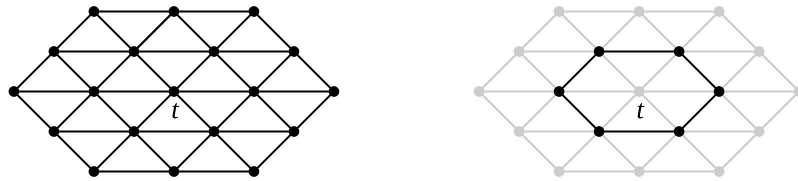
Buildings

We start this chapter by giving the pertinent terminology related to simplicial complexes and buildings. We then consider the affine buildings Ξ_n and Δ_n naturally associated to $\mathrm{SL}_n(K)$ and $\mathrm{Sp}_n(K)$, respectively, where K is a local field. In particular, we use lattices to describe Ξ_n and Δ_n, and we use the fact that Δ_n is a subcomplex of Ξ_{2n} to obtain information about Δ_n.

1.1 Definitions

Following [6, pp. 27 – 30, 63] and [20, p. 32], we review some definitions related to simplicial complexes. A *simplicial complex* Δ with vertex set $V(\Delta)$ is a set of finite subsets of $V(\Delta)$, called *simplices*, such that every singleton $\{v\}$ is a simplex and every subset (including the empty set) of a simplex A is a simplex, called a *face* of A. Note that a simplicial complex is also a poset, ordered by the face relation.[1] Write $A \in \Delta$ if A is a simplex of Δ. If v is a vertex of a simplex A, then v and A are *incident*. Call two (distinct) vertices $v, v' \in \Delta$ *incident* if they are in a common simplex; a vertex is incident to itself. A *maximal simplex* of Δ is a simplex not properly contained in any simplex. A *subcomplex* of Δ is a subset Δ' that contains, for each of its elements A, all the faces of A; hence, Δ' is a simplicial complex with vertex set $V(\Delta') \subseteq V(\Delta)$. The *dimension* of a simplex A, denoted $\dim(A)$, is $\mathrm{Card}(A) - 1$, one less than the number of vertices in A, and an *m-simplex* is an m-dimensional simplex. If A is a face of a simplex B such that $\dim(B) - \dim(A) = 1$, then A is a *codimension-*

[1] In other words, if Δ is a simplicial complex and $A \leq B$ denotes A and B are simplices of Δ with A a face of B, then $\leq$ is reflexive ($A \leq A$ for all simplices A of Δ), antisymmetric ($A \leq B$ and $B \leq A$ imply $A = B$ for all simplices A, B of Δ), and transitive ($A \leq B$ and $B \leq C$ imply $A \leq C$ for all simplices A, B, C of Δ).



(d) A thin chamber complex Δ. (e) Link of central vertex $t \in \Delta$.

Figure 1.1: Chamber complexes.

one face of B. An *isomorphism* between two simplicial complexes Δ and Δ' is a bijection $\varphi : V(\Delta) \to V(\Delta')$ taking an m-simplex of Δ to an m-simplex of Δ', and an *automorphism* of a simplicial complex Δ is an isomorphism of Δ with itself. For two simplicial complexes Δ and Δ', the *join* of Δ and Δ', denoted $\Delta \cup \Delta'$, is the simplicial complex with vertex set the disjoint union of $V(\Delta)$ and $V(\Delta')$ and one simplex $A \cup A'$ for every $A \in \Delta$ and $A' \in \Delta'$.

Example. *Let Δ be the simplicial complex with vertex set $\{a, b\}$ and simplices $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, and let Δ' be the simplicial complex with vertex set $\{c, d\}$ and simplices $\emptyset$, $\{c\}$, $\{d\}$, $\{c, d\}$. Then the join of Δ and Δ' is the simplicial complex $\Delta \cup \Delta'$ with vertex set $\{a, b, c, d\}$ and simplices $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$, $\{a, b, c, d\}$. In particular, the join of two disjoint edges of a 3-simplex is the whole 3-simplex.*

From now on, Δ is a simplicial complex all of whose maximal simplices have the same (finite) dimension, say n. Then Δ is an *n-dimensional simplicial complex*, and a *codimension-one simplex* is an $(n-1)$-simplex. Two maximal simplices are *adjacent* if they share a codimension-one face, and a *gallery of length m* connecting two maximal simplices C and C' is a sequence of maximal simplices $C = C_0, \ldots, C_m = C'$ such that C_i and C_{i+1} are adjacent for all $0 \leq i \leq m - 1$. If any two maximal simplices of Δ can be connected by a gallery, then Δ is a *chamber complex*, and the maximal simplices of Δ are *chambers*. A chamber complex is *thin* if every codimension-one simplex is a face of exactly two chambers, and a chamber complex is *thick* if every codimension-one simplex is a face of at least three chambers. Figure 1.1(d) gives an example of a thin chamber complex; a chamber is a 2-simplex, an "equilateral triangle" together with its interior.

A *labelling* of a chamber complex Δ by a set I is a map $V(\Delta) \to I$ such that the vertices of each chamber of Δ are in bijection with the elements of I; the elements of I are *labels* or *types*. One way to think of a labelling is as a coloring of the elements of $V(\Delta)$, where there are exactly $n + 1$ colors and distinct, incident vertices are different colors. If Δ is a labellable chamber complex, then labelling the vertices of one chamber determines the labels on the other vertices of Δ; i.e., any two labellings of Δ differ by a bijection on the sets of labels. Moreover, a labelling of Δ gives a label to every simplex of Δ—the label of a simplex A is the set of labels of the vertices of A.

Let Δ be a chamber complex and $A \in \Delta$. Then the *link of* A *in* Δ, denoted $\mathrm{lk}_\Delta A$, is the subcomplex of Δ consisting of the simplices $B \in \Delta$ such that $B \cap A = \emptyset$ and there is a chamber of Δ containing both A and B. More concretely, take the set of all chambers of Δ containing A, and delete A, along with any incident simplices. Figure 1.1(e) shows $\mathrm{lk}_\Delta t$, where Δ is the chamber complex in Figure 1.1(d) and t its central vertex. By [6, p. 31], $\mathrm{lk}_\Delta A$ is isomorphic (as a poset) to the subposet of Δ consisting of those simplices containing A; the isomorphism takes $B \in \mathrm{lk}_\Delta A$ to $B \cup A$, where $B \cup A$ is the join of A and B.

Definition. A *(thick) building* is a thick chamber complex Δ that can be written as the union of subcomplexes called *apartments*, each of which is a thin chamber complex, satisfying the following axioms.

1. If A and B are simplices of Δ, then there is an apartment containing both of them.

2. If Σ and Σ' are apartments of Δ containing simplices A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

A collection $\mathcal{A}$ of subcomplexes of Δ satisfying the above axioms is a *system of apartments* for Δ.

Remarks.

1. This definition is arguably the easiest possible definition of a building; for others, see [6, p. 76], [20, p. 52], or [35, p. 27].

2. Taking both A and B to be the empty simplex in the second axiom shows that any two apartments of a building are isomorphic.

3. A building need not be treated as a simplicial complex (see [36, p. 3]).

4. Every building is labellable.

5. By [6, Proposition IV.1.3], if Δ is a building, then $\mathrm{lk}_\Delta A$ is a building for any simplex $A \in \Delta$; in particular, $\mathrm{lk}_\Delta A$ is a chamber complex.

6. When we talk about a building Δ, we do not require that a specific system of apartments for Δ be part of the structure of Δ; as shown in Section IV.4 of [6], a building always admits a canonical system of apartments.

A building is *spherical* if all its apartments are finite; i.e., if all its apartments have finitely many vertices (see [6, p. 92]).

Examples.

1. *The $(q+1)$-regular tree is a building: a chamber is an edge, and an apartment is a "copy" of the 2-regular tree. When q is a power of a prime, the building is naturally associated to $\mathrm{SL}_2(K)$, where K is a local field with residue field isomorphic to $\mathbb{F}_q$.*

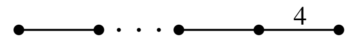
2. *Let q be a power of a prime. A building Δ can be described as follows: a chamber is an "equilateral triangle" (together with its interior), and an apartment is a "copy" of $\mathbb{R}^2$ tiled by chambers. Each side of a triangle (codimension-one face of the building) is a side of exactly $q+1$ triangles.*

3. *Let $n \geq 1$, k a field, and V an $(n+1)$-dimensional k-vector space. A vertex of the spherical $A_n(k)$ building $\Xi_n^s(k)$ is a non-trivial, proper subspace of V, and a nested sequence $V_1 \subsetneq \cdots \subsetneq V_{i+1}$ of non-trivial, proper subspaces of V is an i-simplex of $\Xi_n^s(k)$. In particular, a maximal flag $V_1 \subsetneq \cdots \subsetneq V_n$, where $\dim_k V_i = i$, of non-trivial, proper subspaces of V is a chamber of $\Xi_n^s(k)$. Two chambers $V_1 \subsetneq \cdots \subsetneq V_n$ and $V_1' \subsetneq \cdots \subsetneq V_n'$ are adjacent if $V_j = V_j'$ for all except possibly one value of j. A basis $\{v_1, \ldots, v_{n+1}\}$ of V specifies an apartment of $\Xi_n^s(k)$; the simplices of the apartment are the subspaces spanned by a proper subset of the basis, as well as all nested sequences of such subspaces (see [35, pp. 4 – 5]).*

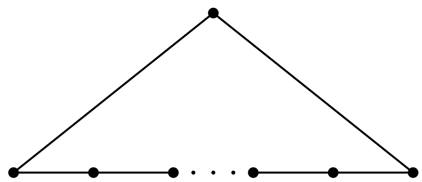
Before we proceed, we give some terminology related to Coxeter groups following [20, pp. 2 – 3, 40] and [36, p. 3]. Fix a finite set[2] S, and let $m : S \times S \to \mathbb{Z}^+ \cup \{\infty\}$ be a function such that $m(s,s) = 1$ for all $s \in S$ and $m(s,t) = m(t,s)$ for all $s,t \in S$. If W is the group generated by S with presentation $s^2 = 1$ for all $s \in S$ and $(st)^{m(s,t)} = 1$ for all $s,t \in S$, then the pair (W,S) is a *Coxeter system* and W is a *Coxeter group*.[3] Note that $m(s,t) = \infty$ means that there is no relation between s and t and $m(s,t) = 2$ implies $st = ts$ (since $s^2 = 1 = t^2$). The *rank* of W is $\mathrm{Card}(S)$. If W is finite and

[2] In general, the set need not be finite.
[3] Calling W a Coxeter group is actually an abuse of language. See [20, p. 3].

(a) Type A_n Coxeter diagram (n vertices).



(b) Type C_n Coxeter diagram (n vertices).



(c) Type $\widetilde{A}_n$ Coxeter diagram ($n+1$ vertices, $n \geq 2$).

(d) Type $\widetilde{C}_n$ Coxeter diagram ($n+1$ vertices).

Figure 1.2: Some Coxeter diagrams.

of rank n, it is the symmetry group of a triangulation of the $(n-1)$-sphere S^{n-1} and is said to be of spherical type. More generally, a Coxeter group of rank n acts non-trivially on $\mathbb{R}^n$.

A *Coxeter diagram* is used to keep track of the $m(s,t)$ describing the Coxeter system (W,S) as follows: there is a vertex corresponding to each $s \in S$, and the s-vertex and the t-vertex are connected by an edge if $m(s,t) \geq 3$, where the edge is labelled with $m(s,t)$ when $m(s,t) \geq 4$. The only Coxeter diagrams that concern us are those of type A_n, C_n, $\widetilde{A}_n$, and $\widetilde{C}_n$ for $n \geq 2$, which are given in Figures 1.2(a), 1.2(b), 1.2(c), and 1.2(d), respectively. A Coxeter diagram is connected if for all $s,t \in S$, there is a sequence $s = s_1, \ldots, s_m = t$ such that $m(s_i, s_{i+1}) \geq 3$ for all $1 \leq i \leq m-1$, in which case the corresponding Coxeter system is irreducible. Let (W,S) be a Coxeter system. For a subset $T \subseteq S$, write $\langle T \rangle$ for the subgroup of W generated by the elements of T. A Coxeter system (W,S) has an associated *Coxeter complex* $\Sigma = \Sigma(W,S)$, which is the simplicial complex with simplices the cosets of W of the form $w\langle T \rangle$ for $w \in W$ and a proper (possibly empty) subset T of S and face relation the opposite of the inclusion relation. By [20, p. 41], a Coxeter complex Σ is a chamber complex; hence, the chambers of Σ have the form $w\langle \emptyset \rangle = \{w\}$ for $w \in W$, and the codimension-one simplices have the form $w\langle s \rangle = \{w, ws\}$ for $w \in W$ and $s \in S$.

Let Δ be a building. By [20, pp. 54 – 55], there is a Coxeter system (W,S) such that every apartment of Δ is isomorphic (as a simplicial complex) to the Coxeter complex $\Sigma(W,S)$, where $\mathrm{Card}(S)$ is the number of vertices in a chamber of Δ, in which case W is the *Weyl group* of Δ. If Δ is a building whose associated Coxeter diagram has type M, then Δ is a building of *type* M. By [51, p. 35], a vertex of the type $\widetilde{A}_n$ (resp., type $\widetilde{C}_n$) Coxeter diagram is *special* if, when we delete it and all incident edges from the Coxeter diagram, we obtain the Coxeter diagram of type A_n

(resp., of type C_n). If Δ is a building of type $\widetilde{A}_n$ (resp., of type $\widetilde{C}_n$), then a vertex of Δ is *special* if it corresponds to a special vertex of its associated Coxeter diagram. In particular, every vertex of a type $\widetilde{A}_n$ building is special, but only two vertices in each chamber of a type $\widetilde{C}_n$ building are special.

Our focus is on a particular type of *affine building*, a building whose apartments can be visualized as a tiling of Euclidean space.[4] We think of a building as a simplicial complex naturally associated to a group such as $\mathrm{GL}_n(K), \mathrm{SL}_n(K)$, or $\mathrm{Sp}_n(K)$ (K a field), though in general, a building need not be associated to a group (see [31, p. 3]). Let Δ be a building and $\mathcal{A}$ a system of apartments for Δ. Suppose G is a group that acts *strongly transitively* on Δ; i.e., suppose

- the action of each $g \in G$ determines a simplicial complex automorphism of Δ such that
  - $t \in \Delta$ a vertex of type a implies gt also a vertex of type a and
  - $\Sigma \in \mathcal{A}$ implies $g\Sigma \in \mathcal{A}$,
- G acts transitively on the set of chambers of Δ, and the stabilizer of a chamber C is transitive on the set of apartments containing C.

By [6, p. 99], the second condition above is equivalent to requiring that G act transitively on $\mathcal{A}$ and that the stabilizer of an apartment Σ is transitive on the set of chambers of Σ. Fix an apartment $\Sigma \in \mathcal{A}$ and a chamber $C \in \Sigma$, and let B be the stabilizer in G of C.[5] Then the simplices of Δ are the cosets in G of the form gP, where P is a subgroup of G containing B. Equivalently, the simplices of Δ are the left cosets in G containing a left coset of B (in G) by [6, p. 111]. A simplex $A \in \Delta$ is a face of a simplex $A' \in \Delta$ if $A \supseteq A'$; hence, the chambers of Δ are the left cosets of B in G (for more details, see [6, Section V.2]). Note that if N is the stabilizer in G of Σ, then the Weyl group W of Δ satifies $W \cong N/(B \cap N)$ (see [6, p. 100]).

Remarks.

[4]The tiles are the chambers of the apartment.

[5]Even though we sometimes use B for a simplex of a simplicial complex, we follow convention and use B (instead of, say, H) for the stabilizer in G of a chamber $C \in \Delta$. The group B is sometimes called a "minimal parabolic" subgroup of G, with the stabilizer (in G) of a simplex of Δ being a "parabolic" subgroup of G. More accurately, if Δ is a spherical building, then B is a *minimal parabolic* or *Borel* subgroup of G, and the stabilizer (in G) of a simplex of Δ is a *parabolic* subgroup of G. On the other hand, if Δ is an affine building, then B is an *Iwahori* subgroup of G, and the stabilizer (in G) of a simplex of Δ is a *parahoric* subgroup of G.

1. There is no one-to-one correspondence between a group such as $\mathrm{GL}_n(K)$ and a building. For example, if K is a local field treated as any other field (i.e., ignore the discrete valuation that makes K a *local* field), then the building associated to $\mathrm{SL}_n(K)$ is the spherical $A_{n-1}(K)$ building. On the other hand, if we make full use of the discrete valuation of K, the building associated to $\mathrm{SL}_n(K)$ is the affine building associated to $\mathrm{SL}_n(K)$.

2. We write "the affine building of G" to refer to a specific ad-hoc construction of an affine building on which G acts strongly transitively.

3. It will be convient for us to describe our buildings in terms of homothety classes of lattices rather than the left cosets of G given previously.

4. In the rest of this thesis, we often use the property that any two simplices of a building are contained in a common apartment of the building. In particular, every simplex of a building is contained in an apartment of the building (recall that the empty set is a simplex).

1.2 The affine building Ξ_n of $\mathrm{SL}_n(K)$

From now on, K is a local field with discrete valuation "ord," valuation ring $\mathcal{O} = \{\alpha \in K : \mathrm{ord}(\alpha) \geq 0\}$, uniformizing parameter π ($\mathrm{ord}(\pi) = 1$), and (finite) residue field[6] $k \cong \mathbb{F}_q$. For any finite-dimensional K-vector space V, a *lattice* of V is a free $\mathcal{O}$-submodule of V generated by a K-basis of V; i.e., a free $\mathcal{O}$-module of rank $\dim_K V$. Two lattices L and L' of V are *homothetic* if there is an $\alpha \in K^\times$ such that $L' = \alpha L$, and $[L]$ denotes the homothety class of the lattice L.

To describe the affine building Ξ_n of $\mathrm{SL}_n(K)$ in terms of lattices (see [35, p. 115]), let V be an n-dimensional K-vector space. The homothety classes of lattices of V are the vertices of Ξ_n. Two vertices t and t' of Ξ_n are *incident* if there are representatives $L \in t$ and $L' \in t'$ such that $\pi L \subseteq L' \subseteq L$; i.e., $L'/\pi L$ is a k-subspace of $L/\pi L$ by the Correspondence Theorem.[7] Note that the incidence relation is reflexive and symmetric (if $\pi L \subseteq L' \subseteq L$, then $\pi L' \subseteq \pi L \subseteq L'$). It is also true that if $t_0, \ldots, t_i$ are distinct vertices of Ξ_n such that any two are incident, then by

[6]We assume that our residue field is finite so that we can count when we drop to the quotient; in general, the residue field of a local field need not be finite. On the other hand, if K is a finite extension of $\mathbb{Q}_p$ for some prime p, then the residue field of K is finite.

[7]More precisely, we use the Correspondence Theorem for Modules. From now on, "the Correspondence Theorem" refers to the Correspondence Theorem for Modules.

renumbering if necessary, there are representatives $L_j \in t_j$ for all $0 \leq j \leq i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_i \subsetneq L_0$ (see [20, pp. 322 – 323]). Since we use the arguments later, we give the details here.

Call two distinct, incident vertices of Ξ_n *adjacent*.

Proposition 1.2.1. *Let $t, t' \in \Xi_n$ be adjacent vertices. If $L \in t$, then there is a unique representative $L' \in t'$ such that $\pi L \subsetneq L' \subsetneq L$.*

Proof. First note that since t and t' are incident, they have representatives $M \in t$ and $M' \in t'$ such that $\pi M \subseteq M' \subseteq M$. Since $t \neq t'$, $\pi M \subsetneq M' \subsetneq M$. Moreover, M and L homothetic implies that there is an $\alpha \in K^\times$ such that $L = \alpha M$; hence, $\pi L = \pi\alpha M \subsetneq \alpha M' \subsetneq \alpha M = L$. Let $L' = \alpha M'$. To see that L' is unique, let $L'' \in t'$ such that $\pi L \subsetneq L'' \subsetneq L$. Since L' and L'' are homothetic, there is a $\beta \in K^\times$ such that $L'' = \beta L'$. Write $\beta = \pi^m u$ for some $m \in \mathbb{Z}$ and $u \in \mathcal{O}^\times$. Then $\pi L \subsetneq L' \subsetneq L$ implies $\pi^{m+1} L \subsetneq \pi^m L' \subsetneq \pi^m L$; i.e., $\pi^{m+1} L \subsetneq L'' \subsetneq \pi^m L$. Since $\pi L \subsetneq L'' \subsetneq L$, $L = \pi^m L$; i.e., $m = 0$, $\beta \in \mathcal{O}^\times$, and $L'' = \beta L' = L'$. ∎

Proposition 1.2.2. *If $t_0, \ldots, t_i \in \Xi_n$ are vertices such that any two are adjacent, then there are representatives $L_j \in t_j$ for all $0 \leq j \leq i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_i \subsetneq L_0$.*

Proof. First note that $i \geq 1$. We proceed by induction on i. If $i = 1$, the last proposition establishes the claim. Suppose $i \geq 1$ and that the claim holds for any set of i vertices of Ξ_n, any two of which are adjacent. Let $t_0, \ldots, t_i$ be distinct vertices of Ξ_n such that any two are adjacent. By the induction hypothesis, there are representatives $L_0 \in t_0$ and $M_j \in t_j$ for all $1 \leq j \leq i-1$ such that $\pi L_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i-1} \subsetneq L_0$. Since t_i is adjacent to t_0, the last proposition implies that t_i has a unique representative M_i such that $\pi L_0 \subsetneq M_i \subsetneq L_0$. By [20, p. 322], either $M_i \subsetneq M_1$ or $M_i \supsetneq M_1$. If $M_i \subsetneq M_1$, set $L_1 = M_i$ and $L_j = M_{j-1}$ for all $2 \leq j \leq i$. Otherwise, let $j \in \{2, \ldots, i-1\}$ be minimal such that $M_i \supsetneq M_j$ (such a j exists since $M_i \subsetneq L_0$). If $j = i-1$, set $L_r = M_r$ for all $1 \leq r \leq i$. Otherwise, $M_i \subsetneq M_{j+1}$, and setting $L_r = M_r$ for all $1 \leq r \leq j$, $L_{j+1} = M_i$, and $L_r = M_{r-1}$ for all $j+2 \leq r \leq i$ finishes the proof. ∎

It follows from the Correspondence Theorem that we can think of an i-simplex of Ξ_n as being given by a nested sequence $U_1 \subsetneq \cdots \subsetneq U_i$ of non-trivial, proper k-subspaces of $L_0/\pi L_0$ for some lattice L_0 of V. In particular, since $L_0/\pi L_0$ is an n-dimensional k-vector space for any lattice L_0 of V, we can think of a maximal simplex or *chamber* of Ξ_n as being given by a maximal flag $U_1 \subsetneq \cdots \subsetneq U_{n-1}$ of non-trivial, proper k-subspaces of $L_0/\pi L_0$ for some lattice L_0 of V.[8] But by the Correspondence Theorem, there are unique $\mathcal{O}$-submodules L_i of L_0 containing πL_0 such that $L_i/\pi L_0 = U_i$ and $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$. Since both L_0 and πL_0 are free $\mathcal{O}$-modules of rank n, L_i is a lattice of V for all $1 \leq i \leq n-1$. Thus, a chamber of Ξ_n has n vertices $t_0, \ldots, t_{n-1}$, which have representatives $L_i \in t_i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ and $L_1/\pi L_0 \subsetneq \cdots \subsetneq L_{n-1}/\pi L_0$ is a maximal flag of non-trivial, proper k-subspaces of $L_0/\pi L_0$. Alternatively, a chamber of Ξ_n corresponds to a chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ such that $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n-1$.

Convention. From now on, we write that a chamber of Ξ_n corresponds to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ only when the lattices $L_0, \ldots, L_{n-1}$ satisfy $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ with $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n-1$.

Remark. If the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0$ corresponds to a chamber of Ξ_n, then the condition $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n-1$ implies $[L_j : \pi L_0] = q^j$ for all $1 \leq j \leq n-1$; in particular, $[L_0 : L_{n-1}] = [L_0 : \pi L_0]/[L_{n-1} : \pi L_0] = q$.

The building Ξ_2 is the $(q+1)$-regular tree (see [42, Chapter II]): a chamber is a 1-simplex, and an apartment is a "copy" of the 2-regular tree. A partial picture of the 3-regular tree is in Figure 1.3(a); Figure 1.3(b) shows a partial picture of the 2-regular tree. In the building Ξ_3, a chamber is a 2-simplex—an "equilateral triangle" together with its interior—and an apartment is a "copy" of the plane. Figure 1.4 shows a partial picture of an apartment of Ξ_3. Note that by [6, p. 137], Ξ_n has type $\widetilde{A}_{n-1}$; hence, all its vertices are special.

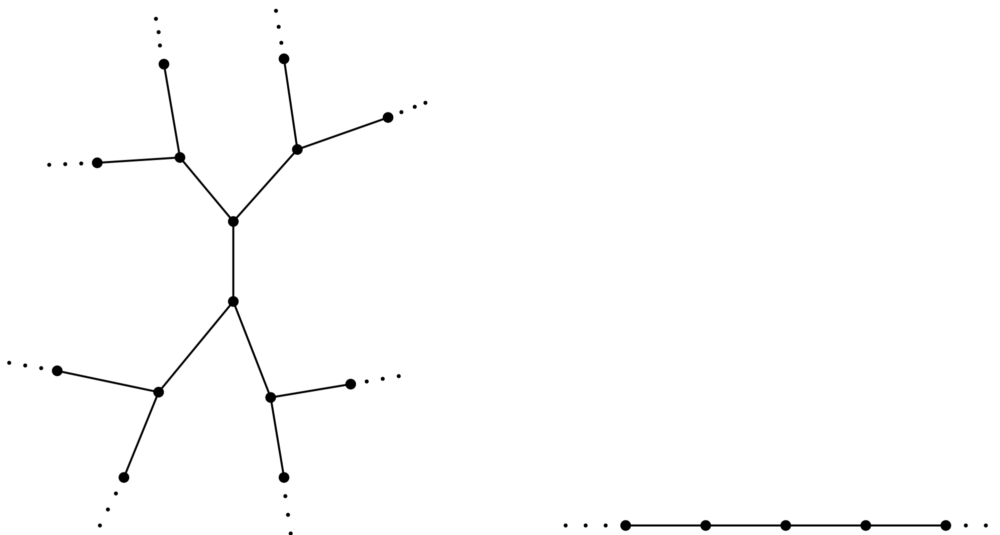
Following [20, p. 323], we describe the apartments of Ξ_n. A *frame in V for Ξ_n* is an unordered n-tuple $\lambda_1, \ldots, \lambda_n$ of lines (1-dimensional K-subspaces) in V such that $V = \lambda_1 + \cdots + \lambda_n$. Then a vertex $t \in \Xi_n$ lies in the apartment specified by the frame $\lambda_1, \ldots, \lambda_n$ if for any representative $L \in t$, $L = M_1 + \cdots + M_n$, where M_i is a lattice of λ_i for $1 \leq i \leq n$.

Lemma 1.2.1.

1. *Every basis of V specifies an apartment of Ξ_n.*

2. *If Σ is an apartment of Ξ_n, then there is a basis $\{v_1, \ldots, v_n\}$ of V such that every vertex of Σ has the form $[\mathcal{O}\pi^{a_1} v_1 + \cdots + \mathcal{O}\pi^{a_n} v_n]$ for some $a_i \in \mathbb{Z}$.*

[8] The condition that $U_1 \subsetneq \cdots \subsetneq U_{n-1}$ be a *flag* of k-subspaces of $L_0/\pi L_0$ means that in addition to the inclusion relation among the U_i, $\dim_k U_i = i$ for all $1 \leq i \leq n-1$.



(a) 3-regular tree. (b) 2-regular tree.

Figure 1.3: Building and apartment.

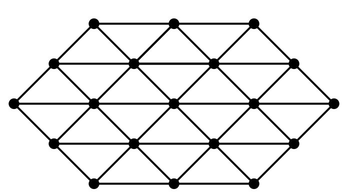


Figure 1.4: Partial picture of apartment of Ξ_3.

Remark. This lemma does not assert that there is a one-to-one correspondence between the set of bases of V and the set of frames in V for Ξ_n. For example, if $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of V, then $\{-v_1, \ldots, -v_n\}$ is a basis of V that determines the same apartment of Ξ_n as does $\mathcal{B}$.

Proof. For part 1, if $\{v_1, \ldots, v_n\}$ is a basis of V, let $\lambda_i = Kv_i$ for all $1 \leq i \leq n$. Then $\{v_1, \ldots, v_n\}$ a basis of V implies $V = \lambda_1 + \cdots + \lambda_n$. It follows that $\lambda_1, \ldots, \lambda_n$ is a frame in V for Ξ_n and hence specifies an apartment of Ξ_n.

For part 2, let Σ be an apartment of Ξ_n, and let $\lambda_1, \ldots, \lambda_n$ be a frame in V for Ξ_n specifying Σ. For $1 \leq i \leq n$, let $v_i \in \lambda_i$ be non-zero. Then $\lambda_i = Kv_i$ for all i and $V = \lambda_1 + \cdots + \lambda_n$ imply $\{v_1, \ldots, v_n\}$ is a basis of V. Furthermore, a lattice of λ_i has the form $\mathcal{O}\alpha_i v_i$ for some $\alpha_i \in K^\times$. Since $\alpha_i \in K^\times$ has the form $\alpha_i = \pi^{a_i} u$ for some $a_i \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, $\mathcal{O}\alpha_i v_i = \mathcal{O}\pi^{a_i} v_i$; i.e., every vertex of Σ has the form $[\mathcal{O}\pi^{a_1} v_1 + \cdots + \mathcal{O}\pi^{a_n} v_n]$ for some $a_i \in \mathbb{Z}$. □

1.3 The affine building Δ_n of $\mathrm{Sp}_n(K)$

We now turn to Δ_n, the affine building of $\mathrm{Sp}_n(K)$.[9] For the rest of this chapter, V is a $2n$-dimensional K-vector space endowed with the non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ defined on the standard unit K-basis vectors $e_1, \ldots, e_{2n}$ of V by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } 1 \leq i \leq n \text{ and } j = n+i, \\ -1 & \text{if } n+1 \leq i \leq 2n \text{ and } j = i-n, \\ 0 & \text{otherwise} \end{cases}$$

and a lattice of V is a free $\mathcal{O}$-module of rank $2n$. A basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V is *symplectic* if $\langle u_i, w_j \rangle = \delta_{ij}$ (Kronecker delta) and $\langle u_i, u_j \rangle = 0 = \langle w_i, w_j \rangle$ for all i, j.[10] Let $f_i = e_{n+i}$ for all $1 \leq i \leq n$ and $\mathcal{B}_0 = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, a symplectic basis of V. A subspace U of V is *totally isotropic* if $\langle u, u' \rangle = 0$ for all $u, u' \in U$.

[9] We do not use the definition of $\mathrm{Sp}_n(K)$ in this section and its subsections, though we refer to the fact that $\mathrm{Sp}_1(K) = \mathrm{SL}_2(K)$ when we look more closely at the special vertices of Δ_n. We give a definition of $\mathrm{Sp}_n(K)$ when we consider the action of $\mathrm{GL}_{2n}(K)$ on Ξ_{2n}.

[10] This is the "standard" definition of a symplectic basis over a field (as in [40, p. 701] and [43, p. 3]). If we relax this definition and call a basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V symplectic if $\langle u_i, w_j \rangle = \alpha\delta_{ij}$ for some $\alpha \in K^\times$ and $\langle u_i, u_j \rangle = 0 = \langle w_i, w_j \rangle$ for all i, j, then $\{\alpha^{-1}u_1, \ldots, \alpha^{-1}u_n, w_1, \ldots, w_n\}$, for example, is a symplectic basis of V as we originally defined it. Although the "relaxed" definition allows us to simplify some of our results (for example, we could combine Proposition 2.2.6 and Theorem 2.2.1 into one theorem), the "normalized" version (i.e., the "relaxed" version with $\alpha = 1$) is more convenient, so we use the "normalized" version throughout this thesis.

We describe Δ_n in terms of lattices following [20, pp. 336 – 337]. A lattice L of V is *primitive* if $\langle L, L\rangle = \{\langle x, y\rangle : x, y \in L\} \subseteq \mathcal{O}$ and the alternating k-bilinear form $\langle\cdot,\cdot\rangle_0$ on $L/\pi L$ given by $\langle x+\pi L, y+\pi L\rangle_0 = \langle x, y\rangle + \pi\mathcal{O}$ is non-degenerate. The vertices of Δ_n are the homothety classes of lattices of V with a representative L such that $\langle L, L\rangle \subseteq \pi\mathcal{O}$ and there is a primitive lattice L_0 with $\pi L_0 \subseteq L \subseteq L_0$; equivalently, $L/\pi L_0$ is a totally isotropic k-subspace of $L_0/\pi L_0$. Two vertices $t, t' \in \Delta_n$ are *incident* if there are representatives $L \in t$ with $\langle L, L\rangle \subseteq \pi\mathcal{O}$ and $L' \in t'$ with $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and a primitive lattice L_0 such that either $\pi L_0 \subseteq L \subseteq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subseteq L \subseteq L_0$; i.e., either $L/\pi L_0 \subseteq L'/\pi L_0$ or $L'/\pi L_0 \subseteq L/\pi L_0$ is a nested sequence of (not necessarily non-trivial) totally isotropic k-subspaces of $L_0/\pi L_0$. Note that this incidence relation is reflexive and symmetric.

Proposition 1.3.1. *The building Δ_n is a subcomplex of the building Ξ_{2n}.*

Proof. Since a vertex of Δ_n is a homothety class of lattices of V, a vertex of Δ_n is also a vertex of Ξ_{2n}. It thus remains to show that if t and t' are incident vertices of Δ_n, then they are incident as vertices of Ξ_{2n}. But t and t' incident in Δ_n implies that there are representatives $L \in t$ and $L' \in t'$ and a primitive lattice L_0 such that either $\pi L_0 \subseteq L \subseteq L' \subseteq L_0$ or $\pi L_0 \subseteq L' \subseteq L \subseteq L_0$. Thus, either $\pi L \subseteq \pi L' \subseteq \pi L_0 \subseteq L$ or $\pi L \subseteq \pi L_0 \subseteq L' \subseteq L$; i.e., if t and t' are incident vertices of Δ_n, then they are incident as vertices of Ξ_{2n}. ☐

Call two distinct, incident vertices of Δ_n *adjacent*.

Lemma 1.3.1. *Let $t, t' \in \Delta_n$ be adjacent vertices such that t has a primitive representative L. Then t' has a unique representative L' such that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$.*

Proof. First note that since t, t' are adjacent vertices of Ξ_{2n} by the last proposition, Proposition 1.2.1 applied to Ξ_{2n} implies that if the representative $L' \in t'$ exists as in the statement of the lemma, it is unique. It thus suffices to show that such an L' exists. Since t and t' are incident vertices of Δ_n, they have representatives $M \in t$ with $\langle M, M\rangle \subseteq \pi\mathcal{O}$ and $M' \in t'$ with $\langle M', M'\rangle \subseteq \pi\mathcal{O}$ and there is a primitive lattice L_0 such that either $\pi L_0 \subseteq M \subseteq M' \subseteq L_0$ or $\pi L_0 \subseteq M' \subseteq M \subseteq L_0$. Since $t \neq t'$, either $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$ or $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$. First suppose $\pi L_0 \subseteq M \subsetneq M' \subseteq L_0$. Then M and πL homothetic implies $\pi L = \pi^r M$ for some $r \in \mathbb{Z}$;[11] hence, $\pi L = \pi^r M \subsetneq \pi^r M' \subseteq \pi^r L_0 \subseteq \pi^{r-1} M = L$. Let $L' = \pi^r M'$. Since L is primitive, $\langle L, L\rangle \subseteq \mathcal{O}$, so $\langle \pi^{r-1} M, \pi^{r-1} M\rangle \subseteq \mathcal{O}$. On the other hand, $\langle M, M\rangle \subseteq \pi\mathcal{O}$

[11] More precisely, M and πL homothetic implies $\pi L = \alpha M$ for some $\alpha \in K^\times$. But we can write

implies $\langle \pi^{r-1}M, \pi^{r-1}M\rangle \subseteq \pi^{2(r-1)+1}\mathcal{O}$. Since $\pi^{2(r-1)+1}\mathcal{O} \subseteq \mathcal{O}$ if and only if $r \in \mathbb{Z}^+$, $\langle M', M'\rangle \subseteq \pi\mathcal{O}$ implies $\langle L', L'\rangle = \langle \pi^r M', \pi^r M'\rangle \subseteq \pi^{2r+1}\mathcal{O} \subseteq \pi\mathcal{O}$. In other words, $L' = \pi^r M' \in t'$ satisfies $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$, where the inclusion $L' \subsetneq L$ follows from the assumption that $t \neq t'$.

Similarly, suppose $\pi L_0 \subseteq M' \subsetneq M \subseteq L_0$. Since M and L are homothetic, $L = \pi^r M$ for some $r \in \mathbb{Z}$, so $\pi L = \pi^{r+1}M \subseteq \pi^{r+1}L_0 \subseteq \pi^r M' \subsetneq \pi^r M = L$. Let $L' = \pi^r M'$. Then L primitive implies $\langle L, L\rangle \subseteq \mathcal{O}$; hence, $\langle \pi^r M, \pi^r M\rangle \subseteq \mathcal{O}$. On the other hand, $\langle M, M\rangle \subseteq \pi\mathcal{O}$, so $\langle \pi^r M, \pi^r M\rangle \subseteq \pi^{2r+1}\mathcal{O}$; i.e., $r \in \mathbb{Z}^{\geq 0}$. Since $\langle M', M'\rangle \subseteq \pi\mathcal{O}$, $\langle L', L'\rangle = \langle \pi^r M', \pi^r M'\rangle \subseteq \pi^{2r+1}\mathcal{O} \subseteq \pi\mathcal{O}$. It follows that $L' = \pi^r M' \in t'$ such that $\langle L', L'\rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$, where we use the assumption that $t \neq t'$ to obtain the inclusion $\pi L \subsetneq L'$. □

By [20, p. 337], a maximal simplex or *chamber* of Δ_n has $n+1$ vertices $t_0, \ldots, t_n$ with representatives $L_i \in t_i$ such that L_0 is primitive, $\pi L_0 \subsetneq L_i \subsetneq L_0$ for all $1 \leq i \leq n$, and $L_1/\pi L_0 \subsetneq \cdots \subsetneq L_n/\pi L_0$ is a maximal flag of non-trivial, totally isotropic k-subspaces of $L_0/\pi L_0$. Alternatively, a chamber of Δ_n corresponds to a chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$, where L_0 is primitive, $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq i \leq n$, and $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n$.

Convention. From now on, we write that a chamber of Δ_n corresponds to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ only when the lattices $L_0, \ldots, L_n$ satisfy $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ with L_0 primitive, $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq i \leq n$, and $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n$.

Remark. If the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ corresponds to a chamber of Δ_n, then the condition $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \leq i \leq n$ implies $[L_j : \pi L_0] = q^j$ for all $1 \leq j \leq n$; in particular, $[L_0 : L_n] = [L_0 : \pi L_0]/[L_n : \pi L_0] = q^n$.

In the building Δ_2, a chamber is a 2-simplex—a "right isosceles triangle" together with its interior—and an apartment is a "copy" of the plane. Figure 1.5 shows a partial picture of an apartment of Δ_2. Note that by [43, p. 3], Δ_n has type $\widetilde{C}_n$, so only two vertices in each chamber of Δ_n are special.

Remarks.

1. This is just one possible construction of Δ_n. Although the construction seems to single out the vertices with a primitive representative, this is only for conve-

$\alpha = \pi^r u$ for some $r \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, and since M is a free $\mathcal{O}$-module, $\pi L = \alpha M = \pi^r M$. This is the argument we use from now on when we write "Since L and M are homothetic, $L = \pi^r M$ for some $r \in \mathbb{Z}$."

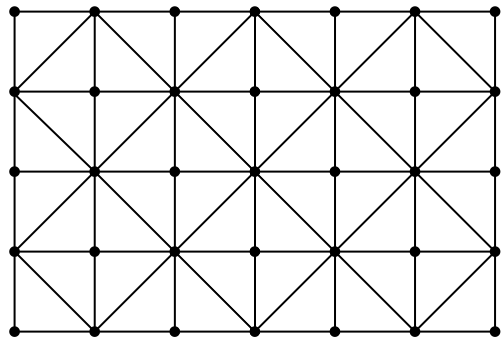


Figure 1.5: Partial picture of apartment of Δ_2.

nience's sake; the primitivity condition allows us to proceed in a natural manner in our counting arguments.

2. Every primitive lattice L of V represents a vertex of Δ_n since $\pi L \subseteq \pi L \subseteq L$ and $\langle \pi L, \pi L\rangle = \pi^2\langle L, L\rangle \subseteq \pi^2\mathcal{O}$.

1.3.1 The buildings Δ_n and Ξ_{2n}

In contrast to Ξ_{2n}, not every lattice of V represents a vertex of Δ_n. Furthermore, if L_0 is a primitive lattice of V, then $L_0/\pi L_0$ is a $2n$-dimensional k-vector space; hence, a flag of k-subspaces of $L_0/\pi L_0$ corresponding to a chamber (n-simplex) of Δ_n is roughly half of a flag of k-subspaces corresponding to a chamber ($(2n-1)$-simplex) of Ξ_{2n}.

Lemma 1.3.2. *Let $C \in \Delta_n$ be a chamber corresponding to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. If $\emptyset \neq A \in \mathrm{lk}_{\Xi_{2n}} C$ with $\dim(A) = i$, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of V such that $L_n \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L_0$.*

Proof. Let $\emptyset \neq A \in \mathrm{lk}_{\Xi_{2n}} C$. Then $A \cup C$ (the join of A and C) is an $(n+i+1)$-simplex of Ξ_{2n} containing C. Note that $n+i+1 \geq n+1$. Then there are lattices $M_0', \ldots, M_{n+i+1}'$ of V such that $[M_0'], \ldots, [M_{n+i+1}']$ are the vertices of $A \cup C$ and $\pi M_0' \subsetneq M_1' \subsetneq \cdots \subsetneq M_{n+i+1}' \subsetneq M_0'$ by Proposition 1.2.2 applied to Ξ_{2n}. If $M_j' \in [L_0]$ for some $1 \leq j \leq n+i+1$, then $\pi M_j' \subsetneq \cdots \subsetneq \pi M_{n+i+1}' \subsetneq \pi M_0' \subsetneq M_1' \subsetneq \cdots \subsetneq M_j'$; i.e., by changing the representative of the homothety classes if necessary, we can assume that $M_0' \in [L_0]$ and $\pi M_0' \subsetneq M_1' \subsetneq \cdots \subsetneq M_{n+i+1}' \subsetneq M_0'$. Since M_0' and L_0 are homothetic, there is an $\alpha \in K^\times$ such that $L_0 = \alpha M_0'$. Then $\pi L_0 = \pi\alpha M_0' \subsetneq \alpha M_1' \subsetneq \cdots \subsetneq \alpha M_{n+i+1}' \subsetneq \alpha M_0' = L_0$. By Proposition 1.2.1 applied to Ξ_{2n}, $L_1 = \alpha M_j'$ for some

$1 \leq j \leq n+i+1$. But $[L_1 : \pi L_0] = q$ implies $L_1 = \alpha M_1'$. Proceeding inductively, suppose $1 \leq j \leq n-1$ and $L_j = \alpha M_j'$. Since $L_{j+1} = \alpha M_\ell'$ for some $j+1 \leq \ell \leq n+i+1$ (by Proposition 1.2.1 applied to Ξ_{2n}) and $[L_{j+1} : L_j] = q$, $L_{j+1} = \alpha M_{j+1}'$. It follows that $L_j = \alpha M_j'$ for all $0 \leq j \leq n$. Let $M_j = \alpha M_{n+j}'$ for all $1 \leq j \leq i+1$. Then $L_n = \alpha M_n' \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq \alpha M_0' = L_0$ and $[M_1], \ldots, [M_{i+1}]$ are the vertices of A. □

Lemma 1.3.3. *Let C be as in the last lemma and $\emptyset \neq A, B \in \mathrm{lk}_{\Xi_{2n}} C$ with A a face of B, $\dim(A) = i$, and $\dim(B) = j$. Then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of V and B corresponds to a chain of lattices $N_1 \subsetneq \cdots \subsetneq N_{j+1}$ of V, where*

$$L_n \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L_0, \qquad L_n \subsetneq N_1 \subsetneq \cdots \subsetneq N_{j+1} \subsetneq L_0,$$

and for all $1 \leq \ell \leq i+1$, there is an $1 \leq r \leq j+1$ such that $M_\ell = N_r$.

Proof. First note that since A is a face of B, $i = \dim(A) \leq \dim(B) = j$. By the last lemma, there are lattices $M_1, \ldots, M_{i+1}, N_1, \ldots, N_{j+1}$ of V such that

- $L_n \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L_0$,
- $L_n \subsetneq N_1 \subsetneq \cdots \subsetneq N_{j+1} \subsetneq L_0$,
- $[M_1], \ldots, [M_{i+1}]$ are the vertices of A, and
- $[N_1], \ldots, [N_{j+1}]$ are the vertices of B.

Since A is a face of B, the vertices of A are also vertices of B; hence, Proposition 1.2.1 applied to Ξ_{2n} implies that for all $1 \leq \ell \leq i+1$, there is an $1 \leq r \leq j+1$ such that $M_\ell = N_r$. □

Proposition 1.3.2. *For any chamber $C \in \Delta_n$, $\mathrm{lk}_{\Xi_{2n}} C$ is isomorphic (as a poset) to the spherical $A_{n-1}(k)$ building $\Xi_{n-1}^s(k)$.*

Proof. Let $C \in \Delta_n$ be a chamber corresponding to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. Then L_0/L_n is an n-dimensional k-vector space. Let $\Xi_{n-1}^s(k)$ be the spherical $A_{n-1}(k)$ building with i-simplices $(0 \leq i \leq n-2)$ given by nested sequences $V_1 \subsetneq \cdots \subsetneq V_{i+1}$ of non-trivial, proper k-subspaces of L_0/L_n. Note that since L_n and L_0 are free $\mathcal{O}$-modules of rank $2n$, any $\mathcal{O}$-submodule of L_0 containing L_n is a lattice of V. Moreover, by the Correspondence Theorem, there is a bijection between $\mathcal{O}$-submodules of L_0 containing L_n and k-subspaces of L_0/L_n. It follows from Lemma 1.3.2 that there is a bijection between 0-simplices (vertices) of $\mathrm{lk}_{\Xi_{2n}}C$ and 0-simplices of $\Xi^s_{n-1}(k)$. But the Correspondence Theorem also gives a bijection between chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of L_0 containing L_n and nested sequences $V_1 \subsetneq \cdots \subsetneq V_{i+1}$ of k-subspaces of L_0/L_n; hence, there is a bijection between i-simplices of $\mathrm{lk}_{\Xi_{2n}}C$ and i-simplices of $\Xi^s_{n-1}(k)$ for all $0 \leq i \leq n-2$ by Lemma 1.3.2. Let $\emptyset \in \mathrm{lk}_{\Xi_{2n}}C$ correspond to $\emptyset \in \Xi^s_{n-1}(k)$. Then we have a bijection between the simplices of $\mathrm{lk}_{\Xi_{2n}}C$ and the simplices of $\Xi^s_{n-1}(k)$. The last lemma shows that this bijection preserves the partial order (face) relation, so is a poset isomorphism. □

Corollary 1.3.1. *Every chamber of Δ_n is contained in exactly $\prod_{m=2}^{n}((q^m-1)/(q-1))$ chambers of Ξ_{2n}.*

Proof. Let $C \in \Delta_n$ be a chamber. Then [6, p. 31] implies that the number of chambers of Ξ_{2n} containing C is the number of chambers in $\mathrm{lk}_{\Xi_{2n}}C$, which is equal to the number of chambers in the spherical $A_{n-1}(k)$ building by the last proposition. Since the number of chambers in the spherical $A_{n-1}(k)$ building is

$$\prod_{m=0}^{n-2} \frac{q^{n-m}-1}{q-1} = \prod_{m=2}^{n} \frac{q^m-1}{q-1}$$

by the proof of Proposition 2.4 of [41], we are done. □

Since a chamber C of Δ_n corresponds to a chain of lattices of the form

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0, \tag{1.1}$$

a codimension-one face A of C corresponds to a chain of lattices of the form (1.1) with either L_j deleted for some $1 \leq j \leq n$ or with both L_0 and πL_0 deleted.

Lemma 1.3.4. *If C, C' is a gallery in Δ_n, then there is a gallery D, D' in Ξ_{2n} such that D (resp., D') contains C (resp., C') and $C \neq C'$ implies $D \neq D'$.*

Proof. If $C = C'$, then Corollary 1.3.1 implies that there is a chamber D in Ξ_{2n} containing C, and setting $D' = D$ finishes the proof. Now suppose $C \neq C'$, with C corresponding to the chain of lattices in (1.1). Since C and C' are adjacent chambers with $C \neq C'$, they share exactly one codimension-one simplex A of Δ_n. Our comments preceding this lemma thus imply that there is a $0 \leq j \leq n$ such that A corresponds to (1.1) with either L_j deleted if $1 \leq j \leq n$ or with both πL_0 and L_0 deleted if $j = 0$. Consequently, if t' is the vertex of C' not in A, then t' has a representative L'

such that C' corresponds to (1.1) with L_j replaced by L'. Note that $C \neq C'$ implies $L_j \neq L'$. We consider three cases, depending on the value of j.

If $1 \leq j \leq n-1$, let $U_{n+1}, \ldots, U_{2n-1}$ be proper k-subspaces of $L_0/\pi L_0$ such that

$$L_n/\pi L_0 \subsetneq U_{n+1} \subsetneq \cdots \subsetneq U_{2n-1}.$$

By the Correspondence Theorem, there are unique $\mathcal{O}$-submodules $L_{n+1}, \ldots, L_{2n-1}$ of L_0 containing πL_0 such that $L_i/\pi L_0 = U_i$ for all $n+1 \leq i \leq 2n-1$ and

$$L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0. \tag{1.2}$$

Furthermore, since L_n and L_0 are free $\mathcal{O}$-modules of rank $2n$, L_i is a lattice for all $n+1 \leq i \leq 2n-1$. Let D (resp., D') be the simplex of Ξ_{2n} with vertices the vertices of C (resp., of C'), together with the vertices $[L_{n+1}], \ldots, [L_{2n-1}]$. Then (1.1), (1.2), and our comments at the beginning of this proof imply that D and D' are chambers of Ξ_{2n} with exactly $2n-1$ vertices in common; i.e., D and D' are distinct, adjacent chambers of Ξ_{2n}.

Now suppose $j = n$, and recall that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L' \subsetneq L_0$ corresponding to C' implies $[L' : L_{n-1}] = q$. In particular, $L_n \cap L' \neq L_n$ and $L_n \cap L' \neq L'$ since $L_n \neq L'$. Furthermore, L_{n-1}, L_n, and L_0 free $\mathcal{O}$-modules of rank $2n$ and $L_{n-1} \subseteq L_n \cap L' \subsetneq L_n \subseteq L_n + L' \subseteq L_0$ imply $L_n \cap L'$ and $L_n + L'$ are lattices of V. Since $[L_n : L_{n-1}] = q$, $L_n \cap L' = L_{n-1}$. It follows from the Second Isomorphism Theorem[12] that $[L_n + L' : L'] = [L_n : L_n \cap L'] = q$, so $[L_n + L' : L_{n-1}] = [L_n + L' : L'][L' : L_{n-1}] = q^2$ and $[L_n + L' : L_n] = [L_n + L' : L_{n-1}]/[L_n : L_{n-1}] = q$. Let $U_{n+2}, \ldots, U_{2n-1}$ be proper k-subspaces of $L_0/\pi L_0$ such that $(L_n + L')/\pi L_0 \subsetneq U_{n+2} \subsetneq \cdots \subsetneq U_{2n-1}$. By the Correspondence Theorem, there are unique $\mathcal{O}$-submodules $L_{n+2}, \ldots, L_{2n-1}$ of L_0 containing πL_0 such that $L_i/\pi L_0 = U_i$ for all $n+2 \leq i \leq 2n-1$ and

$$L_{n-1} \subsetneq L_n, L' \subsetneq L_n + L' \subsetneq L_{n+2} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0. \tag{1.3}$$

As in the last paragraph, L_i is a lattice for all $n+2 \leq i \leq 2n-1$ since both L_n and L_0 are free $\mathcal{O}$-modules of rank $2n$. Let D (resp., D') be the simplex of Ξ_{2n} with vertices the vertices of C (resp., of C'), together with the vertices $[L_n + L'], [L_{n+2}], \ldots, [L_{2n-1}]$. Then (1.1), (1.3), and our previous comments imply that D and D' are chambers of Ξ_{2n} with exactly $2n-1$ vertices in common; i.e., D and D' are distinct, adjacent

[12] More precisely, we use the Second Isomorphism Theorem for Modules. From now on, "the Second Isomorphism Theorem" refers to the Second Isomorphism Theorem for Modules.

chambers of Ξ_{2n}.

Finally, suppose $j = 0$, and recall that $\pi L' \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L'$ corresponding to C' implies $[\pi^{-1}L_1 : L'] = [L_1 : \pi L'] = q$. Thus, $L_0 + L' \neq L_0$ and $L_0 + L' \neq L'$ since $L_0 \neq L'$. Furthermore, L_n, L_0, and $\pi^{-1}L_1$ free $\mathcal{O}$-modules of rank $2n$ and $L_n \subseteq L_0 \cap L' \subseteq L_0 \subsetneq L_0 + L' \subseteq \pi^{-1}L_1$ imply $L_0 \cap L'$ and $L_0 + L'$ are lattices of V. Since $[\pi^{-1}L_1 : L_0] = [L_1 : \pi L_0] = q$, $L_0 + L' = \pi^{-1}L_1$. It follows from the Second Isomorphism Theorem that $[L' : L_0 \cap L'] = [L_0 + L' : L_0] = q$, so $[\pi^{-1}L_1 : L_0 \cap L'] = [\pi^{-1}L_1 : L'][L' : L_0 \cap L'] = q^2$ and $[L_0 : L_0 \cap L'] = [\pi^{-1}L_1 : L_0 \cap L']/[\pi^{-1}L_1 : L_0] = q$. Let $U_{n+1}, \ldots, U_{2n-2}$ be proper k-subspaces of $(L_0 \cap L')/\pi L_0$ such that $L_n/\pi L_0 \subsetneq U_{n+1} \subsetneq \cdots \subsetneq U_{2n-2}$. By the Correspondence Theorem, there are unique $\mathcal{O}$-submodules $L_{n+1}, \ldots, L_{2n-2}$ of $L_0 \cap L'$ containing πL_0 such that $L_i/\pi L_0 = U_i$ for all $n+1 \leq i \leq 2n-2$ and

$$L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-2} \subsetneq L_0 \cap L' \subsetneq L_0, L' \subsetneq \pi^{-1}L_1. \tag{1.4}$$

As in the last paragraph, L_i is a lattice for all $n+1 \leq i \leq 2n-2$ since both L_n and L_0 are free $\mathcal{O}$-modules of rank $2n$. Let D (resp., D') be the simplex of Ξ_{2n} with vertices the vertices of C (resp., of C'), together with the vertices $[L_{n+1}], \ldots, [L_{2n-2}], [L_0 \cap L']$. Then (1.1), (1.4), and our previous comments imply that D and D' are chambers of Ξ_{2n} with exactly $2n-1$ vertices in common; i.e., D and D' are distinct, adjacent chambers of Ξ_{2n}. □

Proposition 1.3.3. *If $C_0, \ldots, C_m$ is a gallery in Δ_n, then there is a gallery $D_0, \ldots, D_r$ in Ξ_{2n} such that for all $0 \leq i \leq r$, D_i contains C_j for some $0 \leq j \leq m$.*

Proof. If $m = 0$, then Corollary 1.3.1 proves the claim with $r = 0$. If $m = 1$, the last lemma proves the claim with $r = 1$. Suppose $m \geq 2$. For $0 \leq i \leq m-1$, applying the last lemma to C_i, C_{i+1} gives adjacent chambers $D'_i, D''_{i+1} \in \Xi_{2n}$ such that D'_i (resp., D''_{i+1}) contains C_i (resp., C_{i+1}). It follows that there are chambers $D'_0, D''_1, D'_1, \ldots, D''_{m-1}, D'_{m-1}, D''_m$ such that

- D'_0 contains C_0,
- D''_i and D'_i contain C_i for all $1 \leq i \leq m-1$,
- D''_m contains C_m, and
- D'_i and D''_{i+1} are adjacent for all $0 \leq i \leq m-1$.

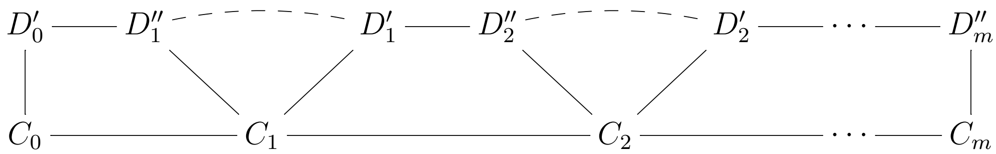


Figure 1.6: Pictorial representation of "lifting" a gallery of Δ_n to a certain gallery of Ξ_{2n}.

Figure 1.6 gives a pictorial representation of the situation, with the solid horizontal lines indicating adjacency and the lines from the top row to the bottom indicating containment. It therefore remains to show that for all $1 \leq i \leq m-1$, there is a gallery in Ξ_{2n} connecting D_i'' and D_i', all of whose chambers contain C_i (indicated by the broken lines in Figure 1.6).

Let $1 \leq i \leq m-1$, and recall that since both D_i'' and D_i' contain C_i, $D_i'' = A_i'' \cup C_i$ and $D_i' = A_i' \cup C_i$ for some chambers $A_i'', A_i' \in \mathrm{lk}_{\Delta_n} C_i$, where $\cup$ denotes the join. Furthermore, $\mathrm{lk}_{\Delta_n} C_i$ a chamber complex implies that there is a gallery $A_i'' = A_0^{(i)}, \ldots, A_{m_i}^{(i)} = A_i'$ in $\mathrm{lk}_{\Delta_n} C_i$ connecting A_i'' and A_i'. Thus, $D_i'' = A_i'' \cup C_i, D_1^{(i)} = A_1^{(i)} \cup C_i, \ldots, D_{m_i-1}^{(i)} = A_{m_i-1}^{(i)} \cup C_i, D_i' = A_i' \cup C_i$ is a gallery in Ξ_{2n} connecting D_i'' and D_i', all of whose chambers contain C_i. $\square$

We now fix an identification of a chamber of Δ_n with a chamber of Ξ_{2n}. Since a chamber C of Δ_n corresponds to a chain of lattices

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0,$$

a chamber of Ξ_{2n} containing C corresponds to a chain of lattices of the form

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0.$$

Recall (see page 21) that since L_0 is primitive, $L_0/\pi L_0$ has a non-degenerate, alternating k-bilinear form $\langle \cdot, \cdot \rangle_0$ given by $\langle v + \pi L_0, v' + \pi L_0 \rangle_0 = \langle v, v' \rangle + \pi \mathcal{O}$. Let $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ be a symplectic basis of $L_0/\pi L_0$ such that for all $1 \leq i \leq n$, $\{x_1, \ldots, x_i\}$ is a basis of $L_i/\pi L_0$. For $1 \leq i \leq n$, let U_{2n-i} be the $(2n-i)$-dimensional k-subspace of $L_0/\pi L_0$ with basis $\{x_1, \ldots, x_n, y_{i+1}, \ldots, y_n\}$, and note that U_{2n-i} is the orthogonal complement of $L_i/\pi L_0$ in $L_0/\pi L_0$. By the Correspondence Theorem, there are unique $\mathcal{O}$-submodules $L_{n-1}^{\perp}, \ldots, L_1^{\perp}$ of L_0 containing πL_0 such that $L_i^{\perp}/\pi L_0 = U_{2n-i}$ and $\pi L_0 \subsetneq L_i^{\perp} \subsetneq L_0$. Since both L_0 and πL_0 are free $\mathcal{O}$-modules of rank $2n$, all the $L_i^\perp$ are lattices of V. Finally, $L_i \subsetneq L_{i+1}$ implies $L_i^\perp \supsetneq L_{i+1}^\perp$; i.e.,

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_{n-1}^\perp \subsetneq \cdots \subsetneq L_1^\perp \subsetneq L_0. \tag{1.5}$$

Note that $[L_1 : \pi L_0] = q$ and $[L_i : L_{i-1}] = q$ for all $2 \leq i \leq n$ since $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$ corresponds to C. Moreover, $[L_{n-1}^\perp : L_n] = q$ since $L_{n-1}^\perp/\pi L_0$ the orthogonal complement of $L_{n-1}/\pi L_0$ in $L_0/\pi L_0$ implies $\dim_k(L_{n-1}^\perp/\pi L_0) = \dim_k(L_0/\pi L_0) - \dim_k(L_{n-1}/\pi L_0) = 2n - (n-1) = n+1$ (the conditions on $L_0, \ldots, L_{n-1}$ imply $[L_{n-1} : \pi L_0] = q^{n-1}$). Similarly, $[L_{n-(i+1)}^\perp : L_{n-i}^\perp] = q$ for all $1 \leq i \leq n-2$. It follows that the chain of lattices in (1.5) corresponds to a chamber of Ξ_{2n}.

Note that this associates distinct chambers of Δ_n with distinct chambers of Ξ_{2n}: if $C' \neq C$ is another chamber of Δ_n corresponding to the chain of lattices

$$\pi M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M_0,$$

then the chamber of Ξ_{2n} that we associate with C' corresponds to

$$\pi M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq M_{n-1}^\perp \subsetneq \cdots \subsetneq M_1^\perp \subsetneq M_0,$$

where $M_i^\perp/\pi M_0$ is the orthogonal complement of $M_i/\pi M_0$ in $M_0/\pi M_0$ for all $1 \leq i \leq n-1$. Since $C \neq C'$, $[L_i] \neq [M_i]$ for some $0 \leq i \leq n$, and the two corresponding chambers of Ξ_{2n} are also different. Call the identification given above the *oc-identification*.[13] If C is a chamber of Δ_n, denote by $\widetilde{C}$ the chamber of Ξ_{2n} obtained by the oc-identification of C; call $\widetilde{C}$ an *oc-chamber*.

1.3.2 The apartments of Δ_n

We give a description of the apartments of Δ_n following [20, p. 337]. A 2-dimensional, totally isotropic subspace U of V is a *hyperbolic plane*, and a symplectic basis $\{x, y\}$ of U is a *hyperbolic pair* in U. A *frame in V for Δ_n* is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines (1-dimensional K-subspaces) in V such that

1. $\lambda_i^1 + \lambda_i^2$ is a hyperbolic plane for all $1 \leq i \leq n$,
2. $\lambda_i^1 + \lambda_i^2$ is orthogonal to $\lambda_j^1 + \lambda_j^2$ for all $i \neq j$, and
3. $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$.

[13] Here, "oc" means "orthogonal complement."

A vertex $t \in \Delta_n$ lies in the apartment specified by the frame $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ if for any representative $L \in t$, $L = M_1^1 + M_1^2 + \cdots + M_n^1 + M_n^2$, where M_i^j is a lattice of λ_i^j for all $1 \leq i \leq n$ and $1 \leq j \leq 2$. Note that a frame in V for Δ_n is also a frame in V for Ξ_{2n}. In fact, if $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ is a frame in V for Δ_n specifying the apartment Σ of Δ_n and $t \in \Sigma$ is a vertex, then t is a vertex of the apartment $\widetilde{\Sigma}$ of Ξ_{2n} specified by the frame $\lambda_1^1, \lambda_1^2, \ldots, \lambda_n^1, \lambda_n^2$. The following is the analogue of Lemma 1.2.1 for Δ_n.

Lemma 1.3.5.

1. *Every symplectic basis of V specifies an apartment of Δ_n.*

2. *If Σ is an apartment of Δ_n, there is a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V such that every vertex of Σ has the form*

$$[\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n]$$

   for some $a_i, b_i \in \mathbb{Z}$.

Remark. As in Lemma 1.2.1, this lemma does not assert that there is a one-to-one correspondence between the set of symplectic bases of V and the set of frames in V for Δ_n. For example, if $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis of V, then $\{-w_1, \ldots, -w_n, u_1, \ldots, u_n\}$ is a symplectic basis of V that determines the same apartment of Δ_n as does $\mathcal{B}$.

Proof. For part 1, if $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis of V, let $\lambda_i^1 = Ku_i$ and $\lambda_i^2 = Kw_i$ for all $1 \leq i \leq n$. Then u_i, w_i is a hyperbolic pair in $\lambda_i^1 + \lambda_i^2$, a 2-dimensional, totally isotropic subspace of V. In addition, $\langle u_i, u_j \rangle = 0 = \langle w_i, w_j \rangle$ and $\langle u_i, w_j \rangle = 0 = \langle u_j, w_i \rangle$ for all $i \neq j$ imply that $\lambda_i^1 + \lambda_i^2$ is orthogonal to $\lambda_j^1 + \lambda_j^2$ if $i \neq j$. Finally, since $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a basis of V, $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$. It follows that $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ is a frame in V for Δ_n and hence specifies an apartment of Δ_n.

For part 2, let Σ be an apartment of Δ_n, and let $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ be a frame in V for Δ_n specifying Σ. For $1 \leq i \leq n$, let u_i, w_i be a hyperbolic pair in $\lambda_i^1 + \lambda_i^2$. Then $V = (\lambda_1^1 + \lambda_1^2) + \cdots + (\lambda_n^1 + \lambda_n^2)$ implies $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a basis of V. Furthermore, $\langle u_i, u_i \rangle = 0 = \langle w_i, w_i \rangle$ and $\langle u_i, w_i \rangle = 1$. In addition, $\lambda_i^1 + \lambda_i^2$ orthogonal to $\lambda_j^1 + \lambda_j^2$ for all $i \neq j$ implies $\langle u_i, u_j \rangle = 0 = \langle w_i, w_j \rangle$ and $\langle u_i, w_j \rangle = 0 = \langle u_j, w_i \rangle$ for all $i \neq j$; i.e., $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis of V. Since u_i (resp., w_i) is a K-basis of λ_i^1 (resp., of λ_i^2), $\lambda_i^1 = Ku_i$ (resp., $\lambda_i^2 = Kw_i$), and a lattice of λ_i^1

(resp., of λ_i^2) has the form $\mathcal{O}\alpha_i u_i$ (resp., $\mathcal{O}\beta_i w_i$) for some $\alpha_i \in K^\times$ (resp., for some $\beta_i \in K^\times$). But every $\alpha \in K^\times$ has the form $\alpha = \pi^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, so $\mathcal{O}\alpha_i u_i = \mathcal{O}\pi^{a_i} u_i$ (resp., $\mathcal{O}\beta_i w_i = \mathcal{O}\pi^{b_i} w_i$) for some $a_i \in \mathbb{Z}$ (resp., for some $b_i \in \mathbb{Z}$); i.e., every vertex of Σ has the form

$$[\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n]$$

for some $a_i, b_i \in \mathbb{Z}$. $\square$

Suppose $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis of V. Since the uniformizing parameter π is fixed, simplify notation and write $(a_1, \ldots, a_n; b_1, \ldots, b_n)_\mathcal{B}$ for the lattice $\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$ and $[a_1, \ldots, a_n; b_1, \ldots, b_n]_\mathcal{B}$ for its homothety class. More generally, for any basis $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ of V, write $(a_1, \ldots, a_n; b_1, \ldots, b_n)_\mathcal{B}$ for the lattice $\mathcal{O}\pi^{a_1}v_1 + \cdots + \mathcal{O}\pi^{a_n}v_n + \mathcal{O}\pi^{b_1}v_{n+1} + \cdots + \mathcal{O}\pi^{b_n}v_{2n}$ and $[a_1, \ldots, a_n; b_1, \ldots, b_n]_\mathcal{B}$ for its homothety class. Note that since every $\alpha \in K^\times$ has the form $\alpha = \pi^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, every representative of $[a_1, \ldots, a_n; b_1, \ldots, b_n]_\mathcal{B}$ has the form $(a_1+m, \ldots, a_n+m; b_1+m, \ldots, b_n+m)_\mathcal{B}$ for some $m \in \mathbb{Z}$. We use this fact from now on without comment.

Remark. Let $\mathcal{B}$ be a symplectic basis of V. In general, $[a_1, \ldots, a_n; b_1, \ldots, b_n]_\mathcal{B}$ is only a vertex of Ξ_{2n}; it need not be a vertex of Δ_n. We characterize when such a vertex is in the apartment of Δ_n specified by $\mathcal{B}$ in Proposition 1.3.4.

Let $\mathcal{B}$ be a symplectic basis of V. Then the lattice $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_\mathcal{B}$ is primitive if and only if $a_i + b_i = 0$ for all i by [43, p. 3]. In particular, if Σ is the apartment of Δ_n specified by $\mathcal{B}$ as in Lemma 1.3.5 and $t \in \Sigma$ is a vertex with primitive representative L, then $L = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_\mathcal{B}$. We characterize the vertices of Δ_n with a primitive representative in terms of types in Proposition 1.5.2. Note that the lattice $L = (a_1, \ldots, a_n; b_1, \ldots, b_n)_\mathcal{B}$ satisfies $\langle L, L\rangle \subseteq \pi\mathcal{O}$ if and only if $a_i + b_i \geq 1$ for all i.

Proposition 1.3.4. *Let $\mathcal{B}$ be a symplectic basis of V and Σ the apartment of Δ_n specified by $\mathcal{B}$ as in Lemma 1.3.5. Then the vertex $t = [a_1, \ldots, a_n; b_1, \ldots, b_n]_\mathcal{B} \in \Xi_{2n}$ is a vertex of Σ if and only if there is an $m \in \mathbb{Z}$ such that for all $1 \leq i \leq n$, $a_i + b_i + 2m \in \{1, 2\}$.*

Remark. In general, the value of $a_i + b_i + 2m$ depends on i; i.e., given i, either $a_i + b_i + 2m = 1$ or $a_i + b_i + 2m = 2$.

Proof. Suppose there is an $m \in \mathbb{Z}$ such that for all $1 \le i \le n$, $a_i + b_i + 2m \in \{1, 2\}$. Let $a_i' = a_i + m$ and $b_i' = b_i + m$, and note that $a_i' + b_i' \in \{1, 2\}$. Define c_i and d_i as follows:

$$c_i = a_i' - 1 \qquad \text{and} \qquad d_i = \begin{cases} b_i' & \text{if } a_i' + b_i' = 1, \\ b_i' - 1 & \text{if } a_i' + b_i' = 2. \end{cases}$$

Then $c_i + d_i = 0$ for all i, so the lattice $(c_1, \ldots, c_n; d_1, \ldots, d_n)_{\mathcal{B}}$ is primitive by [43, p. 3]. Furthermore, $c_i + 1 \ge a_i' \ge c_i$ and $d_i + 1 \ge b_i' \ge d_i$ imply

$$\begin{aligned}(c_1 + 1, \ldots, c_n + 1; d_1 + 1, \ldots, d_n + 1)_{\mathcal{B}} &\subseteq (a_1', \ldots, a_n'; b_1', \ldots, b_n')_{\mathcal{B}} \\ &\subseteq (c_1, \ldots, c_n; d_1, \ldots, d_n)_{\mathcal{B}}.\end{aligned}$$

Finally, since $L = (a_1', \ldots, a_n'; b_1', \ldots, b_n')_{\mathcal{B}} = \pi^m (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ and since $\langle L, L \rangle \subseteq \pi\mathcal{O}$ ($a_i' + b_i' \ge 1$ for all i), $t \in \Sigma$.

Conversely, suppose $t \in \Sigma$. Then t is a vertex of a chamber of Σ, so there is a primitive lattice L_0 of V with $[L_0] \in \Sigma$ and there is a representative $L \in t$ such that $\langle L, L \rangle \subseteq \pi\mathcal{O}$ and $\pi L_0 \subseteq L \subseteq L_0$. By [43, p. 3], $L_0 = (c_1, \ldots, c_n; -c_1, \ldots, -c_n)_{\mathcal{B}}$. Thus, $L = (a_1', \ldots, a_n'; b_1', \ldots, b_n')_{\mathcal{B}}$ implies $a_i' + b_i' \ge 1$ (since $\langle L, L \rangle \subseteq \pi\mathcal{O}$), $c_i + 1 \ge a_i' \ge c_i$, and $-c_i + 1 \ge b_i' \ge -c_i$ for all i. In particular,

$$2 = (c_i + 1) + (-c_i + 1) \ge a_i' + b_i' \ge 1;$$

hence, $a_i' + b_i' \in \{1, 2\}$ for all i. But $L \in t$ implies that there is an $m \in \mathbb{Z}$ such that $a_i' = a_i + m$ and $b_i' = b_i + m$ and $a_i + b_i + 2m \in \{1, 2\}$ for all i. □

1.3.3 The special vertices of Δ_n

We now use the special vertices of Δ_n to derive structural information about Δ_n. Recall that only two vertices in each chamber of Δ_n are special. By [43, p. 6], if $\mathcal{B}$ is a symplectic basis of V and Σ is the apartment of Δ_n specified by $\mathcal{B}$ as in Lemma 1.3.5, then the vertex $[a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}} \in \Sigma$ is special if and only if $a_i + b_i = \mu$ is constant for all i. Let $t \in \Delta_n$ be a special vertex. Then the link $\mathrm{lk}_{\Delta_n} t$ of t in Δ_n is isomorphic to the spherical $C_n(k)$ building $\Delta_n^s(k)$, which we now describe following [35, pp. 5 – 6]. Let V_s be a $2n$-dimensional k-vector space endowed with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle_s$. Then a vertex of $\Delta_n^s(k)$ is a non-trivial, totally isotropic subspace of V_s, and a nested sequence $S_1 \subsetneq \cdots \subsetneq S_{i+1}$

of non-trivial, totally isotropic subspaces of V_s is an i-simplex of $\Delta_n^s(k)$. By [35, p. 6], the dimension of any maximal totally isotropic subspace of V_s is n, so a chamber of $\Delta_n^s(k)$ is a maximal flag $S_1 \subsetneq \cdots \subsetneq S_n$ of non-trivial, totally isotropic subspaces of V_s, where $\dim_k S_i = i$ for all i. Recall that $\mathrm{lk}_{\Delta_n} t$ is isomorphic (as a poset) to the subposet of Δ_n consisting of those simplices containing t. Since $\mathrm{lk}_{\Delta_n} t \cong \Delta_n^s(k)$ is a building, this means that the chambers (resp., codimension-one simplices) of $\mathrm{lk}_{\Delta_n} t$ are in one-to-one correspondence with the chambers (resp., codimension-one simplices) of Δ_n containing t.

Proposition 1.3.5. *Every special vertex of Δ_n is contained in exactly*

$$r(\Delta_n) = \prod_{m=1}^{n} \frac{q^{2m}-1}{q-1}$$

chambers of Δ_n.

Proof. By our previous comments, it suffices to count the number of maximal flags $S_1 \subsetneq \cdots \subsetneq S_n$ of non-trivial, totally isotropic subspaces of $V_s \cong \mathbb{F}_q^{2n}$. Since every 1-dimensional subspace is totally isotropic, the number of S_1 is $(q^{2n}-1)/(q-1)$. Suppose we have chosen $S_1 \subsetneq \cdots \subsetneq S_m$, and let $\{x_1, \ldots, x_m\}$ be a basis of S_m. Extend this to a symplectic basis $\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n, y_1, \ldots, y_n\}$ of V_s. We count the number of vectors in $V_s \backslash S_m$ that we can add to $\{x_1, \ldots, x_m\}$ to form a basis of an $(m+1)$-dimensional, totally isotropic subspace of V_s. Such a vector must have a coefficient of 0 for $y_1, \ldots, y_m$ (to make the resulting space totally isotropic), and at least one of the coefficients of $x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n$ must be non-zero (to make the resulting space not equal to S_m); hence, there are $q^m(q^{2(n-m)}-1)$ possible vectors to add. Two such vectors v and v' satisfy $S_m \oplus \mathrm{Span}(v) = S_m \oplus \mathrm{Span}(v')$ if and only if $v' = x + \lambda v$ for some $x \in S_m$ and some $\lambda \in k^\times$. We therefore overcount by a factor of $q^m(q-1)$, and there are

$$\frac{q^m(q^{2(n-m)}-1)}{q^m(q-1)} = \frac{q^{2(n-m)}-1}{q-1} \tag{1.6}$$

$(m+1)$-dimensional, totally isotropic subspaces of V_s containing S_m. The number of maximal flags of non-trivial, totally isotropic subspaces of V_s is thus

$$\prod_{m=0}^{n-1} \frac{q^{2(n-m)}-1}{q-1} = \prod_{m=1}^{n} \frac{q^{2m}-1}{q-1}.$$

□

Remark. The number $r(\Delta_n)$ in the last proposition corresponds to the number r_n given in Proposition 2.4 of [41]. Since $\mathrm{Sp}_1(K) = \mathrm{SL}_2(K)$, we set $r(\Delta_1) = q + 1$ for completeness.

We now consider the number of chambers of Δ_n containing a given codimension-one simplex of Δ_n.

Proposition 1.3.6. *Every codimension-one simplex of $\Delta_n^s(k)$ is a face of exactly $q+1$ chambers.*

Proof. Note that a codimension-one simplex of $\Delta_n^s(k)$ is given by a nested sequence of non-trivial, totally isotropic subspaces of V_s defining a chamber of $\Delta_n^s(k)$ with exactly one S_i missing. First suppose $i = 1$. Since every subspace of a totally isotropic space is totally isotropic, it suffices to count the number of 1-dimensional subspaces of $S_2 \cong \mathbb{F}_q^2$, which is $q + 1$. If $i = n$, we need to pick an n-dimensional, totally isotropic subspace S_n of V_s containing S_{n-1}. By (1.6), this is

$$\frac{q^{2(n-(n-1))} - 1}{q - 1} = q + 1.$$

Finally, suppose $1 < i < n$, and let $\{x_1, \ldots, x_{i-1}\}$ and $\{x_1, \ldots, x_{i-1}, x_i, x_{i+1}\}$ be bases of S_{i-1} and S_{i+1}, respectively. We want to add a vector of $S_{i+1} \backslash S_{i-1}$ to $\{x_1, \ldots, x_{i-1}\}$ to form a basis of an i-dimensional space S_i. Thus, in any such linear combination of $x_1, \ldots, x_{i+1}$, at least one of the coefficients of x_i and x_{i+1} must be non-zero. As in the proof of Proposition 1.3.5, there are exactly

$$\frac{q^{i-1}(q^2 - 1)}{q^{i-1}(q - 1)} = q + 1$$

vectors satisfying our desired conditions; hence, every codimension-one simplex in $\Delta_n^s(k)$ is contained in exactly $q + 1$ chambers of $\Delta_n^s(k)$. ☐

Lemma 1.3.6. *Let A be a codimension-one simplex of Δ_n containing a special vertex t, and let A' be the corresponding codimension-one simplex of $\mathrm{lk}_{\Delta_n} t \cong \Delta_n^s(k)$. Then the number of chambers of Δ_n containing A equals the number of chambers of $\mathrm{lk}_{\Delta_n} t$ containing A'.*

Proof. First note that we can view a chamber C of Δ_n (resp., of $\mathrm{lk}_{\Delta_n} t$) as the power set of an unordered set, also denoted C, of $n + 1$ (resp., of n) vertices of Δ_n (resp., of $\mathrm{lk}_{\Delta_n} t$), any two of which are adjacent. Write $\mathcal{P}(B)$ for the power set of a set B, and let S (resp., S') be the set of chambers of Δ_n (resp., of $\mathrm{lk}_{\Delta_n} t$) containing A

(resp., A'). Let $\varphi : S \to S'$ be the map $C \mapsto \mathcal{P}(C\backslash\{t\})$ and $\psi : S' \to S$ the map $C' \mapsto \mathcal{P}(C' \cup \{t\})$. Then

$$\psi(\varphi(C)) = \psi(\mathcal{P}(C\backslash\{t\})) = \mathcal{P}((C\backslash\{t\}) \cup \{t\}) = \mathcal{P}(C) = C,$$
$$\varphi(\psi(C')) = \varphi(\mathcal{P}(C' \cup \{t\})) = \mathcal{P}((C' \cup \{t\})\backslash\{t\}) = \mathcal{P}(C') = C';$$

i.e., $\varphi : S \to S'$ is a bijection. ☐

Proposition 1.3.7. *If $A \in \Delta_n$ is a codimension-one simplex, then A is a face of exactly $q+1$ chambers of Δ_n.*

Proof. First note that we obtain a codimension-one simplex of Δ_n from a chamber $C \in \Delta_n$ by removing exactly one of the vertices of C, as well as all incident simplices. Since every chamber of Δ_n contains exactly two special vertices, this means that every codimension-one simplex of Δ_n contains at least one special vertex. Now let $A \in \Delta_n$ be a codimension-one simplex, and let t be a special vertex of A. Let A' be the codimension-one simplex of $\mathrm{lk}_{\Delta_n} t$ corresponding to A. By the last lemma, the number of chambers of Δ_n containing A is the number of chambers of $\mathrm{lk}_{\Delta_n} t$ containing A', which is $q+1$ by Proposition 1.3.6 since $\mathrm{lk}_{\Delta_n} t \cong \Delta_n^s(k)$. ☐

1.4 The action of $\mathrm{GL}_{2n}(K)$ on Ξ_{2n}

For a vertex $t \in \Xi_{2n}$ with representative $L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_{2n}$ and $g \in \mathrm{GL}_{2n}(K)$, define $gt = g[L] = [gL] = [\mathcal{O}(gv_1) + \cdots + \mathcal{O}(gv_{2n})]$. Note that if $L' \in t$, then there is an $\alpha \in K^\times$ such that $L' = \alpha L$, and $gL' = g(\alpha L) = \alpha(gL)$ since g is a linear transformation; i.e., this definition is independent of the choice of representative of t. Since any $g \in \mathrm{GL}_{2n}(K)$ takes a K-basis of V to a K-basis of V, $\mathrm{GL}_{2n}(K)$ acts on the set of lattices of V and thus on the set of vertices of Ξ_{2n}. Finally, $\mathrm{GL}_{2n}(K)$ acts transitively on the lattices of V: if $L = \mathcal{O}v_1 + \cdots + \mathcal{O}v_{2n}$ and if g is the matrix with columns $v_1, \ldots, v_{2n}$ (expressed with respect to the basis $\mathcal{B}_0 = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$), then $g(0, \ldots, 0; 0, \ldots 0)_{\mathcal{B}_0} = L$.

Let

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where 0 is the $n \times n$ zero matrix and I_n the $n \times n$ identity matrix. Then

$$\mathrm{GSp}_n(K) = \left\{ g \in M_{2n}(K) : g^t J_n g = \nu(g) J_n \text{ for some } \nu(g) \in K^\times \right\}$$

and $\mathrm{Sp}_n(K)$ consists of the matrices $g \in \mathrm{GSp}_n(K)$ with $\nu(g) = 1$. Note that J_n is the Gram matrix of $\langle \cdot, \cdot \rangle$ with respect to any symplectic basis $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V; i.e., for all $v_1, v_2 \in V$, $\langle v_i, v_j \rangle = v_i^t J_n v_j$, where v_i (resp., v_j) is the column vector with entries the coordinates of v_i (resp., of v_j) with respect to $\mathcal{B}$. It follows that for all $g \in \mathrm{GSp}_n(K)$ and for all $v_1, v_2 \in V$,

$$\langle gv_1, gv_2 \rangle = (gv_1)^t J_n (gv_2) = v_1^t (g^t J_n g) v_2 = \nu(g) v_1^t J_n v_2 = \nu(g) \langle v_1, v_2 \rangle. \tag{1.7}$$

Note also that

$$\det J_n = (-1)^n \det \left(\begin{smallmatrix} I_n & 0 \\ 0 & -I_n \end{smallmatrix} \right) = 1,$$

so for all $g \in \mathrm{GSp}_n(K)$,

$$\nu(g)^{2n} = \det(\nu(g) J_n) = \det(g^t J_n g) = (\det g)^2; \tag{1.8}$$

i.e., $\mathrm{GSp}_n(K) \leq \mathrm{GL}_{2n}(K)$.

Lemma 1.4.1. *Let* $g \in \mathrm{GSp}_n(K)$. *Then* $\mathrm{ord}(\det g) \equiv 0 \mod 2n$ *if and only if* $\mathrm{ord}(\nu(g)) \equiv 0 \mod 2$ *and* $\mathrm{ord}(\det g) \equiv n \mod 2n$ *if and only if* $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$.

Proof. First note that by (1.8), $\mathrm{ord}(\det g) = n\, \mathrm{ord}(\nu(g))$. If $\mathrm{ord}(\nu(g)) = 2r$ for some $r \in \mathbb{Z}$, then $\mathrm{ord}(\det g) = 2nr \equiv 0 \mod 2n$. If $\mathrm{ord}(\nu(g)) = 2r+1$ for some $r \in \mathbb{Z}$, then $\mathrm{ord}(\det g) = n(2r+1) \equiv n \mod 2n$. Conversely, $\mathrm{ord}(\det g) = 2nr$ for some $r \in \mathbb{Z}$ implies $n\, \mathrm{ord}(\nu(g)) = 2nr$, and $\mathrm{ord}(\nu(g)) \equiv 0 \mod 2$. Similarly, $\mathrm{ord}(\det g) = 2nr + n$ for some $r \in \mathbb{Z}$ implies $n\, \mathrm{ord}(\nu(g)) = 2nr + n$, and $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. ☐

For $g \in \mathrm{GL}_{2n}(K)$ and a basis $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ of V, write $g\mathcal{B}$ for $\{gv_1, \ldots, gv_{2n}\}$. Note that if $\mathcal{B}$ is a symplectic basis of V and $g \in \mathrm{Sp}_n(K)$, then (1.7) implies $g\mathcal{B}$ is a symplectic basis of V.

Proposition 1.4.1. *The group* $\mathrm{Sp}_n(K)$ *acts on the set of primitive lattices of* V.

Proof. Let L be a primitive lattice of V, and recall that $[L]$ is a vertex of Δ_n. Let Σ be an apartment of Δ_n containing $[L]$ and $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5. Then [43, p. 3] implies that $L = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$. Let $g \in \mathrm{Sp}_n(K)$. Since $g\mathcal{B}$ is a symplectic basis of V, $gL = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{g\mathcal{B}}$, and [43, p. 3] implies that gL is primitive. ☐

1.5 Types of vertices in Ξ_{2n}

Recall that $\mathcal{B}_0 = \{e_1, \dots, e_n, f_1, \dots, f_n\}$, where $f_i = e_{n+i}$ for all $1 \leq i \leq n$. Let $L_0 = (0, \dots, 0; 0, \dots, 0)_{\mathcal{B}_0}$ and $t_0 = [L_0]$, a vertex of Ξ_{2n}. Following [35, p. 116], assign type 0 to t_0. For any other vertex $t \in \Xi_{2n}$ with representative L, choose $g \in \mathrm{GL}_{2n}(K)$ such that $L = gL_0$. The *type* of $t = [L]$ is $\mathrm{ord}(\det g) \mod 2n$, which is well-defined, by [35, p. 116].

Let C_0 be the chamber of Δ_n whose vertices are the homothety classes of the lattices

$$L_0 = (0, \dots, 0; 0, \dots, 0)_{\mathcal{B}_0}, L_1 = (0, 1, \dots, 1; 1, \dots, 1)_{\mathcal{B}_0}, \dots, L_n = (0, \dots, 0; 1, \dots, 1)_{\mathcal{B}_0}, \tag{1.9}$$

and identify C_0 with the chamber $\widetilde{C}_0 \in \Xi_{2n}$ whose vertices are the homothety classes of the lattices $L_0, \dots, L_n$ above, along with the homothety classes of the lattices

$$L_{n+1} = (0, \dots, 0; 1, \dots, 1, 0)_{\mathcal{B}_0}, \dots, L_{2n-1} = (0, \dots, 0; 1, 0, \dots, 0)_{\mathcal{B}_0}.$$

Remarks.

1. For all $1 \leq i \leq n$, $L_{n+i}/\pi L_0$ is the orthogonal complement of $L_{n-i}/\pi L_0$ in $L_0/\pi L_0$; i.e., our identification of C_0 with $\widetilde{C}_0$ is the oc-identification, and $\widetilde{C}_0$ is an oc-chamber.

2. This provides a labelling of the vertices of Δ_n: every vertex of Δ_n has a type since Δ_n is a subcomplex of Ξ_{2n} by Proposition 1.3.1. Moreover, the vertices of a chamber of Δ_n have $n + 1$ distinct types since every chamber of Δ_n is contained in a chamber of Ξ_{2n} by Corollary 1.3.1. It thus remains to verify that if C and C' are adjacent chambers of Δ_n with $C \neq C'$, then the vertex $t \in C$ not in C' has the same type as the vertex $t' \in C'$ not in C. But by Lemma 1.3.4, there are adjacent chambers D and D' of Ξ_{2n} such that D contains C, D' contains C', and $D \neq D'$. In particular, t is the vertex of D not in D' and t' is the vertex of D' not in D. Since the above process is a labelling of Ξ_{2n}, t and t' have the same type.

3. For the rest of this section, $L_0, \dots, L_{2n-1}$ denote the above lattices, and C_0 and $\widetilde{C}_0$ denote the above chambers of Δ_n and Ξ_{2n}, respectively.

Note that if the diagonal matrix with (diagonal) entries $a_1, \dots, a_{2n}$ is denoted

$\mathrm{diag}(a_1, \ldots, a_{2n})$, then for all $1 \le i \le n$,

$$L_i = \mathrm{diag}(\underbrace{1, \ldots, 1}_{i}, \pi, \ldots, \pi)L_0.$$

Similarly, for all $n + 1 \le i \le 2n - 1$,

$$L_i = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \pi, \ldots, \pi, \underbrace{1, \ldots, 1}_{i-n})L_0;$$

hence $[L_i]$ has type $2n - i$ for all $1 \le i \le 2n - 1$.

Proposition 1.5.1. *If $t \in \Delta_n$ is a vertex, then t has type i for some $i \equiv n, \ldots, 2n \mod 2n$.*

Proof. It suffices to show that for all $0 \le j \le n$, $[L_j]$ (as in (1.9)) has type i for some $i \equiv n, \ldots, 2n \mod 2n$. But $[L_j]$ a vertex of $\widetilde{C}_0$ implies $[L_j]$ has type $2n - j \in \{n, \ldots, 2n\}$. □

We now use types to characterize the vertices of Δ_n with a primitive representative.

Proposition 1.5.2. *A vertex of Δ_n has a primitive representative if and only if it has type* 0.

Proof. First recall that since Δ_n is the affine building of $\mathrm{Sp}_n(K)$, $\mathrm{Sp}_n(K)$ acts on the vertices of Δ_n in a type-preserving manner and acts transitively on the chambers of Δ_n. Let $t \in \Delta_n$ be a type 0 vertex and $C \in \Delta_n$ a chamber such that $t \in C$. Then we can choose $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$; hence, $[L_0]$ of type 0 implies $gL_0 \in t$. By [43, p. 3], L_0 is primitive, so Proposition 1.4.1 implies that gL_0 is primitive.

Conversely, suppose $t \in \Delta_n$ is a vertex with a primitive representative L, and let $C \in \Delta_n$ be a chamber containing t. Since $\mathrm{Sp}_n(K)$ acts transitively on the chambers of Δ_n, there is a $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$. Then gL is homothetic to one of $L_0, \ldots, L_n$, so there is an $m \in \mathbb{Z}$ such that $gL = \pi^m L_j$ for some $0 \le j \le n$. For this j, let $L_j = (a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}_0}$ as in (1.9). Then $gL = \pi^m L_j = (a_1 + m, \ldots, a_n + m; b_1 + m, \ldots, b_n + m)_{\mathcal{B}_0}$. On the other hand, gL is primitive by Proposition 1.4.1, so [43, p. 3] implies that for all i, $(a_i + m) + (b_i + m) = (a_i + b_i) + 2m = 0$ or $a_i + b_i = -2m$. In particular, $a_i + b_i$ is constant for all i. Since the only lattices in (1.9) with this property are L_0 and L_n, $j = 0$ or $j = n$, and either $a_i + b_i = 0$ for all i or $a_i + b_i = 1$ for all i. Since $a_i + b_i = -2m$ and $m \in \mathbb{Z}$, $m = 0$; i.e., $j = 0$ and $gt = [L_0]$. Since the $\mathrm{Sp}_n(K)$-action preserves types, t has type 0. □

We also use types to characterize the special vertices of Δ_n.

Proposition 1.5.3. *A vertex of Δ_n is special if and only if it has type 0 or n.*

Proof. Let $t \in \Delta_n$ be a type 0 (resp., type n) vertex, and let $C \in \Delta_n$ be a chamber containing t. Since $\mathrm{Sp}_n(K)$ acts transitively on the chambers of Δ_n, we can choose $g \in \mathrm{Sp}_n(K)$ such that $gC_0 = C$. Moreover, $\mathrm{Sp}_n(K) \leq \mathrm{SL}_{2n}(K)$ implies $t = g[L_0]$ (resp., $t = g[L_n]$), so $t = [0, \dots, 0; 0, \dots, 0]_{g\mathcal{B}_0}$ (resp., $t = [0, \dots, 0; 1, \dots, 1]_{g\mathcal{B}_0}$), and t is special by [43, p. 6].

Conversely, let $t \in \Delta_n$ be a special vertex. Let C be a chamber of Δ_n containing t, Σ an apartment of Δ_n containing C, and $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5. By [43, p. 6], $t = [a_1, \dots, a_n; \mu - a_1, \dots, \mu - a_n]_{\mathcal{B}}$ for some $\mu \in \mathbb{Z}$. Since $\mathrm{Sp}_n(K)$ acts transitively on the chambers of Δ_n, we can choose $g \in \mathrm{Sp}_n(K)$ such that $gC = C_0$. Then $gt = [a_1, \dots, a_n; \mu - a_1, \dots, \mu - a_n]_{g\mathcal{B}} = [L_i]$ for some $0 \leq i \leq n$. But gt special, [43, p. 6], and (1.9) imply that $i = 0$ or $i = n$. Since $\mathrm{Sp}_n(K) \leq \mathrm{SL}_{2n}(K)$, t has type 0 or n. ∎

1.6 The action of $\mathrm{GSp}_n(K)$ on Ξ_{2n}

We end this chapter by using the action of $\mathrm{GL}_{2n}(K)$ on the vertices of Ξ_{2n}, as well as the types of the vertices of Ξ_{2n}, to analyze the action of $\mathrm{GSp}_n(K)$ on Δ_n.

Proposition 1.6.1. *If $[L]$ is a type i vertex of Ξ_{2n}, then for any $g \in \mathrm{GL}_{2n}(K)$, the vertex $g[L] \in \Xi_{2n}$ has type $i + \mathrm{ord}(\det g) \mod 2n$.*

Proof. Since $[L]$ has type i, we can write $L = g_i L_0$, where $g_i \in \mathrm{GL}_{2n}(K)$ with $\mathrm{ord}(\det g_i) \equiv i \mod 2n$. Then $gL = gg_iL_0$, and $g[L] = [gL]$ has type $\mathrm{ord}(\det(gg_i)) \mod 2n \equiv \mathrm{ord}(\det g) + \mathrm{ord}(\det g_i) \mod 2n \equiv i + \mathrm{ord}(\det g) \mod 2n$. ∎

Corollary 1.6.1.

1. *An element $g \in \mathrm{GL}_{2n}(K)$ preserves types if and only if $\mathrm{ord}(\det g) \equiv 0 \mod 2n$.*

2. *An element $g \in \mathrm{GSp}_n(K)$ preserves the type of a vertex of Ξ_{2n} if and only if $\mathrm{ord}(\nu(g)) \equiv 0 \mod 2$.*

3. *An element $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$ takes a type i vertex of Ξ_{2n} to a vertex of Ξ_{2n} of type $i + n \mod 2n$.*

Proof. For part 1, suppose g preserves types. If $t \in \Xi_{2n}$ is a type i vertex, then $i + \operatorname{ord}(\det g) \equiv i \mod 2n$ by the last proposition, and $\operatorname{ord}(\det g) \equiv 0 \mod 2n$. Conversely, suppose $\operatorname{ord}(\det g) \equiv 0 \mod 2n$, and let $t \in \Xi_{2n}$ be a type i vertex. Then the last proposition implies that gt has type $i+\operatorname{ord}(\det g) \mod 2n \equiv i \mod 2n$. Part 2 follows from part 1 and Lemma 1.4.1. For part 3, first note that by Lemma 1.4.1, $\operatorname{ord}(\det g) \equiv n \mod 2n$. Then the last proposition implies that g takes a type i vertex of Ξ_{2n} to a vertex of type $i + \operatorname{ord}(\det g) \mod 2n \equiv i + n \mod 2n$. □

Corollary 1.6.2. *If $g \in \mathrm{GSp}_n(K)$ with $\operatorname{ord}(\nu(g)) \equiv 1 \mod 2$, then g maps a non-special vertex of Δ_n to a vertex of Ξ_{2n} that is not in Δ_n.*

Proof. Let t be a non-special vertex of Δ_n. Then Propositions 1.5.3 and 1.5.1 imply that t has type i for some $n+1 \leq i \leq 2n-1$. Thus, by part 3 of the last corollary, gt has type $i+n \mod 2n \in \{1, \dots, n-1\}$. Proposition 1.5.1 finishes the proof. □

Let $\mathcal{B} = \{u_1, \dots, u_n, w_1, \dots, w_n\}$ be a symplectic basis of V. For $g \in \mathrm{GSp}_n(K)$, $g\mathcal{B} = \{gu_1, \dots, gu_n, gw_1, \dots, gw_n\}$ is a basis of V with $\langle gu_i, gw_j\rangle = \nu(g)\langle u_i, w_j\rangle = \nu(g)\delta_{ij}$ and $\langle gu_i, gu_j\rangle = \nu(g)\langle u_i, u_j\rangle = 0 = \nu(g)\langle w_i, w_j\rangle = \langle gw_i, gw_j\rangle$ for all i, j by (1.7); hence, $g\mathcal{B}$ is a symplectic basis of V if and only if $\nu(g) = 1$, which is true if and only if $g \in \mathrm{Sp}_n(K)$.[14] On the other hand, the basis

$$\mathcal{B}_g := \{\nu(g)^{-1}gu_1, \dots, \nu(g)^{-1}gu_n, gw_1, \dots, gw_n\}$$

is always a symplectic basis of V. In particular, since $\nu(g) = \pi^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, $L = (a_1, \dots, a_n; b_1, \dots, b_n)_{\mathcal{B}}$ an $\mathcal{O}$-module implies

$$\begin{aligned}
gL &= \mathcal{O}\pi^{a_1}(gu_1) + \cdots + \mathcal{O}\pi^{a_n}(gu_n) + \mathcal{O}\pi^{b_1}(gw_1) + \cdots + \mathcal{O}\pi^{b_n}(gw_n) \\
&= \mathcal{O}\pi^{a_1+m}(\pi^{-m}gu_1) + \cdots + \mathcal{O}\pi^{a_n+m}(\pi^{-m}gu_n) + \mathcal{O}\pi^{b_1}(gw_1) + \cdots + \mathcal{O}\pi^{b_n}(gw_n) \\
&= \mathcal{O}\pi^{a_1+m}(\nu(g)^{-1}gu_1) + \cdots + \mathcal{O}\pi^{a_n+m}(\nu(g)^{-1}gu_n) \\
&\qquad + \mathcal{O}\pi^{b_1}(gw_1) + \cdots + \mathcal{O}\pi^{b_n}(gw_n) \\
&= (a_1+m, \dots, a_n+m; b_1, \dots, b_n)_{\mathcal{B}_g}.
\end{aligned}$$

We use the basis $\mathcal{B}_g$ in what follows.

Lemma 1.6.1. *If $g \in \mathrm{GSp}_n(K)$ with $\operatorname{ord}(\nu(g)) \equiv 0 \mod 2$, then g maps a chamber of Δ_n to a chamber of Δ_n.*

[14] See footnote 10 on page 20.

Proof. Let C be a chamber of Δ_n with vertices $t_0, \ldots, t_n$, where t_i has type $2n - i \mod 2n$ for all i, and note that gt_i has type $2n - i \mod 2n$ for all $0 \leq i \leq n$ by Corollary 1.6.1 part 2. Let Σ be an apartment of Δ_n containing C and $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5. Since $t_0, \ldots, t_n$ are the vertices of C, they have representatives $L_i \in t_i$ such that L_0 is primitive and $L_1/\pi L_0 \subsetneq \cdots \subsetneq L_n/\pi L_0$ is a maximal flag of non-trivial, totally isotropic k-subspaces of $L_0/\pi L_0$; hence,

$$L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}$$

implies $a_i^{(0)} + b_i^{(0)} = 0$ for all i (by [43, p. 3]), $\langle L_j, L_j \rangle \subseteq \pi\mathcal{O}$ for all $j \neq 0$, and

$$a_i^{(0)} + 1 \geq a_i^{(1)} \geq \cdots \geq a_i^{(n)} \geq a_i^{(0)}, \qquad b_i^{(0)} + 1 \geq b_i^{(1)} \geq \cdots \geq b_i^{(n)} \geq b_i^{(0)}$$

for all i.

Write $\operatorname{ord}(\nu(g)) = 2r$ for some $r \in \mathbb{Z}$. Then for all $0 \leq j \leq n$,

$$g[L_j] = [a_1^{(j)} + 2r, \ldots, a_n^{(j)} + 2r; b_1^{(j)}, \ldots, b_n^{(j)}]_{\mathcal{B}_g},$$

which has representative

$$\pi^{-r} g L_j = (a_1^{(j)} + r, \ldots, a_n^{(j)} + r; b_1^{(j)} - r, \ldots, b_n^{(j)} - r)_{\mathcal{B}_g}.$$

In particular, $\pi^{-r} g L_0$ is primitive by [43, p. 3], $\pi^{-r} g(\pi L_0) \subsetneq \pi^{-r} g L_1 \subsetneq \cdots \subsetneq \pi^{-r} g L_n \subsetneq \pi^{-r} g L_0$, and for all $1 \leq j \leq n$,

$$\langle \pi^{-r} g L_j, \pi^{-r} g L_j \rangle = \pi^{-2r} \nu(g) \langle L_j, L_j \rangle \subseteq \pi\mathcal{O};$$

i.e., $gt_0, \ldots, gt_n$ are the vertices of a chamber of Δ_n. □

Let C be a chamber of Δ_n corresponding to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. Let Σ be an apartment of Δ_n containing C, $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5, and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} specified by $\mathcal{B}$ as in Lemma 1.2.1 applied to Ξ_{2n}. Note that $\widetilde{\Sigma}$ contains C. Let D be any chamber of $\widetilde{\Sigma}$ containing C. Then D corresponds to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ for some lattices $L_{n+1}, \ldots, L_{2n-1}$ of V.

Write $L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}$ for all j. By [43, p. 3], $a_i^{(0)} + b_i^{(0)} = 0$ for all i. The conditions on $L_1, \ldots, L_n$ also imply that $a_i^{(j)} + b_i^{(j)} \geq 1$ for all $1 \leq i, j \leq n$.

Finally,

$$a_i^{(0)} + 1 \geq a_i^{(1)} \geq \cdots \geq a_i^{(2n-1)} \geq a_i^{(0)}, \qquad b_i^{(0)} + 1 \geq b_i^{(1)} \geq \cdots \geq b_i^{(2n-1)} \geq b_i^{(0)},$$

and $2 = a_i^{(0)} + b_i^{(0)} + 2 \geq a_i^{(j)} + b_i^{(j)} \geq a_i^{(0)} + b_i^{(0)} = 0$ for all i, j.

Proposition 1.6.2. *The two special vertices of C are $[L_0]$ and $[L_n]$.*

Proof. The fact that $[L_0]$ is special follows from Propositions 1.5.2 and 1.5.3. To see that $[L_n]$ is special, first recall from [43, p. 6] that a representative of a special vertex of Σ has the form $(a_1, \ldots, a_n; \mu - a_1, \ldots, \mu - a_n)_{\mathcal{B}}$ for some $\mu \in \mathbb{Z}$. Thus, the comments preceding this proposition imply that if L_j represents a special vertex of C, then $a_i^{(j)} + b_i^{(j)} = \mu$ for all i, where $\mu \in \{0, 1, 2\}$. Since $1 \leq j \leq n$, $\mu \in \{1, 2\}$. But $\mu = 2$ implies $a_i^{(j)} = a_i^{(0)} + 1$ and $b_i^{(j)} = b_i^{(0)} + 1$ for all i, which means that $L_j = \pi L_0$, a contradiction of our assumption that $L_j/\pi L_0$ is a non-trivial k-subspace $L_0/\pi L_0$. Thus, $a_i^{(j)} + b_i^{(j)} = 1$ for all i, and $L_j/\pi L_0 \cong k^n$. Since the dimension of a maximal totally isotropic k-subspace of $L_0/\pi L_0$ is n by [35, p. 6], $j = n$. $\square$

Proposition 1.5.3 and the definition of labelling therefore imply that $[L_n]$ has type n. Then by Proposition 1.5.1, the type of $[L_j]$ is in $\{n+1, \ldots, 2n-1\}$ for all $1 \leq j \leq n-1$ and the type of $[L_i]$ is in $\{1, \ldots, n-1\}$ for all $n+1 \leq i \leq 2n-1$. Let $g \in \mathrm{GSp}_n(K)$ such that $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$, and note that by Corollary 1.6.1 part 3, $g[L_0]$ has type n, the type of $g[L_j]$ is in $\{1, \ldots, n-1\}$ for all $1 \leq j \leq n-1$, $g[L_n]$ has type 0, and the type of $g[L_i]$ is in $\{n+1, \ldots, 2n-1\}$ for all $n+1 \leq i \leq 2n-1$.

Lemma 1.6.2. *Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) \equiv 1 \mod 2$. If $L_0, L_n, \ldots, L_{2n-1}$ are lattices of V as above, then the vertices $g[L_n], \ldots, g[L_{2n-1}], g[L_0]$ of Ξ_{2n} are the vertices of a chamber of Δ_n.*

Proof. First note that $g[L_i] \neq g[L_j]$ for all $i \neq j$. Suppose $\mathrm{ord}(\nu(g)) = 2r + 1$ for some $r \in \mathbb{Z}$, and recall that for all $0 \leq j \leq 2n-1$,

$$gL_j = (a_1^{(j)} + 2r + 1, \ldots, a_n^{(j)} + 2r + 1; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}_g}.$$

Then the proof of Proposition 1.6.2 and [43, p. 3] imply that

$$L_n' = \pi^{-(r+1)} gL_n = (a_1^{(n)} + r, \ldots, a_n^{(n)} + r; b_1^{(n)} - r - 1, \ldots, b_n^{(n)} - r - 1)_{\mathcal{B}_g}$$

is a primitive representative of $g[L_n]$ and for all $1 \leq i \leq n$,

$$a_i^{(n)}+r+1 \geq \cdots \geq a_i^{(2n-1)}+r+1 \geq a_i^{(0)}+r+1 \geq a_i^{(1)}+r \geq \cdots \geq a_i^{(n)}+r,$$
$$b_i^{(n)}-r \geq \cdots \geq b_i^{(2n-1)}-r \geq b_i^{(0)}-r \geq b_i^{(1)}-r-1 \geq \cdots \geq b_i^{(n)}-r-1.$$

Furthermore, $(a_i^{(j)}+r+1)+(b_i^{(j)}-r) \geq (a_i^{(0)}+r+1)+(b_i^{(0)}-r) = a_i^{(0)}+b_i^{(0)}+1 = 1$ for all $1 \leq i \leq n$ and for all $n+1 \leq j \leq 2n-1$; i.e., if

$$L'_j = \pi^{-r} g L_j = (a_1^{(j)}+r+1, \ldots, a_n^{(j)}+r+1; b_1^{(j)}-r, \ldots, b_n^{(j)}-r)_{\mathcal{B}_g}$$

for $j = 0, n+1, \ldots, 2n-1$, then $\pi L'_n \subsetneq L'_{n+1} \subsetneq \cdots \subsetneq L'_{2n-1} \subsetneq L'_0 \subsetneq L'_n$ and $\langle L'_j, L'_j \rangle \subseteq \pi\mathcal{O}$ for $j = 0, n+1, \ldots, 2n-1$. It follows that $[L'_n], \ldots, [L'_{2n-1}], [L'_0]$ are the vertices of a chamber of Δ_n. The fact that $[L'_j] = g[L_j]$ for $j = 0$ and for all $n \leq j \leq 2n-1$ finishes the proof. □

Recall that two vertices of Δ_n are adjacent if they are distinct and incident.

Proposition 1.6.3. *The group* $\mathrm{GSp}_n(K)$ *takes adjacent special vertices of* Δ_n *to adjacent special vertices of* Δ_n.

Proof. Let $t, t' \in \Delta_n$ be adjacent special vertices, and let C be a chamber of Δ_n containing t and t'. Suppose C corresponds to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0$. By Proposition 1.6.2, the two special vertices of C are $[L_0]$ and $[L_n]$; i.e., either $L_0 \in t$ and $L_n \in t'$ or $L_0 \in t'$ and $L_n \in t$. In either case, Lemmas 1.6.1 and 1.6.2 imply that for any $g \in \mathrm{GSp}_n(K)$, gt and gt' are adjacent vertices of Δ_n. It thus remains to show that gt and gt' are special.

Let Σ be an apartment of Δ_n containing C and $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5. Since L_0 is primitive, [43, p. 3] implies $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$. Thus, $L_n = (b_1, \ldots, b_n; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}}$ by the proof of Proposition 1.6.2. Let $g \in \mathrm{GSp}_n(K)$ with $\mathrm{ord}(\nu(g)) = m$. Then $gL_0 = (a_1 - m, \ldots, a_n - m; -a_1, \ldots, -a_n)_{\mathcal{B}_g}$ and $gL_n = (b_1 - m, \ldots, b_n - m; 1-b_1, \ldots, 1-b_n)_{\mathcal{B}_g}$. It follows from [43, p. 6] that $[gL_0]$ and $[gL_n]$ are special vertices of Δ_n; i.e., gt and gt' are special. □

Corollary 1.6.3. *The group* $\mathrm{GSp}_n(K)$ *acts transitively on the special vertices of* Δ_n.

Proof. By the last proposition, $\mathrm{GSp}_n(K)$ acts on the special vertices of Δ_n. The fact that this action is transitive follows from [43, Proposition 3.3]. □

Lemma 1.6.3. *If $t, t' \in \Delta_n$ are adjacent special vertices and t has a primitive representative L, then there is a representative $L' \in t'$ with $\pi L \subsetneq L' \subsetneq L$ such that the number of chambers of Δ_n containing both t and t' equals the number of maximal flags of non-trivial, proper k-subspaces of $L'/\pi L$.*

Proof. First note that a chamber C of Δ_n containing both t and t' has $n+1$ vertices $t = t_0, \ldots, t_n = t'$ that have representatives $L_i \in t_i$ such that L_0 is primitive and $L_1/\pi L_0 \subsetneq \cdots \subsetneq L_n/\pi L_0$ is a maximal flag of non-trivial, totally isotropic k-subspaces of $L_0/\pi L_0$. By Proposition 1.6.2, L_0 and L_n represent the two special vertices of C. Let $L = L_0$ and $L' = L_n$. Then $\pi L \subsetneq L' \subsetneq L$ and varying $L_1, \ldots, L_{n-1}$ over all lattices of V contained in L and containing πL such that $L_1/\pi L \subsetneq \cdots \subsetneq L_{n-1}/\pi L$ is a maximal flag of non-trivial, proper k-subspaces of $L'/\pi L = L_n/\pi L_0$ gives all the chambers of Δ_n containing both t and t'. □

Proposition 1.6.4. *If $t \in \Delta_n$ is a special vertex, then t is adjacent to exactly $\prod_{m=1}^{n} (q^m+1)$ distinct special vertices of Δ_n.*

Proof. Let $t \in \Delta_n$ be a special vertex, and note that any special vertex of Δ_n adjacent to t must be in a chamber of Δ_n containing t. By Proposition 1.3.5, the number of chambers of Δ_n containing t is

$$\prod_{m=1}^{n} \frac{q^{2m}-1}{q-1},$$

but counting the chambers of Δ_n containing t may overcount the number of distinct special vertices of Δ_n adjacent to t. Thus, fix a special vertex $t' \in \Delta_n$ adjacent to t. Then we count t' more than once if there is more than one chamber of Δ_n containing both t and t'. But $t, t' \in \Delta_n$ adjacent special vertices implies that exactly one of t and t' has a primitive representative (by Propositions 1.5.3 and 1.5.2). By symmetry, suppose t has a primitive representative L. Then by the last lemma, t' has a representative L' with $\pi L \subsetneq L' \subsetneq L$ such that the number of chambers of Δ_n containing both t and t' equals the number of maximal flags of non-trivial, proper k-subspaces of $L'/\pi L \cong k^n$, which is $\prod_{m=2}^{n} ((q^m-1)/(q-1))$ by [41, Proposition 2.4]. It follows that the number of distinct special vertices of Δ_n adjacent to t is

$$\frac{\prod_{m=1}^{n} \frac{q^{2m}-1}{q-1}}{\prod_{m=2}^{n} \frac{q^m-1}{q-1}} = \frac{\prod_{m=1}^{n} \frac{q^{2m}-1}{q-1}}{\prod_{m=1}^{n} \frac{q^m-1}{q-1}} = \prod_{m=1}^{n} \frac{q^{2m}-1}{q^m-1} = \prod_{m=1}^{n} (q^m+1).$$ □

Chapter 2

Close Vertices

Let Δ be any chamber complex. Recall that if C and D are any two chambers of Δ, there is a gallery in Δ connecting them. The minimal length of a gallery in Δ connecting C and D is the *combinatorial distance* between C and D, and a gallery connecting C and D of minimal length is a *minimal gallery*[1] from C to D (see [6, p. 14]).[2] Following [6, p. 15], we can similarly define the distance between any non-empty simplex $A \in \Delta$ and any chamber $C \in \Delta$ to be the minimal length of a gallery starting at a chamber containing A and ending at a chamber containing C. Taking this one step further, we define the distance between any two non-empty simplices $A, B \in \Delta$ as follows (cf. [41, p. 124]).

Definition. Let Δ be a chamber complex. For any non-empty simplices $A, B \in \Delta$, the *distance* between A and B is the minimal length of a gallery in Δ whose initial chamber contains A and whose ending chamber contains B.

In this chapter, we consider the case in which Δ is either Ξ_n ($n \geq 3$) or Δ_n ($n \geq 2$) and both A and B are vertices distance one apart or *close*. Figure 2.1 shows two close vertices of Ξ_3. Since a simplex of Δ is a finite set of vertices, use the notation of set theory to indicate a face of a given simplex. For example, write $A \cap B$ for the simplex (possibly the empty set) shared by the simplices $A, B \in \Delta$. Note that two vertices $t, t' \in \Delta$ are close if and only if they are in adjacent chambers of Δ but not a common one; i.e., if and only if there are adjacent chambers $C, C' \in \Delta$ such that $t \in C$,

[1] By [6, p. 14], a minimal gallery should be thought of as the combinatorial analogue of a geodesic.

[2] Note that the minimal length of a gallery in Δ connecting C and D exists and is the length of a gallery in Δ connecting C and D: consider the set S of galleries in Δ connecting C and D. For each gallery in S, put its length in the set S'. Then S' is a non-empty subset of $\mathbb{Z}^{\geq 0}$, which is well-ordered; hence, S' has a minimal element m, and, by the way we created S', is the length of a gallery in Δ connecting C and D.

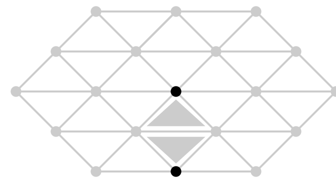


Figure 2.1: Two close vertices of Ξ_3.

$t' \in C'$, and $t, t' \notin C \cap C'$. In particular, if t and t' are close vertices of Δ and C and C' are adjacent chambers of Δ as in the last sentence, then since C and C' share a codimension-one face and $t, t' \notin C \cap C'$, t and t' have the same type.

We start with the case that Δ is Ξ_n for $n \geq 3$.

2.1 The building Ξ_n

Given a vertex t in the affine building Ξ_n of $\mathrm{SL}_n(K)$, Schwartz and Shemanske show in [41, Theorem 3.3] that the number ω_n of vertices of Ξ_n close to t is the number of right cosets of $\mathrm{GL}_n(\mathcal{O})$ in

$$\mathrm{GL}_n(\mathcal{O})\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2)\mathrm{GL}_n(\mathcal{O}).$$

They compute this number for $n = 3, 4, 5$ and conjecture that for all $n \geq 3$, $q \cdot r_n = r_{n-2}\ \omega_n$, where r_n is the number of chambers of Ξ_n containing a given vertex, and $r_1 = 1$ (see the remark following Proposition 3.4 of [41]). We give an explicit formula for ω_n and establish the relationship between ω_n and r_n. We then use the structure of Ξ_n to give a building-theoretic justification for this relationship.

In this section, $n \geq 3$, V is an n-dimensional K-vector space, and a lattice of V is a free $\mathcal{O}$-module of rank n. Recall that $k \cong \mathbb{F}_q$ is the residue field of K and that Ξ_n is an $(n-1)$-dimensional simplicial complex with vertices the homothety classes of lattices of V. Since every $\alpha \in K^\times$ has the form $\alpha = \pi^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, we can think of the homothety class $[L]$ of a lattice L of V as the infinite chain of lattices $\cdots \subsetneq \pi L \subsetneq L \subsetneq \pi^{-1} L \subsetneq \cdots$. We may therefore write a chamber $C \in \Xi_n$ as the infinite chain of lattices

$$\cdots \subsetneq \pi L_{n-1} \subsetneq \pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L_0 \subsetneq \pi^{-1} L_1 \subsetneq \cdots$$

and a codimension-one face of C by the above chain of lattices with $\ldots, \pi L_i, L_i, \pi^{-1} L_i, \ldots$ deleted for some $0 \leq i \leq n-1$.

Suppose $t \in \Xi_n$ is a vertex with representative L. Then a chamber C containing t has the form

$$\cdots \subsetneq \pi L_{n-1} \subsetneq \pi L \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq L \subsetneq \pi^{-1} L_1 \subsetneq \cdots \tag{2.1}$$

(see [20, p. 323]). The codimension-one face of C not containing t is thus

$$\cdots \subsetneq \pi L_{n-1} \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq \pi^{-1} L_1 \subsetneq \cdots,$$

and a vertex of Ξ_n is close to t if and only if it has a representative $M \neq L$ such that

$$\cdots \subsetneq \pi L_{n-1} \subsetneq \pi M \subsetneq L_1 \subsetneq \cdots \subsetneq L_{n-1} \subsetneq M \subsetneq \pi^{-1} L_1 \subsetneq \cdots. \tag{2.2}$$

Given the lattices L_1 and L_{n-1}, the possible L and M satisfy $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1} L_1$. On the other hand, if $t, t' \in \Xi_n$ are close vertices, then they must have representatives $L \in t$ and $M \in t'$ such that there are lattices $L_1, \ldots, L_{n-1}$ as in (2.1) with $L_{n-1} \subsetneq L \neq M \subsetneq \pi^{-1} L_1$. Note that L_{n-1}, L, and $\pi^{-1} L_1$ free $\mathcal{O}$-modules of rank n and $L_{n-1} \subseteq L \cap M \subseteq L, M \subseteq L + M \subseteq \pi^{-1} L_1$ imply that $L \cap M$ and $L + M$ are lattices of V. Furthermore, $L \neq M$, $[L : L_{n-1}] = q = [M : L_{n-1}]$, and $[\pi^{-1} L_1 : L] = [L_1 : \pi L] = q = [L_1 : \pi M] = [\pi^{-1} L_1 : M]$ imply $L \cap M \neq L, M$ and $L + M \neq L, M$. It follows that $L \cap M = L_{n-1}$ and $L + M = \pi^{-1} L_1$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. In other words, if t and t' are close vertices of Ξ_n, there may be two (or more) pairs of adjacent chambers C and C' with $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$. We return to this later.

Before we count the number of vertices of Ξ_n close to a given vertex $t \in \Xi_n$, we make a few observations. Fix a representative $L \in t$. Then $L/\pi L \cong k^n$, and if U_1 is a 1-dimensional k-subspace of $L/\pi L$, the Correspondence Theorem implies that there is a unique $\mathcal{O}$-submodule L_1 of L containing πL such that $L_1/\pi L = U_1$ and $\pi L \subsetneq L_1 \subsetneq L$. Since both πL and L are free $\mathcal{O}$-modules of rank n, L_1 is a lattice of V. Thus, the number of L_1 equals the number of 1-dimensional k-subspaces of $L/\pi L$. Similarly, given L_1 as above, the number of lattices L_{n-1} with $\pi L \subsetneq L_1 \subsetneq L_{n-1} \subsetneq L$ and $L_{n-1}/\pi L$ an $(n-1)$-dimensional k-subspace of $L/\pi L$ equals the number of $(n-2)$-dimensional k-subspaces of $L/L_1 \cong k^{n-1}$. Finally, given L_1 and L_{n-1} as above, the number of lattices $M \neq L$ such that $L_{n-1} \subsetneq M \subsetneq \pi^{-1} L_1$ is one less than the number of non-trivial, proper k-subspaces of $\pi^{-1} L_1 / L_{n-1} \cong k^2$.

Proposition 2.1.1. *If $t \in \Xi_n$ is a vertex, then the number ω_n of vertices of Ξ_n close* *to t is*

$$\frac{q^n-1}{q-1}\cdot\frac{q^{n-1}-1}{q-1}\cdot q.$$

Proof. Our previous comments imply that it suffices to count the number of 1-dimensional subspaces of $k^n \cong \mathbb{F}_q^n$ (the number of L_1), the number of $(n-2)$-dimensional subspaces of $k^{n-1} \cong \mathbb{F}_q^{n-1}$ (the number of L_{n-1} given L_1), and the number of non-trivial, proper subspaces of $k^2 \cong \mathbb{F}_q^2$ (the number of lattices properly contained in $\pi^{-1}L_1$ and properly containing L_{n-1}). Note that duality implies that the number of $(n-2)$-dimensional subspaces of $\mathbb{F}_q^{n-1}$ is the number of 1-dimensional subspaces of $\mathbb{F}_q^{n-1}$. Since there are $(q^n-1)/(q-1)$ 1-dimensional subspaces of $\mathbb{F}_q^n$, $(q^{n-1}-1)/(q-1)$ 1-dimensional subspaces of $\mathbb{F}_q^{n-1}$, and $q+1$ 1-dimensional subspaces of $\mathbb{F}_q^2$, the number of vertices of Ξ_n close to t is

$$\frac{q^n-1}{q-1}\cdot\frac{q^{n-1}-1}{q-1}\cdot q$$

(recall from the remarks preceding this proposition that we select L_1 and L_{n-1} such that $[L_1 : \pi L] = q$ and $[L_{n-1} : L_1] = q^{n-2}$, after which we want to count the number of lattices $M \neq L$ such that $L_{n-1} \subsetneq M \subsetneq \pi^{-1}L_1$, which is one less than the number of 1-dimensional k-subspaces of $\pi^{-1}L_1/L_{n-1} \cong k^2 \cong \mathbb{F}_q^2$ by the Correspondence Theorem; hence, we multiply by q rather than $q+1$). □

Let r_n be the number of chambers of Ξ_n containing t. By [41, Proposition 2.4],

$$r_n = \frac{1}{(q-1)^{n-1}}\prod_{m=0}^{n-2}(q^{n-m}-1) = \frac{1}{(q-1)^{n-1}}\prod_{m=2}^{n}(q^m-1),$$

which, with the last proposition, establishes the conjecture following Proposition 3.4 of [41]:

Theorem 2.1.1. *For all $n \geq 3$, $q \cdot r_n = r_{n-2}\,\omega_n$, where $r_1 = 1$.*

Proof. Note that $r_n/r_{n-2} = (\prod_{m=n-1}^{n}(q^m-1))/(q-1)^2$; hence, by the last proposition,

$$\omega_n = \frac{q^n-1}{q-1}\cdot\frac{q^{n-1}-1}{q-1}\cdot q = \frac{r_n}{r_{n-2}}\cdot q.$$ □

We now use the structure of Ξ_n to give a combinatorial proof for the relationship given in Theorem 2.1.1. Fix a vertex $t \in \Xi_n$. Then we can count the number of vertices of Ξ_n close to t by counting the number of galleries (in Ξ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. Let $C \in \Xi_n$ be a chamber containing t. Since there are exactly r_n chambers in Ξ_n containing a given vertex, there are r_n possible C. A chamber $C' \in \Xi_n$ adjacent to C and not containing t must contain the codimension-one face of C not containing t. By [20, p. 324], there are exactly $q+1$ chambers in Ξ_n containing a given codimension-one simplex of Ξ_n, so there are q possible chambers $C' \in \Xi_n$ adjacent to C not containing t. It follows that there are exactly $r_n \cdot q$ galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber does not contain t. It therefore remains to determine if counting the number of galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber does not contain t overcounts the number of vertices of Ξ_n close to t. To this end, fix a vertex $t' \in \Xi_n$ close to t, and suppose C, C' is a gallery in Ξ_n with $t \in C$ and $t' \in C'$. Then we count t' more than once if there is a chamber $C_1 \neq C$ containing t and a chamber C_1' adjacent to C_1 containing t' (note that since t and t' are close, $C_1' \neq C_1$); i.e., if there is more than one gallery of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t'. Since the number of vertices of Ξ_n close to t is ω_n, this implies $\omega_n = (r_n \cdot q)/m(t,t')$, where $m(t,t')$ is the number of galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t'.

We now investigate the number $m(t,t')$. We fix the following notation for the rest of this section. Let $t, t' \in \Xi_n$ be close vertices, and let $L \in t$, $M \in t'$ be representatives such that there are lattices $L_1, \ldots, L_{n-1}$ as in (2.1) and (2.2). By our comments following (2.2), $L_1 = \pi(L+M)$ and $L_{n-1} = L \cap M$, but we can vary $L_2, \ldots, L_{n-2}$ as long as $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_{n-2} \subsetneq L_{n-1}$. Since any gallery C, C' in Ξ_n such that $C = \{t, [L_1], \ldots, [L_{n-1}]\}$ and $C' = \{t', [L_1], \ldots, [L_{n-1}]\}$ satisfies $C \cap C' = \{[L_1], \ldots, [L_{n-1}]\}$, each such gallery is uniquely determined by the vertices $[L_2], \ldots, [L_{n-2}]$. We therefore consider the following situation. Recall that two vertices of Ξ_n are adjacent if they are distinct and incident. Consider the set of vertices in Ξ_n that are adjacent to $t, t', [L+M]$, and $[L \cap M]$ (note that in the case $n=3$, this set is empty). Define two such vertices to be adjacent if they are adjacent as vertices of Ξ_n.

Proposition 2.1.2. *Let $\Xi_n^c(t,t')$ be the set consisting of*

- *the empty set,*
- *all vertices of Ξ_n adjacent to $t, t', [L+M]$, and $[L \cap M]$, and*
- *all finite sets A of vertices of Ξ_n adjacent to $t, t', [L+M]$, and $[L \cap M]$ such that any two vertices in A are adjacent.*

Then $\Xi_n^c(t,t')$ is a simplicial complex.

Proof. Let A be a finite set of vertices of Ξ_n, each of which is adjacent to $t, t', [L+M]$, and $[L \cap M]$, and suppose that every pair of vertices in A are adjacent. Then $\emptyset \in \Xi_n^c(t,t')$, and every vertex of A is in $\Xi_n^c(t,t')$. Let B be any non-empty subset of A with at least two vertices. Then B is a finite set of vertices of Ξ_n, each of which is adjacent to $t, t', [L+M]$, and $[L \cap M]$, and every pair of vertices in B are adjacent. It follows that every subset (including the empty set) of A is in $\Xi_n^c(t,t')$. □

Corollary 2.1.1. *The simplicial complex $\Xi_n^c(t,t')$ given in the last proposition is a subcomplex of Ξ_n.*

Proof. This follows from the last proposition since a vertex of $\Xi_n^c(t,t')$ is a vertex of Ξ_n. □

Recall from the remarks following (2.2) that $[\pi(L+M) : \pi L] = [L_1 : \pi L] = q$ and $[L : L \cap M] = [L : L_{n-1}] = q$.

Lemma 2.1.1. *If $x \in \Xi_n^c(t,t')$ is a vertex, then x has a representative L' such that $\pi(L+M) \subsetneq L' \subsetneq L \cap M$.*

Proof. First note that since x is adjacent to $[L \cap M]$, x has a unique representative L' such that $\pi(L \cap M) \subsetneq L' \subsetneq L \cap M$ by Proposition 1.2.1. Then [20, p. 322] implies that either $L' \subsetneq \pi(L+M)$ or $L' \supsetneq \pi(L+M)$. In the second case, we are done, so assume $L' \subsetneq \pi(L+M)$. Then $\pi(L \cap M) \subsetneq L' \subsetneq \pi(L+M)$. On the other hand, $\pi(L \cap M) \subsetneq \pi L \subsetneq \pi(L+M)$ and $[\pi(L+M) : \pi(L \cap M)] = [\pi(L+M) : \pi L][\pi L : \pi(L \cap M)] = q^2$. But since x is adjacent to t, [20, p. 322] implies that either $L' \subsetneq \pi L$ or $L' \supsetneq \pi L$. Thus, $\pi(L \cap M) \subsetneq L' \neq \pi L \subsetneq \pi(L+M)$, which is impossible given the previous index computation. □

In particular, if $A \in \Xi_n^c(t,t')$, then A has at most $n-3$ vertices (since $[\pi(L+M) : \pi L] = q = [L : L \cap M]$ implies $(L \cap M)/\pi(L+M) \cong k^{n-2}$).

Lemma 2.1.2. *If $\emptyset \neq A \in \Xi_n^c(t,t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$ and $[M_1], \ldots, [M_{i+1}]$ are the vertices of A.*

Proof. We proceed by induction on i. The last lemma proves the claim if $i = 0$. Suppose $0 \leq i \leq n-5$ and that the claim holds for any i-simplex in $\Xi_n^c(t,t')$. Let $A \in \Xi_n^c(t,t')$ be an $(i+1)$-simplex, and note that $A \neq \emptyset$. Let x be a vertex of A. By the induction hypothesis, the i-simplex $A - \{x\}$ corresponds to a chain of lattices $M'_1 \subsetneq \cdots \subsetneq M'_{i+1}$ such that $\pi(L+M) \subsetneq M'_1 \subsetneq \cdots \subsetneq M'_{i+1} \subsetneq L \cap M$ and

$[M_1'], \ldots, [M_{i+1}']$ are the vertices of $A - \{x\}$. By the last lemma, x has a representative M' such that $\pi(L+M) \subsetneq M' \subsetneq L \cap M$. If $M' \subsetneq M_1'$, set $M_1 = M'$ and $M_j = M_{j-1}'$ for all $2 \leq j \leq i+2$. Otherwise, $M' \supsetneq M_1'$ by [20, p. 322]. Let $j \in \{1, \ldots, i+1\}$ be maximal such that $M' \supsetneq M_j'$ (j exists since $M' \subsetneq L \cap M$). If $j = i+1$, set $M_\ell = M_\ell'$ for all $1 \leq \ell \leq i+1$ and $M_{i+2} = M'$. Otherwise, $M' \subsetneq M_{j+1}'$, and setting $M_\ell = M_\ell'$ for all $1 \leq \ell \leq j$, $M_{j+1} = M'$, and $M_\ell = M_{\ell-1}'$ for all $j+2 \leq \ell \leq i+2$ finishes the proof. ☐

Proposition 2.1.3. *For any vertices $t, t' \in \Xi_n$ that are close, $\Xi_n^c(t,t')$ as defined in Proposition 2.1.2 is isomorphic (as a poset) to the spherical $A_{n-3}(k)$ building $\Xi_{n-3}^s(k)$ (independent of t and t'), where we interpret the spherical $A_0(k)$ building to be $\emptyset$.*

Proof. Let $L \in t, M \in t'$ be as in the paragraph preceding Proposition 2.1.2. Then $(L \cap M)/\pi(L+M)$ is an $(n-2)$-dimensional k-vector space. Let $\Xi_{n-3}^s(k)$ be the spherical $A_{n-3}(k)$ building with i-simplices $(0 \leq i \leq n-4)$ given by the nested sequences $V_1 \subsetneq \cdots \subsetneq V_{i+1}$ of non-trivial, proper k-subspaces of $(L \cap M)/\pi(L+M)$. Note that since $L \cap M$ and $\pi(L+M)$ are free $\mathcal{O}$-modules of rank n, any $\mathcal{O}$-submodule of $L \cap M$ containing $\pi(L+M)$ is a lattice of V. Moreover, by the Correspondence Theorem, there is a bijection between $\mathcal{O}$-submodules of $L \cap M$ containing $\pi(L+M)$ and k-subspaces of $(L \cap M)/\pi(L+M)$. It follows from the last lemma that there is a bijection between 0-simplices (vertices) of $\Xi_n^c(t,t')$ and 0-simplices of $\Xi_{n-3}^s(k)$. But the Correspondence Theorem also gives a bijection between chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of $L \cap M$ containing $\pi(L+M)$ and nested sequences $V_1 \subsetneq \cdots \subsetneq V_{i+1}$ of k-subspaces of $(L \cap M)/\pi(L+M)$; hence, the last lemma implies that for all $0 \leq i \leq n-4$, there is a bijection between the i-simplices of $\Xi_n^c(t,t')$ and the i-simplices of $\Xi_{n-3}^s(k)$. Let $\emptyset \in \Xi_n^c(t,t')$ correspond to $\emptyset \in \Xi_{n-3}^s(k)$. Then we have a bijection between the simplices of $\Xi_n^c(t,t')$ and the simplices of $\Xi_{n-3}^s(k)$. A modification of the proof of Lemma 1.3.3 shows that this bijection preserves the partial order (face) relation, so is a poset isomorphism. ☐

Theorem 2.1.2. *If $t, t' \in \Xi_n$ are close vertices, then the number $m(t,t')$ of galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' equals r_{n-2} (independent of t and t'). In particular, the number ω_n of vertices of Ξ_n close to a given vertex of Ξ_n is $\omega_n = (r_n \cdot q)/r_{n-2}$.*

Proof. By our previous comments, $\omega_n = (r_n \cdot q)/m(t,t')$. The last proposition and our previous comments also imply that the galleries of length 1 in Ξ_n whose initial chamber contains t and whose ending chamber contains t' are in one-to-one correspondence with the chambers of $\Xi^s_{n-3}(k)$. Since the proof of [41, Proposition 2.4] implies that $\Xi^s_{n-3}(k)$ contains exactly r_{n-2} chambers, we are done. $\square$

2.2 The building Δ_n

We now consider close vertices in the affine building Δ_n of $\mathrm{Sp}_n(K)$. We prove an analogue of Theorem 3.3 of [41] to characterize the vertices of Δ_n close to a given *special* vertex of Δ_n (recall that all the vertices of Ξ_n are special) in terms of symplectic divisors, after which we follow our work in the previous section to prove the analogous results for the *type* 0 vertices of Δ_n. For the rest of this chapter, $n \geq 2$, V is a $2n$-dimensional K-vector space endowed with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$, and a lattice of V is a free $\mathcal{O}$-module of rank $2n$. Recall that Δ_n is an n-dimensional simplicial complex whose vertices are certain homothety classes of lattices of V. We first prove the results about Δ_n that we need.

2.2.1 Preliminaries

We start with a fact about group actions and stabilizers.

Lemma 2.2.1. *Let G be a group acting (on the left) on a set A, and let $a \in A$. Write $G_a = \{g \in G : ga = a\}$, the stabilizer in G of a. If $b \in A$ such that $b = ha$ for some $h \in G$, then $G_b = hG_ah^{-1}$.*

Proof. Let $g \in G_a$. Then $(hgh^{-1})b = hga = ha = b$ implies $hG_ah^{-1} \subseteq G_b$. Now let $g' \in G_b$. Since $g' = h(h^{-1}g'h)h^{-1}$ and $(h^{-1}g'h)a = h^{-1}g'b = h^{-1}b = a$, $G_b \subseteq hG_ah^{-1}$. $\square$

Recall that if $\mathcal{B} = \{v_1, \ldots, v_{2n}\}$ is a basis (not necessarily symplectic) of V, then $(a_1, \ldots, a_n; b_1, \ldots, b_n)_{\mathcal{B}}$ denotes the lattice $\mathcal{O}\pi^{a_1}v_1 + \cdots + \mathcal{O}\pi^{a_n}v_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$ of V.

Proposition 2.2.1. *Let L be a lattice of V. Then $\mathrm{GSp}_n(\mathcal{O}) := \{g \in M_{2n}(\mathcal{O}) : g^tJ_ng = \nu(g)J_n,\ \nu(g) \in \mathcal{O}^\times\}$ can be identified with $\{g \in \mathrm{GSp}_n(K) : gL = L\}$, where g acts on L as the matrix of a linear transformation with respect to a fixed basis of L.*

Proof. First recall that if G is a group, $H \leq G$, and $g \in G$, then $gHg^{-1} \cong H$. Then by the last lemma and the fact that $\mathrm{GL}_{2n}(K)$ acts transitively on the lattices of V, it suffices to prove the proposition for any lattice L of V. Let $\mathcal{B} = \{e_1, \ldots, e_{2n}\}$ be the standard unit basis of V, and let $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$, so that $L \cong \mathcal{O}^{2n}$. Let

$g \in \mathrm{GSp}_n(K)$ such that $gL = L$. Then for all $1 \le i \le n$, there are $a_{ji}, b_{ji}, c_{ji}, d_{ji} \in \mathcal{O}$ such that $gu_i = \sum_{j=1}^n a_{ji}u_j + \sum_{j=1}^n c_{ji}w_j$ and $gw_i = \sum_{j=1}^n b_{ji}u_j + \sum_{j=1}^n d_{ji}w_j$; hence, $g \in M_{2n}(\mathcal{O})$. But $g\mathcal{B}$ a basis of $L \cong \mathcal{O}^{2n}$ implies $g \in \mathrm{GL}_{2n}(\mathcal{O})$; i.e., $g \in \mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$. On the other hand, any element of $\mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$ is an element of $\mathrm{GSp}_n(K)$ that takes an $\mathcal{O}$-basis of $\mathcal{O}^{2n}$ to an $\mathcal{O}$-basis of $\mathcal{O}^{2n}$. Since we can identify L with $\mathcal{O}^{2n}$, this implies that any element of $\mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O})$ fixes L, so the stabilizer of L in $\mathrm{GSp}_n(K)$ is

$$\mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O}) = \{g \in M_{2n}(\mathcal{O}) : g^t J_n g = \nu(g) J_n,\ \nu(g) \in K^\times,\ \det g \in \mathcal{O}^\times\}.$$

Since $(\det g)^2 = \nu(g)^{2n}$ by (1.8), $\det g \in \mathcal{O}^\times$ implies $\nu(g) \in \mathcal{O}^\times$; hence, $\mathrm{GSp}_n(K) \cap \mathrm{GL}_{2n}(\mathcal{O}) = \mathrm{GSp}_n(\mathcal{O})$. □

Lemma 2.2.2. *Let $t \in \Delta_n$ be a vertex with a primitive representative L, and let Σ be an apartment of Δ_n containing t. Then there is a symplectic basis $\mathcal{B}$ of V specifying Σ as in Lemma 1.3.5 such that $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$.*

Proof. Let $\mathcal{B}' = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ be a symplectic basis of V specifying Σ as in Lemma 1.3.5. Then [43, p. 3] implies that $L = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}'}$. Let $\mathcal{B} = \{\pi^{a_1}u_1, \ldots, \pi^{a_n}u_n, \pi^{-a_1}w_1, \ldots, \pi^{-a_n}w_n\}$. Then $\mathcal{B}$ is a symplectic basis of V specifying Σ as in Lemma 1.3.5. Since $\mathcal{B}$ is also an $\mathcal{O}$-basis of L, $L = (0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}}$. □

The next lemma is an analogue of Lemma 1.3.4. Recall that if Σ is an apartment of Δ_n and $\mathcal{B}$ is a symplectic basis of V specifying Σ as in Lemma 1.3.5, then $\mathcal{B}$ specifies an apartment $\widetilde{\Sigma}$ of Ξ_{2n}.

Lemma 2.2.3. *Let Σ be an apartment of Δ_n and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} such that $\mathcal{B}$ a symplectic basis of V specifying Σ implies $\mathcal{B}$ specifies $\widetilde{\Sigma}$. If C, C' is a gallery in Σ, then there is a gallery D, D' in $\widetilde{\Sigma}$ such that D (resp., D') contains C (resp., C') and $C \neq C'$ implies $D \neq D'$.*

Proof. If $C = C'$, then $C \in \widetilde{\Sigma}$ implies there is a chamber $D \in \widetilde{\Sigma}$ containing C; setting $D = D'$ finishes the proof. Now suppose $C \neq C'$, with C corresponding to the chain of lattices

$$\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0. \tag{2.3}$$

Let $\mathcal{B}$ be a symplectic basis of V specifying Σ as in Lemma 1.3.5. Then, as in the proof of Lemma 1.3.4, there is a $0 \le j \le n$ such that $C \cap C'$ (the codimension-one simplex of Δ_n shared by C and C') corresponds to (2.3) with either L_j deleted if $1 \le j \le n$ or with both πL_0 and L_0 deleted if $j = 0$. It follows that if t' is the vertex of C' not in $C \cap C'$, then t' has a representative L' such that C' corresponds to (2.3) with L_j replaced by L'. Note that $C \neq C'$ implies $L_j \neq L'$. As in Lemma 1.3.4, we consider three cases, depending on the value of j.

First suppose $1 \leq j \leq n-1$. Then by [43, p. 3], $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ for some $a_i \in \mathbb{Z}$, and the proof of Proposition 1.6.2 implies $L_n = (b_1, \ldots, b_n; 1 - b_1, \ldots, 1 - b_n)_{\mathcal{B}}$ for some $b_i \in \mathbb{Z}$ such that $a_i \leq b_i \leq a_i + 1$ for all $1 \leq i \leq n$. For $1 \leq i \leq n$, let $a_{n+i} = -a_i$ and $b_{n+i} = 1 - b_i$, and note that $b_i - a_i \in \{0, 1\}$ for all $1 \leq i \leq 2n$. Then $b_i = a_i + 1$ for exactly n values of i by Proposition 1.6.2. Let $\{i_1, \ldots, i_n\}$ be those n values, and for each $1 \leq r \leq n-1$, set $L_{n+r} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_\ell = b_\ell - 1 = a_\ell$ if $\ell \in \{i_1, \ldots, i_r\}$ and $c_\ell = b_\ell$ otherwise. Then $L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, and setting D (resp., D') to be the simplex of $\widetilde{\Sigma}$ with vertices the vertices of C (resp., the vertices of C'), together with the vertices $[L_{n+1}], \ldots, [L_{2n-1}]$ finishes the proof in this case (see the proof of Lemma 1.3.4 for the details).

Now suppose $j = n$, and let $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ for some $a_i \in \mathbb{Z}$, $L_n = (b_1, \ldots, b_n; 1 - b_1, \ldots, 1 - b_n)_{\mathcal{B}}$ for some $b_i \in \mathbb{Z}$ such that $a_i \leq b_i \leq a_i + 1$ for all $1 \leq i \leq n$, and $L' = (b'_1, \ldots, b'_n; 1 - b'_1, \ldots, 1 - b'_n)_{\mathcal{B}}$ for some $b'_i \in \mathbb{Z}$ such that $a_i \leq b'_i \leq a_i + 1$ for all $1 \leq i \leq n$ (we can do this because of our comments in the first paragraph of this proof, [43, p. 3], and the proof of Proposition 1.6.2). Note that $L_n \neq L'$ implies $b_i \neq b'_i$ for at least one value of i. Let $L_{n+1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \min\{b_i, b'_i\}$ and $c_{n+i} = \min\{1 - b_i, 1 - b'_i\}$ for all $1 \leq i \leq n$. Then $L_n, L' \subsetneq L_{n+1}$ and by the proof of Lemma 1.3.4, $[L_{n+1} : L_n] = q = [L_{n+1} : L']$; i.e., if $a_{n+i} = -a_i$ for all $1 \leq i \leq n$, then $c_i = a_i + 1$ for exactly $n-1$ values of i and $c_i = a_i$ otherwise (so that $L_{n+1}/\pi L_0 \cong k^{n+1}$). A modification of the second half of the last paragraph (starting from "Let $\{i_1, \ldots, i_n\}$") gives lattices $L_{n+2}, \ldots, L_{2n-1}$ such that $L_{n+1} \subsetneq L_{n+2} \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$, and setting D and D' as in the last paragraph finishes the proof in this case.

Finally, suppose $j = 0$, and let $L_0 = (a_1, \ldots, a_n; -a_1, \ldots, -a_n)_{\mathcal{B}}$ for some $a_i \in \mathbb{Z}$, $L' = (a'_1, \ldots, a'_n; -a'_1, \ldots, -a'_n)_{\mathcal{B}}$ for some $a'_i \in \mathbb{Z}$, and $L_n = (b_1, \ldots, b_n; 1 - b_1, \ldots, 1 - b_n)_{\mathcal{B}}$ for some $b_i \in \mathbb{Z}$ such that $a_i \leq b_i \leq a_i + 1$ and $a'_i \leq b_i \leq a'_i + 1$ for all $1 \leq i \leq n$ (as in the last paragraph, we can do this because of our comments in the first paragraph of this proof, [43, p. 3], and the proof of Proposition 1.6.2). Note that $L_0 \neq L'$ implies $a_i \neq a'_i$ for at least one value of i. Let $L_{2n-1} = (c_1, \ldots, c_n; c_{n+1}, \ldots, c_{2n})_{\mathcal{B}}$, where $c_i = \max\{a_i, a'_i\}$ and $c_{n+i} = \max\{-a_i, -a'_i\}$ for all $1 \leq i \leq n$. Then $L_{2n-1} \subsetneq L_0, L'$ and by the proof of Lemma 1.3.4, $[L_0 : L_{2n-1}] = q = [L' : L_{2n-1}]$; i.e., if $b_{n+i} = 1 - b_i$ for all $1 \leq i \leq n$, then $c_i = b_i + 1$ for exactly 1 value of i and $c_i = b_i$ otherwise (so that $L_{2n-1}/\pi L_0 \cong k^{2n-1} \cong L_{2n-1}/\pi L')$. A modification of the second half of the first paragraph (starting from "Let $\{i_1, \ldots, i_n\}$") gives lattices $L_{n+1}, \ldots, L_{2n-2}$ such that $L_n \subsetneq L_{n+1} \subsetneq \cdots \subsetneq L_{2n-2} \subsetneq L_{2n-1}$, and setting D and D' as in the second paragraph finishes the proof in this case. □

As we will see, it will sometimes be convenient to prove a result about the type 0 vertices of Δ_n and then use the action of $\mathrm{GSp}_n(K)$ on the special vertices (specifically Corollary 1.6.1 part 3 and Lemma 1.6.2) to infer the same result about the type n vertices. For this purpose, we need to know whether $\mathrm{GSp}_n(K)$ acts on the set of galleries of Ξ_{2n} and if so, whether the action preserves the length of a gallery. As we obtained information about the types of the vertices of Δ_n by considering the types of the vertices of Ξ_{2n}, we first consider whether $\mathrm{GL}_{2n}(K)$ acts on the set of galleries of Ξ_{2n} and if so, whether the action preserves the length of a gallery. For $g \in \mathrm{GL}_{2n}(K)$ and a chamber $C \in \Xi_{2n}$, abuse notation and write gC for the image of the vertices of C under the action of g.

Lemma 2.2.4. *The group* $\mathrm{GL}_{2n}(K)$ *acts on the chambers of* Ξ_{2n}.

Proof. Let $g \in \mathrm{GL}_{2n}(K)$, and let C be a chamber of Ξ_{2n}. Let Σ be an apartment of Ξ_{2n} containing C and $\mathcal{B}$ a basis of V specifying Σ as in Lemma 1.2.1 applied to Ξ_{2n}. Suppose C corresponds to the chain of lattices $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$. Write $L_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{\mathcal{B}}$, and note that the inclusion relation among the L_j implies

$$a_i^{(0)} + 1 \geq a_i^{(1)} \geq \cdots \geq a_i^{(2n-1)} \geq a_i^{(0)} \qquad \text{and} \qquad b_i^{(0)} + 1 \geq b_i^{(1)} \geq \cdots \geq b_i^{(2n-1)} \geq b_i^{(0)}$$

for all i. Then $gL_j = (a_1^{(j)}, \ldots, a_n^{(j)}; b_1^{(j)}, \ldots, b_n^{(j)})_{g\mathcal{B}}$, and the above inequalities finish the proof. □

Proposition 2.2.2. *The group* $\mathrm{GL}_{2n}(K)$ *maps a gallery in* Ξ_{2n} *of length* m *to a gallery in* Ξ_{2n} *of length* m. *In particular, if* $C \neq C'$ *are adjacent chambers of* Ξ_{2n} *and* $g \in \mathrm{GL}_{2n}(K)$, *then* $gC \neq gC'$ *are adjacent chambers of* Ξ_{2n}.

Proof. Let $C_0, \ldots, C_m$ be a gallery in Ξ_{2n} and $g \in \mathrm{GL}_{2n}(K)$. If $m = 0$, we are done by the last lemma. If $m = 1$ and $C_0 = C_1$, then $gC_0 = gC_1$, and gC_0, gC_1 is a gallery in Ξ_{2n}. Now suppose $C_0 \neq C_1$, and let $t_0, \ldots, t_{2n-1}$ (resp., $x_0, \ldots, x_{2n-1}$) be the vertices of C_0 (resp., of C_1). Then $C_0 \neq C_1$ implies that $t_j \neq x_j$ for some $0 \leq j \leq 2n-1$ and $t_i = x_i$ for all $0 \leq i \neq j \leq 2n-1$. For $0 \leq i \leq 2n-1$, let $L_i \in t_i$ and $M_i \in x_i$ such that $\pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_{2n-1} \subsetneq L_0$ and $\pi M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{2n-1} \subsetneq M_0$

correspond to C_0 and C_1, respectively. Then $\pi(gL_0) = g(\pi L_0) \subsetneq gL_1 \subsetneq \cdots \subsetneq gL_{2n-1} \subsetneq gL_0$ and $\pi(gM_0) = g(\pi M_0) \subsetneq gM_1 \subsetneq \cdots \subsetneq gM_{2n-1} \subsetneq gM_0$. Note that $t_i = x_i$ implies $[L_i] = [M_i]$. Since L_i is an $\mathcal{O}$-module and every $\alpha \in K^\times$ has the form $\alpha = \pi^m u$ for some $m \in \mathbb{Z}$ and some $u \in \mathcal{O}^\times$, this means that $M_i = \pi^m L_i$ for some $m \in \mathbb{Z}$; i.e., $gM_i = g(\pi^m L_i) = \pi^m(gL_i)$, and $[gL_i] = [gM_i]$. Since gC_0 has vertices $[gL_0], \dots, [gL_{2n-1}]$ and gC_1 has vertices $[gM_0], \dots, [gM_{2n-1}]$, it follows that gC_0, gC_1 is a gallery in Ξ_{2n}. To see that $gC_0 \neq gC_1$, we must show that $[gL_j] \neq [gM_j]$. If $[gL_j] = [gM_j]$, then $gL_j = \pi^m(gM_j)$ for some $m \in \mathbb{Z}$, so $gL_j = \pi^m(gM_j) = g(\pi^m M_j)$, and $L_j = g^{-1}(gL_j) = g^{-1}(g\pi^m M_j) = \pi^m M_j$; i.e., $[L_j] = [M_j]$, a contradiction. If $m \geq 2$, our previous work implies that gC_i, gC_{i+1} is a gallery in Ξ_{2n} for all $0 \leq i \leq m-1$, with $gC_i \neq gC_{i+1}$ if $C_i \neq C_{i+1}$. Thus, $gC_0, \dots, gC_m$ is a gallery in Ξ_{2n}. □

Corollary 2.2.1. *The group $\mathrm{GSp}_n(K)$ maps a gallery in Ξ_{2n} of length m to a gallery in Ξ_{2n} of length m. In particular, if $C \neq C'$ are adjacent chambers of Ξ_{2n} and $g \in \mathrm{GSp}_n(K)$, then $gC \neq gC'$ are adjacent chambers of Ξ_{2n}.*

Proof. This follows from the last proposition since $\mathrm{GSp}_n(K) \leq \mathrm{GL}_{2n}(K)$. □

2.2.2 Symplectic divisors

Following [43, pp. 9 – 10], let S be a set of representatives for $K^\times/\mathcal{O}^\times$; for convenience, we take $S = \{\pi^m : m \in \mathbb{Z}\}$. Let $G^S = \{g \in \mathrm{GSp}_n(K) : \nu(g) \in S\}$ and $\Gamma = \mathrm{Sp}_n(\mathcal{O})$.

Remark. The analogues of the results in this section also hold if we replace G^S and $\mathrm{Sp}_n(\mathcal{O})$ with $\mathrm{GSp}_n(K)$ and $\mathrm{GSp}_n(\mathcal{O})$, respectively; see Appendix A.

A lattice L of V is *symplectic* if it has the form $L = \mathcal{O}u_1 + \cdots + \mathcal{O}u_n + \mathcal{O}w_1 + \cdots + \mathcal{O}w_n$ for some symplectic basis $\{u_1, \dots, u_n, w_1, \dots, w_n\}$ of V. Note that since a symplectic basis of V specifies an apartment of Δ_n by Lemma 1.3.5, [43, p. 3] implies that a lattice of V is symplectic if and only if it is primitive. Fix a symplectic lattice L_0 of V, and let $\mathcal{R} = \mathcal{R}(L_0) = \{gL_0 : g \in G^S\}$.

Proposition 2.2.3. *A lattice L of V is in $\mathcal{R}$ if and only if $[L]$ is a special vertex of Δ_n.*

Proof. First note that since L_0 is symplectic, L_0 is primitive; hence, Propositions 1.5.2 and 1.5.3 imply $[L_0]$ is a special vertex of Δ_n. Let $L \in \mathcal{R}$. Then there is a $g \in G^S$ such that $L = gL_0$; i.e., such that $[L] = g[L_0]$. Since $G^S \subseteq \mathrm{GSp}_n(K)$, the fact that $[L]$ is special follows from Corollary 1.6.3. The converse follows from [43, Proposition 3.3]. □

Since we use use a left action, we prove the analogues of the results in section 4.1 of [43].

Proposition 2.2.4. *Let $L \in \mathcal{R}$. Then we can identify $\Gamma = \mathrm{Sp}_n(\mathcal{O})$ with $\{g \in G^S : gL = L\}$, where g acts on L as the matrix of a linear transformation with respect to a fixed basis of L.*

Proof. This follows from Proposition 2.2.1 since

$$\begin{aligned} \mathrm{GSp}_n(\mathcal{O}) \cap G^S &= \{g \in M_{2n}(\mathcal{O}) : g^t J_n g = \nu(g) J_n,\ \nu(g) = \pi^m \in \mathcal{O}^\times\} \\ &= \{g \in M_{2n}(\mathcal{O}) : g^t J_n g = \nu(g) J_n,\ \nu(g) = 1\} = \mathrm{Sp}_n(\mathcal{O}), \end{aligned}$$

as claimed. □

Lemma 2.2.5. *Let $L, M \in \mathcal{R}$. Then there is a basis $\{x_1, \dots, x_n, y_1, \dots, y_n\}$ of V with $\langle x_i, y_j \rangle = \pi^r \delta_{ij}$ for some $r \in \mathbb{Z}$ and $\langle x_i, x_j \rangle = 0 = \langle y_i, y_j \rangle$ and elements $\alpha_i, \beta_i \in S$ with $\beta_1 \mathcal{O} \subseteq \cdots \subseteq \beta_n \mathcal{O} \subseteq \alpha_n \mathcal{O} \subseteq \cdots \subseteq \alpha_1 \mathcal{O}$ and $\beta_i \alpha_i = \pi^{r'-r} \in S$ for some $r' \in \mathbb{Z}$ such that*

$$\begin{aligned} L &= \mathcal{O}x_1 + \cdots + \mathcal{O}x_n + \mathcal{O}y_1 + \cdots + \mathcal{O}y_n, \\ M &= \mathcal{O}\alpha_1 x_1 + \cdots + \mathcal{O}\alpha_n x_n + \mathcal{O}\beta_1 y_1 + \cdots + \mathcal{O}\beta_n y_n. \end{aligned}$$

Remark. The ideals $\alpha_i \mathcal{O}$ and $\beta_i \mathcal{O}$ are the *symplectic divisors* of M in L and coincide with the standard elementary divisors of M in L. In other words, if we choose two lattices from $\mathcal{R}$ and consider their elementary divisors in the traditional sense, they satisfy the above-stated additional properties. For $L, M \in \mathcal{R}$, write $\{L : M\} = \{\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n\}$ if there are bases of L and M as in the lemma.

Proof. Let $L, M \in \mathcal{R}$. Since $[L]$ and $[M]$ are special vertices of Δ_n by Proposition 2.2.3, there is an apartment Σ of Δ_n containing both $[L]$ and $[M]$. Let $\{u_1, \dots, u_n, w_1, \dots, w_n\}$ be a symplectic basis of V specifying Σ as in Lemma 1.3.5. Then [43, p. 6] implies that there are $a_1, \dots, a_n, b_1, \dots, b_n, r, r' \in \mathbb{Z}$ such that

$$\begin{aligned} L &= \mathcal{O}\pi^{a_1} u_1 + \cdots + \mathcal{O}\pi^{a_n} u_n + \mathcal{O}\pi^{r-a_1} w_1 + \cdots + \mathcal{O}\pi^{r-a_n} w_n, \\ M &= \mathcal{O}\pi^{b_1} u_1 + \cdots + \mathcal{O}\pi^{b_n} u_n + \mathcal{O}\pi^{r'-b_1} w_1 + \cdots + \mathcal{O}\pi^{r'-b_n} w_n. \end{aligned}$$

For all $1 \le j \le n$, let $a_{n+j} = r - a_j$, $b_{n+j} = r' - b_j$, and $u_{n+j} = w_j$. Let $1 \le i_1 \le 2n$

be minimal such that $b_{i_1} - a_{i_1} \geq b_j - a_j$ for all $j \neq i_1$, and set

$$y_1 = \pi^{a_{i_1}} u_{i_1} = \begin{cases} \pi^{a_{i_1}} u_{i_1} & \text{if } 1 \leq i_1 \leq n, \\ \pi^{r-a_{i_1-n}} w_{i_1-n} & \text{if } n+1 \leq i_1 \leq 2n, \end{cases}$$

$$x_1 = \begin{cases} -\pi^{r-a_{i_1}} w_{i_1} & \text{if } 1 \leq i_1 \leq n, \\ \pi^{a_{i_1-n}} u_{i_1-n} & \text{if } n+1 \leq i_1 \leq 2n. \end{cases}$$

Then $\langle x_1, y_1 \rangle = \pi^r$.

Proceeding inductively, let $1 \leq m \leq n-1$, and suppose $i_1, \ldots, i_m$ have been chosen such that for all $1 \leq \ell \leq m$, $b_{i_\ell} - a_{i_\ell} \geq b_j - a_j$ for all $j \not\equiv i_1, \ldots, i_\ell, i_1 + n, \ldots, i_\ell + n \mod 2n$, and suppose $y_1, \ldots, y_m, x_1, \ldots, x_m$ have been set as in the last paragraph. Then $\langle x_\ell, y_\ell \rangle = \pi^r$ for all $1 \leq \ell \leq m$, $\langle x_\ell, y_j \rangle = 0$ if $\ell \neq j$, and $\langle x_\ell, x_j \rangle = 0 = \langle y_\ell, y_j \rangle$ for all $1 \leq \ell, j \leq m$. Let $1 \leq i_{m+1} \leq 2n$ be minimal such that $i_{m+1} \not\equiv i_1, \ldots, i_m, i_1 + n, \ldots, i_m + n \mod 2n$ and $b_{i_{m+1}} - a_{i_{m+1}} \geq b_j - a_j$ for all $j \not\equiv i_1, \ldots, i_m, i_{m+1}, i_1 + n, \ldots, i_m + n \mod 2n$. Set

$$y_{m+1} = \pi^{a_{i_{m+1}}} u_{i_{m+1}} = \begin{cases} \pi^{a_{i_{m+1}}} u_{i_{m+1}} & \text{if } 1 \leq i_{m+1} \leq n, \\ \pi^{r-a_{i_{m+1}-n}} w_{i_{m+1}-n} & \text{if } n+1 \leq i_{m+1} \leq 2n, \end{cases}$$

$$x_{m+1} = \begin{cases} -\pi^{r-a_{i_{m+1}}} w_{i_{m+1}} & \text{if } 1 \leq i_{m+1} \leq n, \\ \pi^{a_{i_{m+1}-n}} u_{i_{m+1}-n} & \text{if } n+1 \leq i_{m+1} \leq 2n. \end{cases}$$

Then $\langle x_{m+1}, y_{m+1} \rangle = \pi^r$, $\langle x_{m+1}, y_j \rangle = 0$ for all $1 \leq j \leq m$, and $\langle x_{m+1}, x_j \rangle = 0 = \langle y_{m+1}, y_j \rangle$ for all $1 \leq j \leq m+1$. Continuing this process until y_n and x_n have been set gives $2n$ vectors $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ in V such that $\langle x_i, y_j \rangle = \pi^r \delta_{ij}$ and $\langle x_i, x_j \rangle = 0 = \langle y_i, y_j \rangle$. Note that $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is an $\mathcal{O}$-basis of L by construction, so it is also a K-basis of V. It follows that

$$M = \mathcal{O}\pi^{c_1} x_1 + \cdots + \mathcal{O}\pi^{c_n} x_n + \mathcal{O}\pi^{d_1} y_1 + \cdots + \mathcal{O}\pi^{d_n} y_n,$$

where for all $1 \leq j \leq n$,

$$d_j = b_{i_j} - a_{i_j}, \qquad c_j = (r' - b_{i_j}) - (r - a_{i_j}).$$

Let $\alpha_j = \pi^{c_j}$ and $\beta_j = \pi^{d_j}$ for all $1 \leq j \leq n$. Then $\beta_i \mathcal{O} \subseteq \beta_j \mathcal{O}$ for all $i+1 \leq j \leq n$ and $\beta_i \mathcal{O} \subseteq \alpha_i \mathcal{O}$ for all i by construction. In particular, $\beta_1 \mathcal{O} \subseteq \cdots \subseteq \beta_n \mathcal{O}$. It therefore remains to show that $\alpha_n \mathcal{O} \subseteq \cdots \subseteq \alpha_1 \mathcal{O}$. But $c_j + d_j = r' - r = c_{j+1} + d_{j+1}$ and

$d_j \geq d_{j+1}$ for all $1 \leq j \leq n-1$ imply

$$c_j + d_j = c_{j+1} + d_{j+1} \leq c_{j+1} + d_j;$$

hence, $\alpha_j\mathcal{O} = \pi^{c_j}\mathcal{O} \supseteq \pi^{c_{j+1}}\mathcal{O} = \alpha_{j+1}\mathcal{O}$ for all $1 \leq j \leq n-1$. $\square$

Lemma 2.2.6. *Let $L \in \mathcal{R}$. For $h_1, h_2 \in G^S$, the left cosets $h_1\Gamma$ and $h_2\Gamma$ are equal if and only if $h_1L = h_2L$.*

Proof. Recall that $h_1\Gamma = h_2\Gamma$ if and only if $h_1^{-1}h_2 \in \Gamma$, which is true if and only if $h_1^{-1}h_2L = L$ by Proposition 2.2.4. $\square$

Lemma 2.2.7. *Let $L, M, M' \in \mathcal{R}$ with L symplectic. Then $\{L : M\} = \{L : M'\}$ if and only if there is a $g \in \Gamma$ such that $gM = M'$.*

Proof. Suppose there is a $g \in \Gamma$ such that $gM = M'$. Then $\{L : M'\} = \{L : gM\} = \{g^{-1}L : g^{-1}gM\} = \{g^{-1}L : M\} = \{L : M\}$ by Proposition 2.2.4. Conversely, suppose $\{L : M\} = \{L : M'\}$. Since L is symplectic and hence primitive, [43, p. 3] and Lemma 2.2.5 imply that there are symplectic bases $\{u_1^{(j)}, \ldots, u_n^{(j)}, w_1^{(j)}, \ldots, w_n^{(j)}\}$ of V for $j = 1, 2$ and $\alpha_i, \beta_i \in S$ with $\beta_1\mathcal{O} \subseteq \cdots \subseteq \beta_n\mathcal{O} \subseteq \alpha_n\mathcal{O} \subseteq \cdots \subseteq \alpha_1\mathcal{O}$ such that

$$\begin{aligned} L &= \mathcal{O}u_1^{(1)} + \cdots + \mathcal{O}u_n^{(1)} + \mathcal{O}w_1^{(1)} + \cdots + \mathcal{O}w_n^{(1)}, \\ M &= \mathcal{O}\alpha_1u_1^{(1)} + \cdots + \mathcal{O}\alpha_nu_n^{(1)} + \mathcal{O}\beta_1w_1^{(1)} + \cdots + \mathcal{O}\beta_nw_n^{(1)}, \\ L &= \mathcal{O}u_1^{(2)} + \cdots + \mathcal{O}u_n^{(2)} + \mathcal{O}w_1^{(2)} + \cdots + \mathcal{O}w_n^{(2)}, \\ M' &= \mathcal{O}\alpha_1u_1^{(2)} + \cdots + \mathcal{O}\alpha_nu_n^{(2)} + \mathcal{O}\beta_1w_1^{(2)} + \cdots + \mathcal{O}\beta_nw_n^{(2)}. \end{aligned}$$

Let g be the matrix of the linear transformation (with respect to either symplectic basis) that takes $u_i^{(1)}$ to $u_i^{(2)}$ and $w_i^{(1)}$ to $w_i^{(2)}$. Since g takes a symplectic basis to a symplectic basis, $g \in \mathrm{Sp}_n(K) \subseteq G^S$. On the other hand, $gL = L$, so Proposition 2.2.4 implies $g \in \mathrm{Sp}_n(\mathcal{O})$. Finally, $gM = M'$. $\square$

Proposition 2.2.5. *Let $L \in \mathcal{R}$ be a symplectic lattice, and identify $\Gamma = \mathrm{Sp}_n(\mathcal{O})$ with the stabilizer of L in G^S. Let $g \in G^S$ with*

$$\Gamma g\Gamma = \Gamma\mathrm{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\Gamma.$$

Then $h\Gamma \mapsto hL$ gives a one-to-one correspondence between the cosets $h\Gamma$ in $\Gamma g\Gamma$ and the lattices $M \in \mathcal{R}$ with $\{L : M\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$.

Proof. We may assume that $g = \mathrm{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. If $h\Gamma = h_1 g\Gamma$ with $h_1 \in \Gamma$ (so that $h\Gamma$ is a coset in $\Gamma g\Gamma$), then Proposition 2.2.4 implies $hL \in \mathcal{R}$ with $\{L : hL\} = \{L : h_1 gL\} = \{h_1^{-1}L : gL\} = \{L : gL\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$. Conversely, if $M \in \mathcal{R}$ and $\{L : M\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$, then Lemma 2.2.7 implies that there is an $h_1 \in \Gamma$ such that $M = h_1 gL$ ($\{L : gL\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$). Since $h_1 \in \Gamma$, $h_1 g\Gamma \subseteq \Gamma g\Gamma$. By Lemma 2.2.6, the left cosets $h\Gamma$ and $h_1\Gamma$ are equal if and only if $hL = h_1 L$, so the correspondence is one-to-one. □

We now follow [41, p. 123]. Fix an apartment Σ of Δ_n and a chamber $C_0 \in \Sigma$. As in [41, p. 119], the chambers of Σ are in one-to-one correspondence with the elements of W, the Weyl group of Δ_n. If $g \in W$ corresponds to the chamber C of Σ, write $g \leftrightarrow C$. We also associate the generators of W with reflections across the codimension-one faces of C_0. Following [43, p. 4], if $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ is a symplectic basis of V specifying Σ as in Lemma 1.3.5, the reflections across the codimension-one faces of C_0 are given by their action on $\mathcal{B}$ as follows (any basis vector not specified is fixed):

s_1: $u_n \leftrightarrow w_n$,

s_i: $u_{n-i+1} \leftrightarrow u_{n-i+2}$ and $w_{n-i+1} \leftrightarrow w_{n-i+2}$ for all $2 \leq i \leq n$,

s_{n+1}: $u_1 \mapsto \pi w_1$ and $w_1 \mapsto \pi^{-1} u_1$.

Note that the first n reflections just interchange one or two pairs of basis vectors.

Let $C_0 \in \Sigma$ be the chamber with vertices

$$t_0 = [0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}}, t_1 = [0, 1, \ldots, 1; 1, \ldots, 1]_{\mathcal{B}}, \ldots, t_n = [0, \ldots, 0; 1, \ldots, 1]_{\mathcal{B}}. \tag{2.4}$$

Then s_i fixes t_j for all $j \neq n + 1 - i$. Associate s_i with the generator r_i of W such that if $g \in W$ with $g \leftrightarrow C_0$, then $gr_i \leftrightarrow C$, where C is the chamber of Σ with vertices determined by the action of s_i on the vertices of C_0. Let $\langle r_i : 1 \leq i \leq n \rangle$ be the subgroup of W generated by $r_1, \ldots, r_n$. Write s_{r_i} for s_i. If $w = r_{i_1} \cdots r_{i_m} \in W$, let $s_w = s_{r_{i_1}} \circ \cdots \circ s_{r_{i_m}}$. Finally (abusing notation), write $s_w C_0$ for the chamber of Σ with vertices determined by the action of s_w on the vertices of C_0. The proof of Proposition 2.3 of [41] also proves the following lemma.

Lemma 2.2.8. *Let Σ and C_0 be as above and $g \in W$ such that $g \leftrightarrow C_0$.*

1. *If $w \in W$, then $gw \leftrightarrow s_w C_0$.*

2. *If $h, w \in W$ and $h \leftrightarrow C_h$, then $hw \leftrightarrow s_{vwv^{-1}} C_h$, where $h = gv$.*

We now prove an analogue of Theorem 3.3 of [41] for the type 0 vertices of Δ_n.

Proposition 2.2.6. *If $t \in \Delta_n$ is a type 0 vertex, then the number of vertices of Δ_n close to t is the number of left cosets of Γ in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. Let $t \in \Delta_n$ be a type 0 vertex and $t' \in \Delta_n$ a vertex close to t. Then there are adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$. Let Σ be an apartment of Δ_n containing C and C' and $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 2.2.2 applied to a primitive representative $L \in t$. Then $t = [0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}} \in C_0$, where C_0 has vertices as in (2.4) and $t_0 = t$. Since a reflection s_{r_i} fixes t if and only if $1 \leq i \leq n$, the chambers of Σ containing t are the chambers $s_\xi C_0$, where $\xi \in \langle r_i : 1 \leq i \leq n \rangle \leq W$; hence, $C = s_\xi C_0$ for some $\xi \in \langle r_i : 1 \leq i \leq n \rangle \leq W$. Let $g \in W$ such that $g \leftrightarrow C_0$. Then by Lemma 2.2.8 part 1, $C = s_\xi C_0 \leftrightarrow g\xi$. Moreover, C and C' adjacent with $t \notin C'$ implies $C' \leftrightarrow g\xi r_{n+1}$; i.e., $C' = s_{\xi r_{n+1}} C_0$ by Lemma 2.2.8 part 1. It follows that

$$\begin{aligned} s_{\xi r_{n+1}} t = s_{\xi r_{n+1}}[0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}} &= s_\xi[-1, 0, \ldots, 0; 1, 0, \ldots, 0]_{\mathcal{B}} \\ &= s_\xi[0, 1, \ldots, 1; 2, 1, \ldots, 1]_{\mathcal{B}} = t'. \end{aligned}$$

Since s_{r_i} permutes one or two pairs of coordinates for all $1 \leq i \leq n$, we can write $t' = [a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}}$, where $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ is a permutation of $\{0, 1, \ldots, 1, 2\}$ such that $a_i + b_i = 2$ for all i (recall that t' close to t implies that t' has the same type as t, hence is special by Proposition 1.5.3). By reordering the elements of $\mathcal{B}$ if necessary, we can assume that $t' = [0, 1, \ldots, 1; 2, 1, \ldots, 1]_{\mathcal{B}}$.

Thus, to find the vertices of Δ_n close to $t = [L]$, it suffices to determine the homothety classes of lattices with a representative M such that with respect to some $\mathcal{O}$-basis of L, $\{L : M\} = \{0, 1, \ldots, 1; 2, 1, \ldots 1\}$. But Proposition 2.2.5 implies that the number of such classes is the number of left cosets of Γ in $\Gamma \mathrm{diag}(1, \pi, \ldots, \pi, \pi^2, \pi, \ldots, \pi)\Gamma$, where $\mathrm{diag}(1, \pi, \ldots, \pi, \pi^2, \pi, \ldots, \pi) \in \mathrm{GSp}_n(K)$. Finally, suppose M_1 and M_2 are distinct lattices with $\{L : M_1\} = \{L : M_2\}$. If $M_1 = \alpha M_2$ for some $\alpha \in K^\times$ with $\alpha = \pi^m u$, then $\{L : M_1\} = \{m, m+1, \ldots, m+1; m+2, m+1, \ldots, m+1\} = \{L : M_2\}$ if and only if $m = 0$; i.e., if and only if $\alpha \in \mathcal{O}^\times$. It follows that $M_1 = \alpha M_2 = M_2$, contradicting our assumption that $M_1 \neq M_2$; hence, $[M_1] \neq [M_2]$. ◻

Theorem 2.2.1. *If $t \in \Delta_n$ is a special vertex, then the number of vertices of Δ_n*

close to t is the number of left cosets of Γ *in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. Let $t \in \Delta_n$ be a special vertex. Then by Proposition 1.5.3, t has type either 0 or n. If t has type 0, the last proposition proves the claim, so suppose t has type n. Let $t' \in \Delta_n$ be a vertex close to t, and let $C, C' \in \Delta_n$ be adjacent chambers such that $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$. Let Σ be an apartment of Δ_n containing C and C', $\mathcal{B}$ a symplectic basis of V specifying Σ as in Lemma 1.3.5, and $\widetilde{\Sigma}$ the apartment of Ξ_{2n} specified by $\mathcal{B}$ as in Lemma 1.2.1 applied to Ξ_{2n}. Since $C \neq C'$, Lemma 2.2.3 implies that there are adjacent chambers $D, D' \in \widetilde{\Sigma}$ such that $D \neq D'$, D contains C, D' contains C', and $t, t' \notin D \cap D'$. Let $g \in \mathrm{GSp}_n(K)$ with $\nu(g) \equiv 1 \mod 2$. Since t has type n, Corollary 1.6.1 part 3 implies gt has type 0. By Lemma 1.6.2, gD (resp., gD') contains a chamber $C_1 \in \Delta_n$ (resp., a chamber $C_1' \in \Delta_n$) with $gt \in C_1$ (resp., with $gt' \in C_1'$). Furthermore, Corollary 2.2.1 implies $gD \neq gD'$ are adjacent chambers of Ξ_{2n}. Since $t, t' \notin D \cap D'$, the proof of Proposition 2.2.2 implies $gt, gt' \notin gD \cap gD'$. Thus, C_1 and C_1' are adjacent chambers of Δ_n with $gt \in C_1$, $gt' \in C_1'$, and $gt, gt' \notin C_1 \cap C_1'$; i.e., gt and gt' are close vertices of Δ_n.

Let S_t be the set of vertices of Δ_n close to t and S_{gt} the set of vertices of Δ_n close to gt. Then the last paragraph implies that we can define functions $\varphi : S_t \to S_{gt}$ and $\psi : S_{gt} \to S_t$ by $\varphi(t') = gt'$ and $\psi(t') = g^{-1}t'$. Then $\varphi(\psi(t')) = t'$ and $\psi(\varphi(t')) = t'$; i.e., φ is a bijection. Thus, $\mathrm{Card}(S_t) = \mathrm{Card}(S_{gt})$. The last proposition finishes the proof. $\square$

Remark. Note that the last theorem shows that the number of vertices of Δ_n close to a given special vertex of Δ_n is independent of the type of special vertex we are given.

2.2.3 Counting close vertices

We now count the number of vertices of Δ_n close to a given special vertex of Δ_n. Note that by Theorem 2.2.1, it suffices to do the count in the case that the given special vertex has type 0. As in Section 2.1, we can think of the homothety class $[L]$ of a lattice L of V as the infinite chain of lattices $\cdots \subsetneq \pi L \subsetneq L \subsetneq \pi^{-1}L \subsetneq \cdots$ and a chamber C of Δ_n as the infinite chain of lattices

$$\cdots \subsetneq \pi L_n \subsetneq \pi L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L_0 \subsetneq \pi^{-1}L_1 \subsetneq \cdots,$$

where L_0 is primitive, $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \le i \le n$, and $[L_1 : \pi L_0] = q = [L_i : L_{i-1}]$ for all $2 \le i \le n$. A codimension-one face of C is thus given by the above chain of lattices with $\ldots, \pi L_i, L_i, \pi^{-1}L_i, \ldots$ deleted for some $0 \le i \le n$. Recall that if $t, t' \in \Delta_n$ are close vertices, they have the same type.

Fix a type 0 vertex $t \in \Delta_n$. By Proposition 1.5.2, t has a primitive representative L, so a chamber $C \in \Delta_n$ containing t is given by the chain of lattices

$$\cdots \subsetneq \pi L_n \subsetneq \pi L \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq L \subsetneq \pi^{-1}L_1 \subsetneq \cdots, \tag{2.5}$$

where $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $1 \le i \le n$ and $[L_1 : \pi L] = q = [L_i : L_{i-1}]$ for all $2 \le i \le n$. The codimension-one face of C not containing t is thus

$$\cdots \subsetneq \pi L_n \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq \pi^{-1}L_1 \subsetneq \cdots,$$

and a vertex of Δ_n is close to t if and only if it has a primitive representative $M \neq L$ such that

$$\cdots \subsetneq \pi L_n \subsetneq \pi M \subsetneq L_1 \subsetneq \cdots \subsetneq L_n \subsetneq M \subsetneq \pi^{-1}L_1 \subsetneq \cdots, \tag{2.6}$$

where $[L_1 : \pi M] = q$. Given the lattice L_1, the possible L and M satisfy $L \neq M \subsetneq \pi^{-1}L_1$ with $[\pi^{-1}L_1 : L] = [L_1 : \pi L] = q = [L_1 : \pi M] = [\pi^{-1}L_1 : M]$ and both L and M primitive. On the other hand, if $t' \in \Delta_n$ is a vertex close to t, then there must be primitive representatives $L \in t$ and $M \in t'$ such that there are lattices $L_1, \ldots, L_n$ as in (2.5) with $L \neq M \subsetneq \pi^{-1}L_1$ and $[\pi^{-1}L_1 : L] = q = [\pi^{-1}L_1 : M]$. The same argument as in Section 2.1 (after (2.2)) shows that $\pi^{-1}L_1 = L + M$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i\rangle \subseteq \pi\mathcal{O}$ for all $2 \le i \le n$ and

$$\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L, \qquad \pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$$

correspond to chambers of Δ_n. In other words (as in the case of Ξ_n), if $t, t' \in \Delta_n$ are close type 0 vertices, there may be more than one pair of adjacent chambers $C, C' \in \Delta_n$ such that $t \in C$, $t' \in C'$, and $t, t' \notin C \cap C'$. We return to this later.

Before we count the number of vertices of Δ_n close to a given type 0 vertex $t \in \Delta_n$, we make a few observations similar to those preceding Proposition 2.1.1. Fix a primitive representative $L \in t$. Recall that $L/\pi L \cong k^{2n}$ and is endowed with the non-degenerate, alternating bilinear form $\langle\cdot,\cdot\rangle_0$. If U_1 is a 1-dimensional k-subspace of $L/\pi L$, the Correspondence Theorem implies that there is a unique $\mathcal{O}$-submodule L_1 of L containing πL such that $L_1/\pi L = U_1$ and $\pi L \subsetneq L_1 \subsetneq L$. Since both πL and L are free $\mathcal{O}$-modules of rank $2n$, L_1 is a lattice of V. Moreover, $[L_1 : \pi L] = q$ and

$\langle L_1, L_1 \rangle \subseteq \pi\mathcal{O}$ since every 1-dimensional k-subspace of $L/\pi L$ is totally isotropic (see [35, p. 5]). Thus, the number of L_1 equals the number of 1-dimensional k-subspaces of $L/\pi L$. Given L_1, let C be a chamber of Δ_n containing $[L_1]$ and t. This gives lattices $L_2, \ldots, L_n$ such that the chain of lattices $\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L$ corresponds to C. Since $[L_1], \ldots, [L_n]$ are the vertices of the codimension-one face A of C not containing t, the number of primitive lattices $M \neq L$ of V such that $M \subsetneq \pi^{-1}L_1$ and $[\pi^{-1}L_1 : M] = q$ is one less than the number of chambers of Δ_n containing A.

Proposition 2.2.7. *If $t \in \Delta_n$ is a special vertex of type 0, the number $\omega(\Delta_n)$ of vertices of Δ_n close to t is*

$$\frac{q^{2n}-1}{q-1} \cdot q.$$

Proof. By Proposition 1.5.2, t has a primitive representative L, and our previous comments imply that it suffices to count the number of 1-dimensional subspaces of $k^{2n} \cong \mathbb{F}_q^{2n}$ (the number of L_1) and the number of chambers of Δ_n containing a given codimension-one simplex of Δ_n. Since there are $(q^{2n}-1)/(q-1)$ 1-dimensional subspaces of $\mathbb{F}_q^{2n}$ and $q+1$ chambers of Δ_n containing a given codimension-one simplex by Proposition 1.3.7, the number of vertices of Δ_n close to t is

$$\frac{q^{2n}-1}{q-1} \cdot q$$

(recall from the remarks preceding this proposition that we select L_1 such that $[L_1 : \pi L] = q$, after which we want to count the number of lattices $M \neq L$ such that $[\pi^{-1}L_1 : M] = q$ with M primitive; we thus take a chamber $C \in \Delta_n$ containing $[L_1]$ and $t = [L]$ and count the number of chambers $C' \neq C$ in Δ_n adjacent to C and not containing t, which is one less than the number of chambers of Δ_n containing a given codimension-one simplex of Δ_n, so that we multiply by q rather than $q+1$). □

Proposition 2.2.8. *If $t \in \Delta_n$ is a special vertex, the number of vertices of Δ_n close to t is*

$$\omega(\Delta_n) = \frac{q^{2n}-1}{q-1} \cdot q.$$

Proof. This follows from the last proposition and Theorem 2.2.1. □

Let $r(\Delta_n)$ be the number of chambers of Δ_n containing a special vertex $t \in \Delta_n$. By Proposition 1.3.5,

$$r(\Delta_n) = \prod_{m=1}^{n} \frac{q^{2m}-1}{q-1},$$

which, with the last proposition, proves the following (cf. Theorem 2.1.1).

Theorem 2.2.2. *Let $r(\Delta_n)$ be as in Proposition 1.3.5 and $\omega(\Delta_n)$ as in Proposition 2.2.8. Then for all $n \geq 2$, $q \cdot r(\Delta_n) = r(\Delta_{n-1})\ \omega(\Delta_n)$, where $r(\Delta_1) = q+1$.*

Proof. Since $r(\Delta_2) = (q^2-1)(q^4-1)/(q-1)^2 = (q+1)(q^4-1)/(q-1)$,

$$\omega(\Delta_2) = \frac{q^4-1}{q-1} \cdot q = \frac{r(\Delta_2)}{q+1} \cdot q = \frac{r(\Delta_2)}{r(\Delta_1)} \cdot q.$$

For $n \geq 3$, note that $r(\Delta_n) = \prod_{m=1}^{n} ((q^{2m}-1)/(q-1)) = (q^{2n}-1)r(\Delta_{n-1})/(q-1)$. Then Proposition 2.2.8 implies

$$\omega(\Delta_n) = \frac{q^{2n}-1}{q-1} \cdot q = \frac{r(\Delta_n)}{r(\Delta_{n-1})} \cdot q. \qquad \square$$

As in Section 2.1, we use the structure of Δ_n to give an alternative proof for the relationship given in Theorem 2.2.2 for type 0 vertices.

Remark. Since the special vertices of Δ_n should be indistinguishable if we ignore types, we believe that one should be able to use the structure of Δ_n to prove the relationship given in Theorem 2.2.2 for the type n vertices of Δ_n, but our attempts so far have been unsuccessful. The trick that we use in the proof of Theorem 2.2.1 does not work in this case in light of Corollary 1.6.2, and the method we use below relies heavily on the fact that if L is a primitive lattice, then $L/\pi L$ has a non-degenerate, alternating k-bilinear form. On the other hand, our construction of Δ_n singles out the vertices with a primitive representative, which one need not do (see, for example, [1]).

Fix a type 0 vertex $t \in \Delta_n$. Then, as in Section 2.1, we can count the number of vertices of Δ_n close to t by counting the number of galleries (in Δ_n) of length 1 starting at a chamber containing t and ending at a chamber not containing t. Let $C \in \Delta_n$ be a chamber containing t. Since there are exactly $r(\Delta_n)$ chambers in Δ_n containing a given special vertex, there are $r(\Delta_n)$ possible C. A chamber $C' \in \Delta_n$ adjacent to C and not containing t must contain the codimension-one face of C not containing t. By Proposition 1.3.7, there are exactly $q+1$ chambers in Δ_n containing a given codimension-one simplex of Δ_n, so there are q possible chambers $C' \in \Delta_n$ adjacent to C not containing t. It follows that there are $r(\Delta_n) \cdot q$ galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber does not contain t. It therefore remains to determine if counting the number of galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber does not contain t overcounts the number of vertices of Δ_n close to t. To this end, fix a vertex $t' \in \Delta_n$

close to t, and suppose C, C' is a gallery in Δ_n with $t \in C$ and $t' \in C'$. Then we count t' more than once if there is a chamber $C_1 \neq C$ in Δ_n containing t and a chamber $C_1' \in \Delta_n$ adjacent to C_1 containing t' (note that since t and t' are close, $C_1' \neq C_1$); i.e., if there is more than one gallery of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t'. Since the number of vertices of Δ_n close to t is $\omega(\Delta_n)$, this implies $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/m(\Delta_n, t, t')$, where $m(\Delta_n, t, t')$ is the number of galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t'.

Remark. Nothing in the above paragraph requires that t have type 0, but our analysis of $m(\Delta_n, t, t')$ below uses the fact that a type 0 vertex has a primitive representative (see Proposition 1.5.2).

We now investigate the number $m(\Delta_n, t, t')$. We fix the following notation for the rest of this section. Let $t, t' \in \Delta_n$ be close special vertices with t of type 0, and let $L \in t$, $M \in t'$ be primitive representatives (recall that a type 0 vertex has a primitive representative by Proposition 1.5.2 and that t' close to t implies t' has the same type as t) such that there are lattices $L_1, \ldots, L_n$ as in (2.5) and (2.6). By our comments following (2.6), $L_1 = \pi(L + M)$, but we can vary $L_2, \ldots, L_n$ as long as $\langle L_i, L_i \rangle \subseteq \pi\mathcal{O}$ for all $2 \leq i \leq n$ and

$$\pi L \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq L, \qquad \pi M \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n \subsetneq M$$

correspond to chambers of Δ_n. As in section 2.1, since any gallery C, C' in Δ_n such that $C = \{t, [L_1], \ldots, [L_n]\}$ and $C' = \{t', [L_1], \ldots, [L_n]\}$ satisfies $C \cap C' = \{[L_1], \ldots, [L_n]\}$, each such gallery is uniquely determined by the vertices $[L_2], \ldots, [L_n]$. We therefore consider the following situation. Recall that two vertices of Δ_n are adjacent if they are distinct and incident. Consider the set of vertices in Δ_n that are adjacent to t, t', and $[L + M]$. Define two such vertices to be adjacent if they are adjacent as vertices of Δ_n.

Proposition 2.2.9. *Let $\Delta_n^c(t, t')$ be the set consisting of*

- *the empty set,*
- *all vertices of Δ_n adjacent to t, t', and $[L + M]$, and*
- *all finite sets A of vertices of Δ_n adjacent to t, t', and $[L + M]$ such that any two vertices in A are adjacent.*

Then $\Delta_n^c(t,t')$ is a simplicial complex.

Proof. Let A be a finite set of vertices of Δ_n, each of which is adjacent to t, t', and $[L+M]$, and suppose that every pair of vertices in A are adjacent. Then $\emptyset \in \Delta_n^c(t,t')$, and every vertex of A is in $\Delta_n^c(t,t')$. Let B be any non-empty subset of A with at least two vertices. Then B is a finite set of vertices of Δ_n, each of which is adjacent to t, t', and $[L+M]$, and every pair of vertices in B are adjacent. It follows that every subset (including the empty set) of A is in $\Delta_n^c(t,t')$. ☐

Corollary 2.2.2. *The simplicial complex $\Delta_n^c(t,t')$ given in the last proposition is a subcomplex of Δ_n.*

Proof. This follows from the last proposition since a vertex of $\Delta_n^c(t,t')$ is a vertex of Δ_n. ☐

Recall from the remarks following (2.6) that $[L+M : L] = [\pi(L+M) : \pi L] = q = [\pi(L+M) : \pi M] = [L+M : M]$. Note also that since $L_n \subseteq L \cap M \subseteq L$ and both L_n and L are free $\mathcal{O}$-modules of rank $2n$, $L \cap M$ is a lattice of V.

Lemma 2.2.9. *If $x \in \Delta_n^c(t,t')$ is a vertex, then x has a representative L' such that $\langle L', L' \rangle \subseteq \pi\mathcal{O}$ and $\pi(L+M) \subsetneq L' \subsetneq L \cap M$.*

Proof. Since L is primitive and x is adjacent to t, Lemma 1.3.1 implies that x has a unique representative L' such that $\langle L', L' \rangle \subseteq \pi\mathcal{O}$ and $\pi L \subsetneq L' \subsetneq L$. Since x and $[L+M]$ are adjacent vertices of Ξ_{2n} by Proposition 1.3.1, either $L' \subsetneq \pi(L+M)$ or $L' \supsetneq \pi(L+M)$ by [20, p. 322]. If $L' \subsetneq \pi(L+M)$, then $\pi L \subsetneq L' \subsetneq \pi(L+M)$, which is impossible since $[\pi(L+M) : \pi L] = q$. It follows that $L' \supsetneq \pi(L+M)$. Similarly, since x and t' are adjacent vertices of Ξ_{2n} by Proposition 1.3.1, [20, p. 322] implies that either $L' \subsetneq M$ or $L' \supsetneq M$. If $L' \supsetneq M$, then $M \subsetneq L' \subsetneq L$, which is impossible as $[M : \pi(L+M)] = [M : \pi M]/[\pi(L+M) : \pi M] = [L : \pi L]/[\pi(L+M) : \pi L] = [L : \pi(L+M)]$. Thus, $L' \subsetneq M$. But $L' \subsetneq L$ and $L' \subsetneq M$ imply $L' \subseteq L \cap M$. Moreover, $\langle L', L' \rangle \subseteq \pi\mathcal{O}$ means $L'/\pi L$ is a totally isotropic k-subspace of $L/\pi L$. Since the dimension of any maximal totally isotropic k-subspace of $L/\pi L$ is n by [35, p. 6], $[L' : \pi L] \leq q^n$. On the other hand, $[L \cap M : \pi L] = [L : \pi L]/[L : L \cap M] = [L : \pi L]/[L+M : M]$ by the Second Isomorphism Theorem. Since $[L : \pi L] = q^{2n}$ and $[L+M : M] = q$, $[L \cap M : \pi L] = q^{2n-1}$; i.e., $L' \subsetneq L \cap M$. ☐

To prove an analogue of Lemma 2.1.2 for Δ_n, we need some preliminary lemmas. Recall (see page 21) that since L is primitive, $L/\pi L$ has a non-degenerate, alternating k-bilinear form $\langle \cdot, \cdot \rangle_0$ given by $\langle x + \pi L, y + \pi L \rangle_0 = \langle x, y \rangle + \pi\mathcal{O}$.

Remark. The choice of L rather than M in the last paragraph and in the next two lemmas is arbitrary; one can just as well use the primitive lattice M.

Lemma 2.2.10. *With respect to $\langle \cdot, \cdot \rangle_0$, $(L \cap M)/\pi L$ is the orthogonal complement of $\pi(L+M)/\pi L$ in $L/\pi L$.*

Proof. Note that by the definition of $\langle \cdot, \cdot \rangle_0$, it suffices to show that for all $x \in L \cap M$ and for all $y \in \pi(L+M)$, $\langle x, y \rangle \in \pi\mathcal{O}$. Let $x \in L \cap M$ and $y \in \pi(L+M)$. Write $y = \pi(x' + y')$ for some $x' \in L$ and some $y' \in M$. Then $\langle L, L \rangle \subseteq \mathcal{O}$ and $\langle M, M \rangle \subseteq \mathcal{O}$ imply $\langle x, y \rangle = \langle x, \pi(x' + y') \rangle = \pi\langle x, x' \rangle + \pi\langle x, y' \rangle \in \pi\mathcal{O}$. $\square$

Lemma 2.2.11. *The non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$ induces a non-degenerate, alternating k-bilinear form $\langle \cdot, \cdot \rangle_1$ on the $2(n-1)$-dimensional k-vector space $(L \cap M)/\pi(L+M)$.*

Proof. By the proof of Lemma 2.2.9, $[L \cap M : \pi L] = q^{2n-1}$; hence, $[L \cap M : \pi(L+M)] = [L \cap M : \pi L]/[\pi(L+M) : \pi L] = q^{2n-2}$. Define $\langle \cdot, \cdot \rangle_1$ on $(L \cap M)/\pi(L+M)$ by $\langle x + \pi(L+M), y + \pi(L+M) \rangle_1 = \langle x, y \rangle + \pi\mathcal{O}$. To see that this is well-defined, let $z_1, z_2 \in \pi(L+M)$. Then by the last lemma,

$$\begin{aligned}\langle x + z_1 + \pi(L+M), y + z_2 + \pi(L+M) \rangle_1 &= \langle x + z_1, y + z_2 \rangle + \pi\mathcal{O} \\ &= \langle x, y \rangle + \langle x, z_2 \rangle + \langle z_1, y \rangle + \langle z_1, z_2 \rangle + \pi\mathcal{O} \\ &= \langle x, y \rangle + \langle z_1, z_2 \rangle + \pi\mathcal{O},\end{aligned}$$

so it remains to show that $\langle z_1, z_2 \rangle \in \pi\mathcal{O}$. But $[\pi(L+M) : \pi L] = q$ implies $\pi(L+M)/\pi L$ is a 1-dimensional k-subspace of $L/\pi L$, hence is totally isotropic by [35, p. 5]. Thus, $\langle \pi(L+M), \pi(L+M) \rangle \subseteq \pi\mathcal{O}$ by [20, p. 336], and $\langle \cdot, \cdot \rangle_1$ is well-defined. Note that $\langle \cdot, \cdot \rangle_1$ is k-bilinear and alternating.

To prove that $\langle \cdot, \cdot \rangle_1$ is non-degenerate, let Σ be an apartment of Δ_n containing $[L]$ and $[M]$ and $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ a symplectic basis of V specifying Σ as in Lemma 2.2.2 applied to L. Note that by reordering this basis if necessary, we can assume that $\pi(L+M)$ is generated by $u_1, \pi u_2, \ldots, \pi u_n, \pi w_1, \ldots, \pi w_n$. Since $(L \cap M)/\pi L$ is the orthogonal complement of $\pi(L+M)/\pi L$ in $L/\pi L$ by the last lemma, $L \cap M$ is generated by $u_1, \ldots, u_n, \pi w_1, w_2, \ldots, w_n$. Let $x + \pi(L+M) \in (L \cap M)/\pi(L+M)$ with $x + \pi(L+M) \neq 0 + \pi(L+M)$. Then $x = \sum_{i=1}^n a_i u_i + \pi b_1 w_1 + \sum_{i=2}^n b_i w_i$ for some $a_j, b_j \in \mathcal{O}$. If $a_j, b_j \in \pi\mathcal{O}$ for all $2 \leq j \leq n$, then $x \in \pi(L+M)$, contradicting the fact that $x + \pi(L+M) \neq 0 + \pi(L+M)$. It follows that there is a $2 \leq j \leq n$ such that either $a_j \in \mathcal{O}^\times$ or $b_j \in \mathcal{O}^\times$. If $a_j \in \mathcal{O}^\times$ for some $2 \leq j \leq n$, then $w_j \in L \cap M$ with

$\langle x + \pi(L + M), w_j + \pi(L + M)\rangle_1 = \langle x, w_j\rangle + \pi\mathcal{O} = a_j + \pi\mathcal{O} \neq 0 + \pi\mathcal{O}$. Similarly, if $b_j \in \mathcal{O}^\times$ for some $2 \leq j \leq n$, then $u_j \in L \cap M$ with $\langle x+\pi(L+M), u_j+\pi(L+M)\rangle_1 = \langle x, u_j\rangle + \pi\mathcal{O} = -b_j + \pi\mathcal{O} \neq 0 + \pi\mathcal{O}$. □

Lemma 2.2.12. *There is a bijection between nested sequences* $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ *of totally isotropic* k*-subspaces of* $(L \cap M)/\pi(L + M)$ *and chains of* $\mathcal{O}$*-submodules* $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ *of* $L \cap M$ *containing* $\pi(L + M)$ *with* $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ *for all* $1 \leq j \leq i + 1$.

Proof. Let T be the set of nested sequences $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ of totally isotropic k-subspaces of $(L \cap M)/\pi(L + M)$ and T' the set of chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of $L \cap M$ containing $\pi(L+M)$ with $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$. Then if $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ is a nested sequence of totally isotropic k-subspaces of $(L \cap M)/\pi(L + M)$, the Correspondence Theorem implies that there are unique $\mathcal{O}$-submodules $M_1, \ldots, M_{i+1}$ of $L\cap M$ containing $\pi(L+M)$ such that $M_j/\pi(L+M) = S_j$ for all $1 \leq j \leq i + 1$ and $\pi(L + M) \subseteq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subseteq L \cap M$. Moreover, for all $1 \leq j \leq i + 1$, S_j totally isotropic implies $\langle x + \pi(L + M), y + \pi(L + M)\rangle_1 = 0 + \pi\mathcal{O}$ for all $x, y \in M_j$. But $\langle x + \pi(L + M), y + \pi(L + M)\rangle_1 = \langle x, y\rangle + \pi\mathcal{O}$, so $\langle x, y\rangle \in \pi\mathcal{O}$ for all $x, y \in M_j$; i.e., $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$. Then $\varphi : T \to T'$ given by

$$\varphi : S_1 \subsetneq \cdots \subsetneq S_{i+1} \mapsto M_1 \subsetneq \cdots \subsetneq M_{i+1},$$

where M_j is the unique $\mathcal{O}$-submodule of $L \cap M$ containing $\pi(L + M)$ such that $M_j/\pi(L + M) = S_j$ for all $1 \leq j \leq i + 1$, is well-defined.

Let $\psi : T' \to T$ be given by

$$\psi : M_1 \subsetneq \cdots \subsetneq M_{i+1} \mapsto M_1/\pi(L + M) \subsetneq \cdots \subsetneq M_{i+1}/\pi(L + M).$$

Note that $M_1/\pi(L+M) \subsetneq \cdots \subsetneq M_{i+1}/\pi(L+M)$ is a nested sequence of k-subspaces of $(L \cap M)/\pi(L + M)$. Moreover, since $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i + 1$, $\langle x + \pi(L + M), y + \pi(L + M)\rangle_1 = \langle x, y\rangle + \pi\mathcal{O} = 0 + \pi\mathcal{O}$ for all $x, y \in M_j$; i.e., $M_j/\pi(L+M)$ is a totally isotropic k-subspace of $(L\cap M)/\pi(L+M)$ for all $1 \leq j \leq i+1$. Thus, ψ is also well-defined. Finally, if id_T (resp., $\mathrm{id}_{T'}$) denotes the identity map on T (resp., on T'), then $\psi \circ \varphi = \mathrm{id}_T$ and $\varphi \circ \psi = \mathrm{id}_{T'}$; i.e., $\varphi : T \to T'$ is a bijection. □

In particular, if $A \in \Delta_n^c(t, t')$, then A has at most $n - 1$ vertices (recall that $[L \cap M : \pi(L + M)] = q^{2(n-1)}$ by Lemma 2.2.11 and that the dimension of any maximal totally isotropic k-subspace of $(L\cap M)/\pi(L+M)$ is $n-1$ by [35, p. 6]). We can now prove an analogue of Lemma 2.1.2 (cf. the proof of Lemma 2.1.2).

Lemma 2.2.13. *If $\emptyset \neq A \in \Delta_n^c(t,t')$ is an i-simplex, then A corresponds to a chain of lattices $M_1 \subsetneq \cdots \subsetneq M_{i+1}$, where $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$, $\pi(L+M) \subsetneq M_1 \subsetneq \cdots \subsetneq M_{i+1} \subsetneq L \cap M$, and $[M_1], \ldots, [M_{i+1}]$ are the vertices of A.*

Proof. We proceed by induction on i. Lemma 2.2.9 proves the claim if $i = 0$. Suppose $0 \leq i \leq n-3$ and that the claim holds for any i-simplex in $\Delta_n^c(t,t')$. Let $A \in \Delta_n^c(t,t')$ be an $(i+1)$-simplex, and note that $A \neq \emptyset$. Let x be a vertex of A. By the induction hypothesis, the i-simplex $A - \{x\}$ corresponds to a chain of lattices $M_1' \subsetneq \cdots \subsetneq M_{i+1}'$ such that $\langle M_j', M_j'\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$, $\pi(L+M) \subsetneq M_1' \subsetneq \cdots \subsetneq M_{i+1}' \subsetneq L \cap M$, and $[M_1'], \ldots, [M_{i+1}']$ are the vertices of $A - \{x\}$. By Lemma 2.2.9, x has a representative M' such that $\langle M', M'\rangle \subseteq \pi\mathcal{O}$ and $\pi(L+M) \subsetneq M' \subsetneq L \cap M$. If $M' \subsetneq M_1'$, set $M_1 = M'$ and $M_j = M_{j-1}'$ for all $2 \leq j \leq i+2$. Otherwise, $M' \supsetneq M_1'$ by Proposition 1.3.1 and [20, p. 322]. Let $j \in \{1, \ldots, i+1\}$ be minimal such that $M' \supsetneq M_j'$ (j exists since $M' \subsetneq L \cap M$). If $j = i+1$, set $M_\ell = M_\ell'$ for all $1 \leq \ell \leq i+1$ and $M_{i+2} = M'$. Otherwise, $M' \subsetneq M_{j+1}'$, and setting $M_\ell = M_\ell'$ for all $1 \leq \ell \leq j$, $M_{j+1} = M'$, and $M_\ell = M_{\ell-1}'$ for all $j+2 \leq \ell \leq i+2$ finishes the proof. ☐

Proposition 2.2.10. *For any close special vertices $t, t' \in \Delta_n$ with t of type 0, $\Delta_n^c(t,t')$ as defined in Proposition 2.2.9 is isomorphic (as a poset) to the spherical $C_{n-1}(k)$ building $\Delta_{n-1}^s(k)$ (independent of t and t' with t of type 0).*

Proof. Let $L \in t, M \in t'$ be primitive representatives as in the paragraph preceding Proposition 2.2.9. Then $(L \cap M)/\pi(L+M)$ is a $2(n-1)$-dimensional k-vector space endowed with the non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle_1$ given in Lemma 2.2.11. Let $\Delta_{n-1}^s(k)$ be the spherical $C_{n-1}(k)$ building with i-simplices ($0 \leq i \leq n-2$) given by the nested sequences $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ of non-trivial, totally isotropic k-subspaces of $(L \cap M)/\pi(L+M)$. Note that since $L \cap M$ and $\pi(L+M)$ are free $\mathcal{O}$-modules of rank $2n$, any $\mathcal{O}$-submodule of $L \cap M$ containing $\pi(L+M)$ is a lattice of V. Moreover, by Lemma 2.2.12, there is a bijection between chains of $\mathcal{O}$-submodules $M_1 \subsetneq \cdots \subsetneq M_{i+1}$ of $L \cap M$ containing $\pi(L+M)$ with $\langle M_j, M_j\rangle \subseteq \pi\mathcal{O}$ for all $1 \leq j \leq i+1$ and nested sequences $S_1 \subsetneq \cdots \subsetneq S_{i+1}$ of totally isotropic k-subspaces of $(L \cap M)/\pi(L+M)$; hence, the last lemma implies that for all $0 \leq i \leq n-2$, there is a bijection between the i-simplices of $\Delta_n^c(t,t')$ and the i-simplices of $\Delta_{n-1}^s(k)$. Let $\emptyset \in \Delta_n^c(t,t')$ correspond to $\emptyset \in \Delta_{n-1}^s(k)$. Then we have a bijection between the simplices of $\Delta_n^c(t,t')$ and the simplices of $\Delta_{n-1}^s(k)$. A modification of the proof of Lemma 1.3.3 shows that this bijection preserves the partial order (face) relation, so is a poset isomorphism. ☐

Proposition 2.2.11. *If $t, t' \in \Delta_n$ are close special vertices with t of type 0, then the number $m(\Delta_n, t, t')$ of galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t' equals $r(\Delta_{n-1})$ (independent of t and t'). In particular, the number $\omega(\Delta_n)$ of vertices of Δ_n close to a given type 0 vertex of Δ_n is $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/r(\Delta_{n-1})$.*

Proof. By our previous comments, $\omega(\Delta_n) = (r(\Delta_n) \cdot q)/m(\Delta_n, t, t')$. The last proposition and our previous comments also imply that the galleries of length 1 in Δ_n whose initial chamber contains t and whose ending chamber contains t' are in one-to-one correspondence with the chambers of $\Delta^s_{n-1}(k)$. Since the proof of Proposition 1.3.5 implies that $\Delta^s_{n-1}(k)$ contains exactly $r(\Delta_{n-1})$ chambers, we are done. ∎

Chapter 3

Non-Amenable Graphs

We start this chapter with some definitions from graph theory based on [4, pp. 1 – 3], and [30, pp. 1 – 2]. A *graph* X is a set of *vertices* together with a set $E(X)$ of edges, where an *edge* is an unordered pair $\{x, y\}$ of distinct vertices.[1] A graph X is *finite* if $\mathrm{Card}(X)$ and $\mathrm{Card}(E(X))$ are both finite; otherwise, X is *infinite*. If $\{x, y\} \in E(X)$, then x and y are *adjacent*, denoted $x \sim y$, and $\{x, y\}$ is *incident* to x and y. A *subgraph* of a graph X is a graph Y such that $Y \subseteq X$ and two vertices of Y adjacent implies they are adjacent as vertices of X. If Y is a subgraph of X such that for all $x, y \in Y$, x and y are adjacent as vertices of Y if and only if they are adjacent as vertices of X, then Y is the subgraph of X *induced* by Y. Let X be a graph. For $x \in X$, the *degree* of x, denoted $\deg(x)$, is the number of vertices of X adjacent to x. If $\deg(x) < \infty$ for all $x \in X$, then X is *locally finite*. If there is an $N \in \mathbb{Z}^{\geq 0}$ such that for all $x \in X$, $\deg(x) \leq N$, then X has *bounded degree*. If there is an $m \in \mathbb{Z}^{\geq 0}$ such that for all $x \in X$, $\deg(x) = m$, then X is *m-regular*, or just *regular* if we are not concerned with the specific value of m. A *walk of length m* in X connecting two vertices $x, y \in X$ is a sequence of vertices $x = v_0, \ldots, v_m = y$ of X such that for all $0 \leq i \leq m - 1$, $v_i \sim v_{i+1}$.[2] The graph X is *connected* if any two of its vertices can be connected by a walk in X.

For the rest of this section, X is a connected graph of bounded degree. Following [18, p. 2480], for a connected, induced subgraph X' of X with at least one edge, let $\sigma(X')$ be the set of vertices of X' adjacent to a vertex of X not in X'.

[1] More generally, the set $E(X)$ can be a multiset, so that a pair $\{x, y\}$ can be in $E(X)$ more than once, in which case X has *multiple edges*. In addition, $E(X)$ may have an edge of the form $\{x, x\}$, which is a *loop*. We consider only *simple* graphs; i.e., graphs with neither loops nor multiple edges.

[2] In the literature, a walk is sometimes called a path. Graph theorists distinguish between a walk and a *path* (a walk with no repeated vertex), but according to [30, p. 2], number theorists make no such distinction.

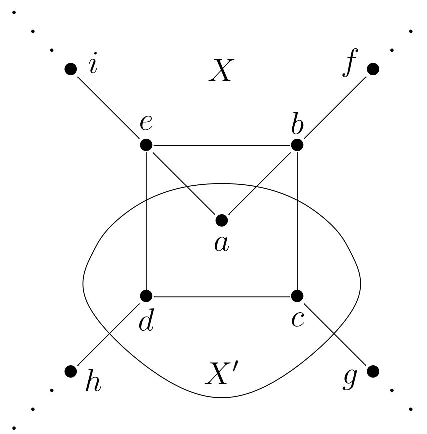


Figure 3.1: A finite, induced subgraph X' of a connected graph X of bounded degree and with infinitely many vertices.

Definition. A connected graph X with infinitely many vertices and of bounded degree is *amenable* if

$$\inf\left\{\frac{\text{Card}(\sigma(X'))}{\text{Card}(X')}\right\} = 0,$$

where the infimum is over all finite, connected, induced subgraphs X' of X with at least one edge.

Remark. If X is a finite, connected graph with at least one edge and of bounded degree, then $\sigma(X) = \emptyset$, and X is trivially amenable.

Now suppose X has infinitely many vertices, and let $N \in \mathbb{Z}^+$ such that $\deg(x) \leq N$ for all $x \in X$ (X connected and with at least two vertices implies $\deg(x) \geq 1$ for all $x \in X$). As in [3, p. 116], if X' is a subset of X, let $\partial X'$ be the set of edges of X incident to exactly one vertex of X'. Let X' be a finite, connected, induced subgraph of X with at least one edge. Then in general, $\text{Card}(\sigma(X')) \leq \text{Card}(\partial X')$ since a vertex $x \in \sigma(X')$ may be adjacent to more than one vertex of X not in X'. For example, for the induced subgraph X' of X given in Figure 3.1, $\sigma(X') = \{a, c, d\}$ but $\partial X' = \{\{a, b\}, \{a, e\}, \{d, e\}, \{d, h\}, \{c, b\}, \{c, g\}\}$. On the other hand, any vertex $x \in \sigma(X')$ can be adjacent to at most $\deg(x) \leq N$ vertices of X not in X'; hence, $\text{Card}(\partial X') \leq N\text{Card}(\sigma(X'))$, and

$$\frac{\text{Card}(\partial X')}{\text{Card}(X')} \leq \frac{N\text{Card}(\sigma(X'))}{\text{Card}(X')}. \tag{3.1}$$

Let A_X be the set of all finite, non-empty subgraphs of X and B_X the set of all finite, connected, induced subgraphs of X with at least one edge. Note that $B_X \subseteq A_X$. Let

$$S(X) = \left\{ \frac{\mathrm{Card}(\partial X')}{\mathrm{Card}(X')} : X' \in A_X \right\}, \qquad S_1(X) = \left\{ \frac{\mathrm{Card}(\partial X')}{\mathrm{Card}(X')} : X' \in B_X \right\},$$
$$S_2(X) = \left\{ \frac{N\mathrm{Card}(\sigma(X'))}{\mathrm{Card}(X')} : X' \in B_X \right\}, \qquad S'(X) = \left\{ \frac{\mathrm{Card}(\sigma(X'))}{\mathrm{Card}(X')} : X' \in B_X \right\}.$$

Since $S(X)$, $S_1(X)$, $S_2(X)$, and $S'(X)$ are all subsets of $\mathbb{R}^{\geq 0}$, $\inf S(X)$, $\inf S_1(X)$, $\inf S_2(X)$, and $\inf S'(X)$ all exist. Furthermore, $B_X \subseteq A_X$ implies $S_1(X) \subseteq S(X)$; hence, $\inf S_1(X) \geq \inf S(X)$. Together with (3.1), this implies $\inf S(X) \leq \inf S_1(X) \leq \inf S_2(X) = N \inf S'(X)$ (since $N > 0$). It follows that if X is a connected graph with infinitely many vertices and of bounded degree such that $\inf S(X) > 0$ (with $S(X)$ as above), then X is *non-amenable*. By [3, p. 116], $i(X) := \inf S(X)$ is the *isoperimetric constant* of X and is a measure of expansion in X.

The *one-complex* of any simplicial complex Δ is the graph with vertices the vertices of Δ and edges the 1-simplices of Δ. In [55, Example (12.20)], Woess considers the one-complex X_n of the affine building Ξ_n of $\mathrm{SL}_n(K)$. Together with [55, Theorem (12.10)] and [3, Theorem 3.1], [55, Example (12.20)] implies that $i(X_n) > 0$; hence, X_n is non-amenable for all $n \geq 2$. Let Y_n be the graph with vertices the *special* vertices of the affine building Δ_n of $\mathrm{Sp}_n(K)$ (recall that all the vertices of Ξ_n are special) and edges the 1-simplices of Δ_n containing only special vertices. Note that $x, y \in Y_n$ are adjacent if and only if x and y are adjacent (distinct and incident) special vertices of Δ_n. In this chapter, we show that for all $n \geq 2$, $i(Y_n) > 0$, so that Y_n is non-amenable.

3.1 Preliminaries

For the rest of this chapter, $n \geq 2$, K is a local field with valuation ring $\mathcal{O}$, uniformizing parameter π, and (finite) residue field isomorphic to $\mathbb{F}_q$, V is a $2n$-dimensional K-vector space endowed with the non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$ given on page 20, and a lattice is a free $\mathcal{O}$-module of rank $2n$. By Proposition 1.6.4, Y_n is $(\prod_{m=1}^{n} (q^m + 1))$-regular, and hence has bounded degree. Recall that every building is a chamber complex and that each chamber of Δ_n contains exactly two special vertices. For chambers $C, C' \in \Delta_n$, write $C \cap C'$ for the simplex (possibly the empty set) shared by C and C'.

Proposition 3.1.1. *The graph Y_n is connected.*

Proof. Let $t \neq t' \in Y_n$. If there is a chamber of Δ_n containing both t and t', then t and t' are adjacent special vertices of Δ_n, and hence are adjacent in Y_n. Otherwise, let $C, C' \in \Delta_n$ be chambers such that $t \in C$ and $t' \in C'$. Since Δ_n is a chamber complex, there is a gallery $C = C_0, \ldots, C_m = C'$ in Δ_n connecting C and C'. For all $0 \leq i \leq m-1$, let S_i be the set of special vertices contained in $C_i \cap C_{i+1}$. Note that since C_i and C_{i+1} have a common codimension-one face and each chamber of Δ_n has exactly two special vertices, $\text{Card}(S_i) \in \{1, 2\}$ for all i.

Let $t_0 = t$, $t_{m+1} = t'$, and $t_i \in S_{i-1}$ for all $1 \leq i \leq m$. Then t_0 and t_1 are incident in Δ_n since $t, t_1 \in C_0 = C$, and t_m and t_{m+1} are incident because $t_m, t' \in C_m = C'$. Finally, for all $1 \leq i \leq m-1$, $t_i \in S_{i-1}$ and $t_{i+1} \in S_i$ imply $t_i, t_{i+1} \in C_i$. It follows that for all $0 \leq i \leq m$, t_i and t_{i+1} are incident vertices of Δ_n; hence, for each $0 \leq i \leq m$, either $t_i = t_{i+1}$ or t_i and t_{i+1} are adjacent vertices of Y_n. Note that if $t_i = t_{i+1}$ for some $0 \leq i \leq m-1$, then t_i and t_{i+2} are incident vertices of Δ_n. It follows that after removing the vertices t_i such that $t_i = t_{i-1}$, we are left with a sequence of vertices $t = t_{i_0}, \ldots, t_{i_{r_1}} = t'$ of Y_n with t_{i_j} and $t_{i_{j+1}}$ incident vertices of Δ_n for all $0 \leq j \leq r_1 - 1$. If the resulting sequence has a vertex t_i such that $t_i = t_{i+1}$, repeat the process in the last sentence. Eventually, we will be left with a sequence of vertices $t = t_{j_0}, \ldots, t_{j_r} = t'$ of Y_n such that t_{j_ℓ} and $t_{j_{\ell+1}}$ are adjacent vertices of Y_n for all $0 \leq \ell \leq r-1$; i.e., we will be left with a walk in Y_n connecting t and t'. □

Note also that by [43, p. 7], there is a one-to-one correspondence between the special vertices in an apartment of Δ_n and the elements of $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$, an infinite group; hence, Y_n has infinitely many vertices. To show that Y_n is non-amenable, it thus suffices to show that $i(Y_n) > 0$. To proceed, we give some background on adjacency operators (cf. [25, p. 158] and [26, p. 246]). Let X be a locally finite graph with countably many vertices and of bounded degree, and let $\ell^2(X)$ be the space of functions $f : X \to \mathbb{C}$ such that $\sum_{x \in X} |f(x)|^2 < \infty$. Then $\ell^2(X)$ is a Hilbert space with inner product

$$(f, g) = \sum_{x \in X} f(x)\overline{g(x)}$$

(see [15, Example I.1.7]). For $f \in \ell^2(X)$, let $\|f\| = (f, f)^{1/2}$. For $x \in X$, let $e_x : X \to \mathbb{C}$ be given by

$$e_x(t) = \begin{cases} 1 & \text{if } t = x, \\ 0 & \text{if } t \neq x. \end{cases}$$

Note that $\|e_x\|^2 = (e_x, e_x) = \sum_{t \in X} e_x(t)\overline{e_x(t)} = 1$ and for $x \neq y \in X$, $(e_x, e_y) = \sum_{t \in X} e_x(t)\overline{e_y(t)} = \overline{e_y(x)} = 0$; i.e., $\{e_x\}_{x \in X}$ is an orthonormal subset of $\ell^2(X)$. In fact, $\{e_x\}_{x\in X}$ is an orthonormal basis of $\ell^2(X)$ (see [15, Example I.4.5]); hence, for any $f \in \ell^2(X)$, $f = \sum_{x\in X} (f, e_x)e_x$ by [15, Theorem 4.13(d)]. The *adjacency operator* $A(X)$ of X is defined on $\ell^2(X)$ by

$$(A(X)f)(x) = \sum_{t\in X} a_{xt} f(t),$$

where a_{xt} is the number of edges from x to t in X. Since X is locally finite and of bounded degree, [26, Theorem 3.2] implies that $A(X)$ is bounded; more precisely, the norm of $A(X)$, which is $\|A(X)\| := \sup\{\|A(X)f\| : f \in \ell^2(X),\ \|f\| \leq 1\}$ (see [15, p. 27]), satisfies $\|A(X)\| \leq N$, where $\deg(x) \leq N$ for all $x \in X$. Let $f \in \ell^2(X)$. Then $\|A(X)f\| \leq \|A(X)\|\ \|f\|$ (see [15, p. 27]) implies

$$\begin{aligned}\sum_{x\in X} |(A(X)f)(x)|^2 &= \sum_{x\in X} (A(X)f)(x)\overline{(A(X)f)(x)} = (A(X)f, A(X)f)\\ &= \|A(X)f\|^2 \leq \|A(X)\|^2\ \|f\|^2 \leq N^2\ \|f\|^2 = N^2\ (f,f)\\ &= N^2 \sum_{x\in X} f(x)\overline{f(x)} = N^2 \sum_{x\in X} |f(x)|^2 < \infty;\end{aligned}$$

i.e., $A(X) : \ell^2(X) \to \ell^2(X)$. Furthermore, $(A(X)e_x)(y) = \sum_{t\in X} a_{yt}e_x(t) = a_{yx}$; hence,

$$(A(X)e_x, e_y) = \sum_{t\in X} (A(X)e_x)(t)\overline{e_y(t)} = (A(X)e_x)(y) = a_{yx}.$$

Since $a_{xy} \in \mathbb{R}$ and $a_{xy} = a_{yx}$ for all $x, y \in X$,

$$\begin{aligned}(A(X)e_x, e_y) = a_{yx} = a_{xy} = \overline{a_{xy}} = \overline{(A(X)e_y)(x)} = \sum_{t\in X} e_x(t)\overline{(A(X)e_y)(t)}\\ = (e_x, A(X)e_y);\end{aligned}$$

i.e., $A(X)$ is self-adjoint. The *spectrum* of $A(X)$ is the set $\{\lambda \in \mathbb{C} : A(X) - \lambda I \text{ is not invertible}\}$, where $I : X \to \mathbb{C}$ is given by $I(x) = 1$ for all $x \in X$. By [25, p. 158], the *spectral radius* of X, denoted $\rho(X)$, is

$$\rho(X) = \sup\{|\lambda| : \lambda \text{ is in the spectrum of } A(X)\}.$$

Since $A(X)$ is self-adjoint and bounded, $\rho(X) = \|A(X)\| = \sup\{|A(X)f| : (f,f) \leq 1\}$ (the first equality follows from [26, p. 252]). It is also well-known that if $x_0 \in X$ is fixed and b_m is the number of walks of length m in X that start and end at x_0, then

$\rho(X) = \limsup_{m\to\infty} b_m^{1/m}$ (see [25, p. 158]).

By [3, Theorem 3.1], to show that $i(Y_n) > 0$, it suffices to compute the spectral radius $\rho(Y_n)$ of Y_n and show that $\rho(Y_n) < \prod_{m=1}^{n} (q^m + 1)$. Following [4, pp. 3, 157], an *isomorphism* between two graphs X and Y is a bijection $\varphi : X \to Y$ such that $x, y \in X$ adjacent implies $\varphi(x), \varphi(y) \in Y$ adjacent, and an *automorphism* of a graph X is an isomorphism of X with itself. If G is a group acting on a set S and $a \in S$, write G_a for $\mathrm{Stab}_G(a) = \{g \in G : ga = a\}$, the stabilizer of a in G. For $a, b \in S$, write $G_a b$ for $\{gb : g \in G_a\}$, the orbit of b under the action of G_a. The main tool that we use is the following result, which is a reformulation of a special case of [55, Theorem (12.10)].

Theorem 3.1.1. *Let X be a locally finite, regular graph. If there is a group Q that*

1. *is solvable,*
2. *acts transitively on X as a group of automorphisms, and*
3. *has a left Haar measure $\mu(\cdot)$,*

then the spectral radius $\rho(X)$ of X is

$$\rho(X) = \sum_{t' \sim t_0} \sqrt{\frac{\mathrm{Card}(Q_{t'} t_0)}{\mathrm{Card}(Q_{t_0} t')}}, \tag{3.2}$$

where t_0 is any vertex of X.

Remark. We show how the statement given above is related to [55, Theorem (12.10)] in Appendix B.

Proof. See the proof of [55, Theorem (12.10)]. ☐

Let $\mathrm{PGSp}_n(K) = \mathrm{GSp}_n(K)/K^\times$, where we identify $K^\times$ with the non-zero scalar matrices of $\mathrm{GSp}_n(K)$, and note that the definition of $\mathrm{GSp}_n(K)$ given on page 35 is equivalent to

$$\mathrm{GSp}_n(K) = \{g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(K) : A^t D - C^t B = \nu(g) I_n \text{ for some } \nu(g) \in K^\times, \\ A^t C = C^t A, B^t D = D^t B\}.$$

Our candidate for Q is the image in $\mathrm{PGSp}_n(K)$ of Q', where

$$Q' = \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{GSp}_n(K) : A \text{ is upper triangular}\}.$$

Note that Q' is the stabilizer in $\mathrm{GSp}_n(K)$ of a maximal flag of non-trivial, totally isotropic subspaces of V.[3]

Before we verify that Q satisfies the hypotheses of Theorem 3.1.1 with X the graph Y_n, we make a few observations and give some notation. If c_i denotes the ith column of $g \in \mathrm{GSp}_n(K)$, then $ge_i = c_i$ for all i, and (by abusing notation slightly) (1.7) implies

$$\langle c_i, c_j \rangle = \langle ge_i, ge_j \rangle = \nu(g)\langle e_i, e_j \rangle = \begin{cases} \nu(g) & \text{if } 1 \leq i \leq n \text{ and } j = n+i, \\ -\nu(g) & \text{if } n+1 \leq i \leq 2n \text{ and } j = i-n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Thus, if

$$g = (g_{ij}) = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in Q',$$

then cofactor expansion along the first columns of the appropriate matrices gives

$$\begin{aligned} \det(g) &= g_{11} \cdots g_{nn}(\det D) = (\det A)(\det D) = (\det A^t)(\det D) \\ &= \det(A^t D - C^t B) = \det(\nu(g) I_n) = \nu(g)^n \in K^\times. \end{aligned}$$

It follows that $g_{11}, \ldots, g_{nn} \in K^\times$. Furthermore, (3.3) implies $g_{11}g_{n+1,n+1} = \nu(g)$ and $g_{11}g_{n+1,i} = 0$ for all $i > n+1$; hence, $g_{n+1,n+1} = \nu(g)g_{11}^{-1}$ and $g_{n+1,i} = 0$ for all $i > n+1$. Similarly, $g_{n+2,n+2} = \nu(g)g_{22}^{-1}$ and $g_{n+2,i} = 0$ for all $i > n+2$. Continuing this process shows that D is lower triangular and $g_{ii} = \nu(g)g_{i-n,i-n}^{-1}$ for all $i \geq n+1$; in particular, $g_{ii} \in K^\times$ for all $1 \leq i \leq 2n$. In addition, for all $2n \geq i > j \geq n+1$ (resp., for all $1 \leq i < i+n < j \leq 2n$), g_{ij}—the elements strictly below the diagonal in D (resp., the elements strictly above the diagonal in B)—can be expressed in terms of the other (non-zero) entries of g.

Write g as a matrix in $\mathrm{GSp}_n(K)$ while thinking of it as an element of $\mathrm{PGSp}_n(K)$ consisting of all its non-zero multiples. Recall that for a symplectic basis $\mathcal{B} = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V and integers $a_1, \ldots, a_n, b_1, \ldots, b_n$, $[a_1, \ldots, a_n; b_1, \ldots, b_n]_{\mathcal{B}}$ denotes the homothety class of the lattice $\mathcal{O}\pi^{a_1}u_1 + \cdots + \mathcal{O}\pi^{a_n}u_n + \mathcal{O}\pi^{b_1}w_1 + \cdots + \mathcal{O}\pi^{b_n}w_n$. Let $o = [\mathcal{O}^{2n}] = [0, \ldots, 0; 0, \ldots, 0]_{\mathcal{B}_0}$, where $\mathcal{B}_0 = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$.

We now verify that Q satisfies the hypotheses of Theorem 3.1.1 (we have already shown that Y_n is locally finite and regular). Since we do not use any of the following in our computation, the reader who is willing to assume that Q does indeed satisfy

[3] In the language of buildings, the group Q' is a *minimal parabolic subgroup* of $\mathrm{GSp}_n(K)$; the other minimal parabolic subgroups of $\mathrm{GSp}_n(K)$ are conjugate to Q'.

the hypotheses of Theorem 3.1.1 (at least temporarily) may skip to the next section.

Claim. *The group Q is solvable.*

Proof. First note that since Q is a homomorphic image of Q', part (2) of Proposition 6.1.10 in [17] implies that it suffices to show that Q' is solvable. Let P_n be the group of upper triangular matrices in $\mathrm{GL}_n(K)$ and $h = (h_{ij}) \in \mathrm{GL}_n(K)$ with $h_{ij} = 1$ if $i + j = n + 1$ and $h_{ij} = 0$ otherwise; i.e.,

$$h = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Then $h^{-1} = h$ and for $g \in P_n$, $hgh^{-1} \in \mathrm{GL}_n(K)$ is lower triangular. Moreover, $g' \in \mathrm{GL}_n(K)$ lower triangular implies $h^{-1}g'h \in P_n$. It follows that if

$$\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in P_{2n}$$

with $A, B, D \in M_n(K)$, then $A, D \in P_n$ and

$$\left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} A & B \\ 0 & hD \end{smallmatrix}\right)\left(\begin{smallmatrix} I_n & 0 \\ 0 & h^{-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & Bh^{-1} \\ 0 & hDh^{-1} \end{smallmatrix}\right),$$

with $hDh^{-1} \in \mathrm{GL}_n(K)$ lower triangular. On the other hand, for

$$\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in Q',$$

$h^{-1}Dh \in P_n$ ($D \in \mathrm{GL}_n(K)$ by previous remarks) and

$$\left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)\left(\begin{smallmatrix} A & Bh \\ 0 & h^{-1}Dh \end{smallmatrix}\right)\left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right);$$

hence,

$$Q' = \left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right) P_{2n} \left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)^{-1} \cap \mathrm{GSp}_n(K). \tag{3.4}$$

By [55, p. 134], the image of P_{2n} in $\mathrm{PGL}_{2n}(K)$ is solvable. Since this image is $P_{2n}/\{\alpha I_{2n} : \alpha \in K^\times\}$ and $\{\alpha I_{2n} : \alpha \in K^\times\}$ is abelian and hence solvable, P_{2n} is solvable by Part 3 of Proposition 6.1.10 in [17] (for a direct proof that P_{2n} is solvable, see Appendix C). Futhermore, homomorphic images of solvable groups are solvable by part (2) of Proposition 6.1.10 in [17], so

$$\left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right) P_{2n} \left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)^{-1}$$

is solvable. But subgroups of solvable groups solvable (by part (1) of Proposition 6.1.10 in [17]), (3.4), and the last sentence imply

$$Q' \leq \left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right) P_{2n} \left(\begin{smallmatrix} I_n & 0 \\ 0 & h \end{smallmatrix}\right)^{-1}$$

is solvable. □

Claim. *The group Q acts transitively on Y_n as a group of automorphisms.*

Proof. First note that Proposition 2.2.1 implies that for all $g \in \mathrm{GSp}_n(\mathcal{O}) = \{g \in M_{2n}(\mathcal{O}) : g^t J_n g = \nu(g) J_n,\ \nu(g) \in \mathcal{O}^\times\}$, $go = o$. If $h \in \mathrm{GSp}_n(K)$, then a modification of the proof of Proposition 1.3.7 of [2] shows that there is a $g \in \mathrm{Sp}_n(\mathcal{O})$ such that $gh^{-1} \in Q'$. Then $hg^{-1} = (gh^{-1})^{-1} \in Q'$, and $g^{-1} \in \mathrm{Sp}_n(\mathcal{O}) \subseteq \mathrm{GSp}_n(\mathcal{O})$ implies $(hg^{-1})o = ho$. Since every element of Y_n has the form ho for some $h \in \mathrm{GSp}_n(K)$ by Corollary 1.6.3, Q acts transitively on the vertices of Y_n.

To show that Q acts on Y_n as a group of automorphisms, we must show that Q takes adjacent vertices of Y_n to adjacent vertices. Since we write g as a matrix in $\mathrm{GSp}_n(K)$ while thinking of it as an element of $\mathrm{PGSp}_n(K)$ consisting of all its non-zero multiples, it suffices to show that the action of Q' on the vertices of Y_n preserves adjacency. But this follows from Proposition 1.6.3 since $Q' \leq \mathrm{GSp}_n(K)$. □

Claim. *The group Q has a left Haar measure.*

Proof. Note that by [19, Theorem 11.8], to show that Q has a left Haar measure, it suffices to show that Q is Hausdorff and locally compact. We first show that Q is a topological group. Recall that K a local field with a finite residue field implies that the discrete valuation ord of K defines a topology on K, with respect to which K is Hausdorff and locally compact (see [23, pp. 4 – 5, Proposition 2.1]). Identify $M_{2n}(K)$ with $K^{(2n)^2}$ to give $M_{2n}(K)$ a topology. Then $K^{(2n)^2}$ Hausdorff ([29, Theorem 17.11]) and locally compact ([44, p. 26]) implies $M_{2n}(K)$ Hausdorff and locally compact. By [29, p. 146 Exercise 2(e)], $\mathrm{GL}_{2n}(K)$ is a topological group. It follows from [29, p. 146 Exercise 3] that $\mathrm{GSp}_n(K) \leq \mathrm{GL}_{2n}(K)$ is a topological group; hence, $Q' \leq \mathrm{GSp}_n(K)$ is a topological group. By [29, p. 146 Exercise 5(d)], to show that Q is a topological group, it remains to show that $\{\alpha I_{2n} : \alpha \in K^\times\}$ is closed in Q'. But a sequence of matrices $g_i \in M_{2n}(K)$ converges to $g \in M_{2n}(K)$ if for all $1 \leq \ell, j \leq 2n$, the sequence consisting of the (ℓ, j)-entries of the g_i converges to the (ℓ, j)-entry of g. Thus, $\{\alpha I_{2n} : \alpha \in K\}$ is closed in $M_{2n}(K)$; hence, $\{\alpha I_{2n} : \alpha \in K^\times\}$ is closed in Q' by [29, Theorem 17.2].

Since Q is a topological group, [29, p. 146 Exercise 7(b)] implies Q is Hausdorff, so it remains to show that Q is locally compact. But $Q = Q'/\{\alpha I_{2n} : \alpha \in K^\times\}$ implies that it suffices to show Q' is a locally compact topological group by [29, p. 186 Exericse 9]. We saw in the last paragraph that Q' is a topological group, so we show that Q' is locally compact. First note that since $M_{2n}(K)$ is a topological ring, the determinant map is continuous, so $\mathrm{GL}_{2n}(K)$ is open in $M_{2n}(K)$. It follows from [29, Corollary 29.3] that $\mathrm{GL}_{2n}(K)$ is locally compact (note that $\mathrm{GL}_{2n}(K)$ is also Hausdorff since $M_{2n}(K)$ is Hausdorff). Now $\{\alpha J_n : \alpha \in K\}$ is closed in $M_{2n}(K)$, so $\{\alpha J_n : \alpha \in K^\times\}$ is closed in $\mathrm{GL}_{2n}(K)$. Let $\{g_i\}$ be a sequence of matrices in $\mathrm{GSp}_n(K)$ that converges to a matrix $g \in \mathrm{GL}_{2n}(K)$. Then $g^t J_n g = \lim_{i\to\infty} g_i^t J_n g_i = \lim_{i\to\infty} \nu(g_i) J_n = \alpha J_n$ for some $\alpha \in K^\times$ (the first equality holds since the (ℓ, j)-entry of $g_i^t J_n g_i$ is a polynomial in the entries of g_i and J_n, which is continuous); i.e., $g \in \mathrm{GSp}_n(K)$, and $\mathrm{GSp}_n(K)$ is closed in $\mathrm{GL}_{2n}(K)$. It follows that $\mathrm{GSp}_n(K)$ is also locally compact (by [29, Corollary 29.3]) and Hausdorff ($\mathrm{GL}_{2n}(K)$ is Hausdorff). Finally,

$$\{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in M_{2n}(K) : A \text{ is upper triangular}\}$$

closed in $M_{2n}(K)$ implies Q' is closed in $\mathrm{GSp}_n(K)$, so that Q' is locally compact by [29, Corollary 29.3], and Q has a left Haar measure $\mu(\cdot)$. $\square$

3.2 Computation

Take $t_0 = o$ in (3.2). Then

$$\rho(Y_n) = \sum_{t' \sim o} \sqrt{\frac{\mathrm{Card}(Q_{t'} o)}{\mathrm{Card}(Q_o t')}}. \tag{3.5}$$

It therefore remains to determine $\mathrm{Card}(Q_{t'} o)$ and $\mathrm{Card}(Q_o t')$ for vertices $t' \in Y_n$ with $t' \sim o$. For a symplectic basis $\mathcal{B}$ of V and Σ the apartment of Δ_n specified by $\mathcal{B}$ as in Lemma 1.3.5, we follow [55, Example (12.20)] and write $U(\mathcal{B})$ for the subgraph of Y_n induced by the special vertices of Σ. Figure 3.2 shows a partial picture of $U(\mathcal{B})$ when $n = 2$. Let $\mathcal{E}(n) = \{0, 1\}^n$. Then the neighbors of o in $U(\mathcal{B}_0)$ (i.e., the vertices

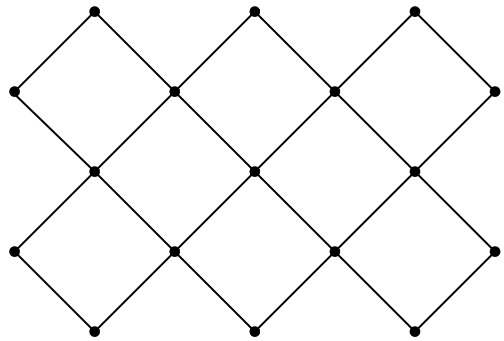


Figure 3.2: Subgraph of Y_2.

of $U(\mathcal{B}_0)$ adjacent to o) are the vertices

$$x_{\underline{\varepsilon}} := [\varepsilon_1, \ldots, \varepsilon_n; 1-\varepsilon_1, \ldots, 1-\varepsilon_n]_{\mathcal{B}_0},$$

where $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{E}(n)$ (by the proof of Proposition 1.6.2). Note that by Proposition 2.2.1, the stabilizer in $\mathrm{GSp}_n(K)$ of $(0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0} \in o$ can be identified with $\mathrm{GSp}_n(\mathcal{O})$; hence, we can identify the stabilizer of o in $\mathrm{PGSp}_n(K)$ with $\mathrm{PGSp}_n(\mathcal{O}) = \mathrm{GSp}_n(\mathcal{O})/\mathcal{O}^\times$, where we identify $\mathcal{O}^\times$ with the set of non-zero scalar matrices of the form αI_{2n} for $\alpha \in \mathcal{O}^\times$. Thus, if $g = (g_{ij}) \in Q_o = Q \cap \mathrm{PGSp}_n(\mathcal{O})$, then

$$g_{ii} \in \mathcal{O}^\times, \qquad g_{ij} \in \mathcal{O} \text{ if } i < j \text{ (with } 1 \leq i \leq n) \text{ or } 2n \geq i > j \geq n+1, \tag{3.6}$$

and $g_{ij} = 0$ otherwise. Furthermore, for all $2n \geq i \geq j \geq n+1$ and for all $1 \leq i < i+n < j \leq 2n$, g_{ij} is completely determined by the other (non-zero) entries of g (using an analysis similar to that used in looking at Q' on starting on page 78). Alternatively, if $|\cdot|_K$ is the absolute value of K, normalized such that $|\pi|_K = 1/q$, then

$$|g_{ii}|_K = 1, \qquad |g_{ij}|_K \leq 1 \text{ if } i < j \text{ (with } 1 \leq i \leq n) \text{ or } 2n \geq i > j \geq n+1, \tag{3.7}$$

and $g_{ij} = 0$ otherwise. As above, for all $2n \geq i \geq j \geq n+1$ and for all $1 \leq i < i+n < j \leq 2n$, $|g_{ij}|_K$ is completely determined by the absolute values of the other (non-zero) entries of g. Finally, for $\underline{\varepsilon} \in \mathcal{E}(n)$, let

$$g_{\underline{\varepsilon}} := \mathrm{diag}(\pi^{\varepsilon_1}, \ldots, \pi^{\varepsilon_n}, \pi^{1-\varepsilon_1}, \ldots, \pi^{1-\varepsilon_n}).$$

Then $g_{\underline{\varepsilon}} \in Q'$ and $g_{\underline{\varepsilon}} o = x_{\underline{\varepsilon}}$, so Lemma 2.2.1 implies

$$Q_{x_{\underline{\varepsilon}}} = g_{\underline{\varepsilon}} Q_o g_{\underline{\varepsilon}}^{-1}. \tag{3.8}$$

Let $b = (b_{ij}) \in Q_{x_{\underline{\varepsilon}}}$ with $b = g_{\underline{\varepsilon}} g g_{\underline{\varepsilon}}^{-1}$ for $g = (g_{ij}) \in Q_o$. Since the (m, j)-entry of $g g_{\underline{\varepsilon}}^{-1}$ is

$$\sum_{\ell=1}^{2n} g_{m\ell} (g_{\underline{\varepsilon}}^{-1})_{\ell j} = \begin{cases} g_{mj} \pi^{-\varepsilon_j} & \text{if } 1 \leq j \leq n, \\ g_{mj} \pi^{-(1-\varepsilon_{j-n})} & \text{if } n+1 \leq j \leq 2n, \end{cases}$$

and $g_{mj} = 0$ if $m > j$ with $1 \leq j \leq n$ or if $n+1 \leq m < j \leq 2n$,

$$b_{ij} = \sum_{m=1}^{2n} (g_{\underline{\varepsilon}})_{im} (g g_{\underline{\varepsilon}}^{-1})_{mj} = \begin{cases} g_{ij} \pi^{\varepsilon_i - \varepsilon_j} & \text{if } 1 \leq i \leq j \leq n, \\ g_{ij} \pi^{\varepsilon_i - (1-\varepsilon_{j-n})} & \text{if } 1 \leq i \leq n < j \leq 2n, \\ g_{ij} \pi^{-\varepsilon_{i-n} + \varepsilon_{j-n}} & \text{if } n+1 \leq j \leq i \leq 2n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.9}$$

and

$$|b_{ij}|_K = 1 \text{ if } i = j, \tag{3.10}$$

$$|b_{ij}|_K \leq \begin{cases} q^{-(\varepsilon_i - \varepsilon_j)} & \text{if } 1 \leq i < j \leq n, \\ q^{-[\varepsilon_i - (1-\varepsilon_{j-n})]} & \text{if } 1 \leq i \leq n < j \leq 2n, \\ q^{-(-\varepsilon_{i-n} + \varepsilon_{j-n})} & \text{if } n+1 \leq j < i \leq 2n, \end{cases}$$

and $b_{ij} = 0$ otherwise. Finally, let $h = (h_{ij}) \in Q_o \cap Q_{x_{\underline{\varepsilon}}}$. Then (3.7) and (3.10) imply $|h_{ii}|_K = 1$ for all i. For all $1 \leq i < j \leq n$, (3.6) and (3.9) imply $h_{ij} \in \mathcal{O} \cap \mathcal{O}\pi^{\varepsilon_i - \varepsilon_j}$. Since $\mathcal{O} \cap \mathcal{O}\pi^{\varepsilon_i - \varepsilon_j} = \mathcal{O}$ if and only if $\varepsilon_i - \varepsilon_j \in \{-1, 0\}$ and $\mathcal{O} \cap \mathcal{O}\pi^{\varepsilon_i - \varepsilon_j} = \pi\mathcal{O}$ if and only if $\varepsilon_i - \varepsilon_j = 1$, $h_{ij} \in \mathcal{O}\pi^{\max\{0, \varepsilon_i - \varepsilon_j\}}$; i.e., $|h_{ij}|_K \leq q^{-\max\{0, \varepsilon_i - \varepsilon_j\}}$. Similarly, $1 \leq i \leq n < j \leq 2n$ implies $|h_{ij}|_K \leq q^{-\max\{0, \varepsilon_i - (1-\varepsilon_{j-n})\}}$ and $n+1 \leq j < i \leq 2n$ implies $|h_{ij}|_K \leq q^{-\max\{0, -\varepsilon_{i-n} + \varepsilon_{j-n}\}}$. In summary,

$$|h_{ij}|_K = 1 \text{ if } i = j, \tag{3.11}$$

$$|h_{ij}|_K \leq \begin{cases} q^{-\max\{0, \varepsilon_i - \varepsilon_j\}} & \text{if } 1 \leq i < j \leq n, \\ q^{-\max\{0, \varepsilon_i - (1-\varepsilon_{j-n})\}} & \text{if } 1 \leq i \leq n < j \leq 2n, \\ q^{-\max\{0, -\varepsilon_{i-n} + \varepsilon_{j-n}\}} & \text{if } n+1 \leq j < i \leq 2n, \end{cases} \tag{3.12}$$

and $h_{ij} = 0$ otherwise.

Lemma 3.2.1. *For* $\underline{\varepsilon} \in \mathcal{E}(n)$, $\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) = \mu(Q_o)/\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})$ *and* $\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o) = \mathrm{Card}(Q_o x_{\underline{1}-\underline{\varepsilon}})$.

Proof. Recall that by the orbit-stabilizer theorem,

$$\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) = [Q_o : \mathrm{Stab}_{Q_o}(x_{\underline{\varepsilon}})] = [Q_o : Q_o \cap Q_{x_{\underline{\varepsilon}}}].$$

On the other hand, if $Q_o = \coprod_i \; g_i(Q_o \cap Q_{x_{\underline{\varepsilon}}})$ for $g_i \in Q_o$, then

$$\mu(Q_o) = \sum_i \; \mu(g_i(Q_o \cap Q_{x_{\underline{\varepsilon}}})) = \sum_i \; \mu(Q_o \cap Q_{x_{\underline{\varepsilon}}}) = [Q_o : Q_o \cap Q_{x_{\underline{\varepsilon}}}]\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}}),$$

and

$$\frac{\mu(Q_o)}{\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})} = [Q_o : Q_o \cap Q_{x_{\underline{\varepsilon}}}] = \mathrm{Card}(Q_o x_{\underline{\varepsilon}}).$$

Since $\pi g_{\underline{\varepsilon}}^{-1}(0, \ldots, 0; 0, \ldots, 0)_{\mathcal{B}_0} = (1-\varepsilon_1, \ldots, 1-\varepsilon_n; \varepsilon_1, \ldots, \varepsilon_n)_{\mathcal{B}_0}$, $g_{\underline{\varepsilon}}^{-1} o = x_{\underline{1}-\underline{\varepsilon}}$, and (3.8) implies $Q_{x_{\underline{\varepsilon}}} o = g_{\underline{\varepsilon}} Q_o g_{\underline{\varepsilon}}^{-1} o = g_{\underline{\varepsilon}} Q_o x_{\underline{1}-\underline{\varepsilon}}$. But $\underline{\varepsilon} \in \mathcal{E}(n)$ is fixed, so $\mathrm{Card}(g_{\underline{\varepsilon}} Q_o x_{\underline{1}-\underline{\varepsilon}}) = \mathrm{Card}(Q_o x_{\underline{1}-\underline{\varepsilon}})$.[4] □

Consequently, to determine $\mathrm{Card}(Q_o x_{\underline{\varepsilon}})$ and $\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o)$, it suffices to find $\mu(Q_o)$ and $\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})$. If we think of Q_o and $Q_o \cap Q_{x_{\underline{\varepsilon}}}$ as being parametrized by a product of lines with the product measure from that product of lines, it suffices to transport Q_o and $Q_o \cap Q_{x_{\underline{\varepsilon}}}$ back to the parametrizing space and find their measures there. Since we can completely characterize both Q_o and $Q_o \cap Q_{x_{\underline{\varepsilon}}}$ in terms of $|\cdot|_K$, we can think of Q_o and $Q_o \cap Q_{x_{\underline{\varepsilon}}}$ as set-theoretic products, with measure a product measure. In other words, it suffices to determine the "volume" of each "free," non-constant entry in Q_o (resp., in $Q_o \cap Q_{x_{\underline{\varepsilon}}}$); the product of these volumes is then $\mu(Q_o)$ (resp., $\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})$).

Write $\mathrm{vol}(g_{ij})$ for the "volume" of the (i, j)-entry of $g = (g_{ij})$, with vol normalized such that $\mathrm{vol}(\mathcal{O}) = 1$. Then $\mathrm{vol}(\pi\mathcal{O}) = 1/q$ and $\mathrm{vol}(\mathcal{O}^\times) = 1 - (1/q)$. Recall that if

$$g = \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in Q' \leq \mathrm{GSp}_n(K),$$

then the entries strictly above the diagonal of B, as well as all the entries of D, are completely determined by the other (non-zero) entries of g; i.e., the "free," non-zero entries of g are g_{ij}, where $1 \leq i \leq n$ and $i \leq j \leq i + n$. Thus, (3.7) implies

$$\mu(Q_o) = \left(1 - \frac{1}{q}\right)^n. \tag{3.13}$$

[4] If G is a group acting on a set S, then for all $g \in G$, $\mathrm{Card}(gS) = \mathrm{Card}(S)$: clearly, $\mathrm{Card}(gS) \leq \mathrm{Card}(S)$. If $a, b \in S$ such that $ga = gb$, then $a = b$.

For $\underline{\varepsilon} \in \mathcal{E}(n)$, let $|\underline{\varepsilon}|_n = \sum_{i=1}^n \varepsilon_i$, the number of 1s in $\underline{\varepsilon}$. Note that

$$M(\underline{\varepsilon}, n) := \sum_{1 \le i < j \le n} \max\{0, \varepsilon_i - \varepsilon_j\}$$

counts the number of 0s that follow each 1 in $\underline{\varepsilon}$ ($\max\{0, \varepsilon_i - \varepsilon_j\}$ gives a positive contribution to the right-hand side if and only if $\varepsilon_i - \varepsilon_j = 1$). In addition,

$$\begin{aligned}\sum_{1 \le i \le n < j \le i+n} \max\{0, \varepsilon_i - (1 - \varepsilon_{j-n})\} &= \sum_{1 \le i \le n < j \le i+n} \max\{0, (\varepsilon_i + \varepsilon_{j-n}) - 1\} \\ &= \sum_{1 \le j \le i \le n} \max\{0, (\varepsilon_i + \varepsilon_j) - 1\}\end{aligned}$$

adds the number of 1s in $\underline{\varepsilon}$ (when $i = j$ and $\varepsilon_i = 1$) to the number of 1s that follow each 1 in $\underline{\varepsilon}$ (when $j < i$ and $\varepsilon_i = 1 = \varepsilon_j$); i.e., if $|\underline{\varepsilon}|_n = m$, then the last sum is $m(m+1)/2$.

Let $\underline{\varepsilon} \in \mathcal{E}(n)$ with $|\underline{\varepsilon}|_n = m$ and $h = (h_{ij}) \in Q_o \cap Q_{x_{\underline{\varepsilon}}}$. Then (3.11) and (3.12) imply $\mathrm{vol}(h_{ii}) = 1 - (1/q)$ for all $1 \le i \le n$, $\mathrm{vol}(h_{ij}) = q^{-\max\{0, \varepsilon_i - \varepsilon_j\}}$ for all $1 \le i < j \le n$, and $\mathrm{vol}(h_{ij}) = q^{-\max\{0, \varepsilon_i - (1 - \varepsilon_{j-n})\}}$ for all $1 \le i \le n < j \le i+n$. This, together with the last paragraph, implies that $\prod_{1 \le i < j \le n} \mathrm{vol}(h_{ij}) = q^{-M(\underline{\varepsilon}, n)}$ and $\prod_{1 \le i \le n < j \le i+n} \mathrm{vol}(h_{ij}) = q^{-m(m+1)/2}$; hence,

$$\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}}) = \left(1 - \frac{1}{q}\right)^n q^{-M(\underline{\varepsilon}, n)} q^{-m(m+1)/2}.$$

Together with (3.13) and Lemma 3.2.1, this gives

$$\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) = \frac{\mu(Q_o)}{\mu(Q_o \cap Q_{x_{\underline{\varepsilon}}})} = q^{M(\underline{\varepsilon}, n)} q^{m(m+1)/2}, \tag{3.14}$$

and, since $|\underline{1} - \underline{\varepsilon}|_n = n - m$,

$$\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o) = \mathrm{Card}(Q_o x_{\underline{1} - \underline{\varepsilon}}) = q^{M(\underline{1} - \underline{\varepsilon}, n)} q^{(n-m)(n-m+1)/2}. \tag{3.15}$$

Finally, note that

$$M(\underline{1} - \underline{\varepsilon}, n) = \sum_{1 \le i < j \le n} \max\{0, (1 - \varepsilon_i) - (1 - \varepsilon_j)\} = \sum_{1 \le i < j \le n} \max\{0, \varepsilon_j - \varepsilon_i\}$$

counts the number of 1s that follow each 0 in $\underline{\varepsilon}$. Equivalently, $M(\underline{1} - \underline{\varepsilon}, n)$ counts the number of 0s that precede each 1 in $\underline{\varepsilon}$. Since $\underline{\varepsilon}$ has $n - m$ 0s and $M(\underline{\varepsilon}, n)$ counts the number of 0s that follow each 1 in $\underline{\varepsilon}$, for each 1 in $\underline{\varepsilon}$, $M(\underline{\varepsilon}, n) + M(\underline{1} - \underline{\varepsilon}, n)$ counts the number of zeros in $\underline{\varepsilon}$; i.e., $M(\underline{\varepsilon}, n) + M(\underline{1} - \underline{\varepsilon}, n) = m(n - m)$. This, together with (3.14) and (3.15), implies that the exponent of q in $\mathrm{Card}(Q_o x_{\underline{\varepsilon}})\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o)$ is

$$\begin{aligned} M(\underline{\varepsilon}, n) + M(\underline{1} - \underline{\varepsilon}, n) &+ \frac{1}{2}m(m+1) + \frac{1}{2}(n-m)(n-m+1) \\ &= m(n-m) + \frac{1}{2}[m(m+1) + (n-m)(n-m+1)] \\ &= \frac{1}{2}[2m(n-m) + m(m+1) + (n-m)(n-m+1)] \\ &= \frac{1}{2}[mn + n(n-m+1)] = \frac{n(n+1)}{2}; \end{aligned}$$

i.e.,

$$\mathrm{Card}(Q_o x_{\underline{\varepsilon}})\mathrm{Card}(Q_{x_{\underline{\varepsilon}}} o) = q^{n(n+1)/2}. \tag{3.16}$$

Recall that the action of Q_o on Y_n, defines an equivalence relation on Y_n in which the vertices $t, t' \in Y_n$ are equivalent if and only if $t' = gt$ for some $g \in Q_o$.

Proposition 3.2.1. *Let $\underline{\varepsilon} \neq \underline{\varepsilon}' \in \mathcal{E}(n)$. Then for all $g \in Q_o$, $x_{\underline{\varepsilon}'} \neq g x_{\underline{\varepsilon}}$.*

Proof. Let $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ and $\underline{\varepsilon}' = (\varepsilon'_1, \ldots, \varepsilon'_n)$, and recall that $g_{\underline{\varepsilon}} o = x_{\underline{\varepsilon}}$ and $g_{\underline{\varepsilon}'} o = x_{\underline{\varepsilon}'}$; hence, $x_{\underline{\varepsilon}} = g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} x_{\underline{\varepsilon}'}$. If $x_{\underline{\varepsilon}'} = g x_{\underline{\varepsilon}}$ for some $g = (g_{ij}) \in Q_o$, then $x_{\underline{\varepsilon}} = g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} g x_{\underline{\varepsilon}}$; i.e., $g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} g \in Q_{x_{\underline{\varepsilon}}}$. On the other hand, the (i, j)-entry of $g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1} g$ is

$$\sum_{m=1}^{2n} (g_{\underline{\varepsilon}} g_{\underline{\varepsilon}'}^{-1})_{im} g_{mj} = \begin{cases} g_{ij} \pi^{\varepsilon_i - \varepsilon'_i} & \text{if } 1 \leq i \leq n, \\ g_{ij} \pi^{-\varepsilon_i + \varepsilon'_i} & \text{if } n+1 \leq i \leq 2n. \end{cases}$$

Since $\underline{\varepsilon} \neq \underline{\varepsilon}'$, we can choose $1 \leq j \leq n$ such that $\varepsilon'_j \neq \varepsilon_j$. Then (3.7) implies

$$\left| g_{jj} \pi^{\varepsilon_j - \varepsilon'_j} \right|_K = q^{-(\varepsilon_j - \varepsilon'_j)} \in \{q^{-1}, q\},$$

contradicting (3.10). Thus, there is no $g \in Q_o$ such that $x_{\underline{\varepsilon}'} = g x_{\underline{\varepsilon}}$. ☐

It follows that for $\underline{\varepsilon} \in \mathcal{E}(n)$, the $x_{\underline{\varepsilon}}$ are representatives of distinct orbits of Q_o; i.e., the sets $Q_o x_{\underline{\varepsilon}}$ and $Q_o x_{\underline{\varepsilon}'}$ are disjoint if $\underline{\varepsilon} \neq \underline{\varepsilon}'$. We now explore the relationship between $N(o)$, the set of vertices of Y_n adjacent to o, and $\cup_{\underline{\varepsilon} \in \mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}$. By the last proposition and (3.14),

$$\begin{aligned}\operatorname{Card}\left(\bigcup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}\right) &= \sum_{\underline{\varepsilon}\in\mathcal{E}(n)} \operatorname{Card}(Q_o x_{\underline{\varepsilon}}) = \sum_{m=0}^{n}\left(\sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} \operatorname{Card}(Q_o x_{\underline{\varepsilon}})\right)\\ &= \sum_{m=0}^{n}\left(\sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} q^{M(\underline{\varepsilon},n)} q^{m(m+1)/2}\right)\\ &= \sum_{m=0}^{n} q^{m(m+1)/2}\left(\sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} q^{M(\underline{\varepsilon},n)}\right). \end{aligned} \tag{3.17}$$

Let $W(n,m) := \sum_{\underline{\varepsilon}\in\mathcal{E}(n),\ |\underline{\varepsilon}|_n=m} q^{M(\underline{\varepsilon},n)}$, and note that $W(n,0) = q^{M(\underline{0},n)} = 1$ and $W(n,n) = q^{M(\underline{1},n)} = 1$. Furthermore, for a fixed $0 \le m \le n$, we can partition the set of all $\underline{\varepsilon} \in \mathcal{E}(n)$ with $|\underline{\varepsilon}|_n = m$ into two disjoint sets (for $m = 0$ and $m = n$, one of these sets is the empty set): the set consisting of those $\underline{\varepsilon}$ with $\varepsilon_n = 0$ and those with $\varepsilon_n = 1$. If $\underline{\varepsilon} = \underline{\varepsilon}'1$ with $\underline{\varepsilon}' \in \mathcal{E}(n-1)$, then $|\underline{\varepsilon}'|_{n-1} = m-1$ and $M(\underline{\varepsilon}, n) = M(\underline{\varepsilon}', n-1)$. On the other hand, if $\underline{\varepsilon} = \underline{\varepsilon}'0$ with $\underline{\varepsilon}' \in \mathcal{E}(n-1)$, then $|\underline{\varepsilon}'|_{n-1} = m$ and $M(\underline{\varepsilon}, n) = M(\underline{\varepsilon}', n-1) + m$ since the extra zero adds a factor of 1 for each 1 in $\underline{\varepsilon}'$. It follows that

$$\begin{aligned}W(n,m) = \sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n),\\ |\underline{\varepsilon}|_n=m}} q^{M(\underline{\varepsilon},n)} &= \sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n-1),\\ |\underline{\varepsilon}|_{n-1}=m-1}} q^{M(\underline{\varepsilon},n-1)} + \sum_{\substack{\underline{\varepsilon}\in\mathcal{E}(n-1),\\ |\underline{\varepsilon}|_{n-1}=m}} q^{M(\underline{\varepsilon},n-1)+m}\\ &= W(n-1,m-1) + q^m\, W(n-1,m).\end{aligned}$$

Recall (for example, [17, p. 392]) that the number of m-dimensional subspaces of an n-dimensional $\mathbb{F}_q$-vector space is

$$\binom{n}{m}_q = \frac{\varphi_n(q)}{\varphi_m(q)\varphi_{n-m}(q)},$$

where

$$\varphi_0(q) = 1 \qquad \text{and} \qquad \varphi_r(q) = \prod_{i=1}^{r} (q^i - 1) \text{ if } r \ge 1.$$

Since

$$\binom{n-1}{m-1}_q + q^m \binom{n-1}{m}_q = \frac{\varphi_{n-1}(q)}{\varphi_{m-1}(q)\varphi_{n-1-(m-1)}(q)} + q^m \frac{\varphi_{n-1}(q)}{\varphi_m(q)\varphi_{n-1-m}(q)}$$
$$= \frac{\varphi_{n-1}(q)}{\varphi_m(q)\varphi_{n-m}(q)}[(q^m-1)+q^m(q^{n-m}-1)]$$
$$= \frac{\varphi_n(q)}{\varphi_m(q)\varphi_{n-m}(q)} = \binom{n}{m}_q,$$

$W(n,m) = \binom{n}{m}_q$ for all $0 \leq m \leq n$. Thus, (3.17) implies

$$\text{Card}\left(\bigcup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}\right) = \sum_{m=0}^{n} q^{m(m+1)/2}\binom{n}{m}_q. \tag{3.18}$$

Lemma 3.2.2. *For all $n \geq 1$,*

$$\prod_{m=1}^{n}(1+tx^m) = \sum_{i=0}^{n} t^i \frac{x^{i(i+1)/2}\varphi_n(x)}{\varphi_i(x)\varphi_{n-i}(x)}.$$

Proof. We proceed by induction on n. When $n=1$, the right-hand side is

$$\frac{\varphi_1(x)}{\varphi_0(x)\varphi_1(x)} + t\frac{x\varphi_1(x)}{\varphi_1(x)\varphi_0(x)} = 1 + tx.$$

Let $n > 1$, and assume

$$\prod_{m=1}^{n-1}(1+tx^m) = \sum_{i=0}^{n-1} t^i \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-1-i}(x)}.$$

Then

$$\begin{aligned}
\prod_{m=1}^{n}(1+tx^m) &= (1+tx^n)\prod_{m=1}^{n-1}(1+tx^m) = (1+tx^n)\sum_{i=0}^{n-1} t^i \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-1-i}(x)} \\
&= \sum_{i=0}^{n-1} t^i \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-1-i}(x)} + \sum_{i=0}^{n-1} t^{i+1}x^n \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-1-i}(x)} \\
&= \sum_{i=0}^{n-1} t^i \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-1-i}(x)} + \sum_{i=1}^{n} t^i x^n \frac{x^{(i-1)i/2}\varphi_{n-1}(x)}{\varphi_{i-1}(x)\varphi_{n-i}(x)} \\
&= 1 + \sum_{i=1}^{n-1} t^i \frac{x^{i(i+1)/2}\varphi_{n-1}(x)}{\varphi_i(x)\varphi_{n-i}(x)}[(x^{n-i}-1) + x^{n-i}(x^i-1)] + t^n x^{n(n+1)/2} \\
&= \sum_{i=0}^{n} t^i \frac{x^{i(i+1)/2}\varphi_n(x)}{\varphi_i(x)\varphi_{n-i}(x)},
\end{aligned}$$

as claimed. □

Proposition 3.2.2. *The set of vertices of Y_n adjacent to o is $N(o) = \cup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}$.*

Proof. By Proposition 1.6.4, the last lemma (with $t = 1$ and $x = q$), and (3.18),

$$\begin{aligned}
\deg(o) &= \prod_{m=1}^{n}(q^m+1) = \sum_{i=0}^{n} q^{i(i+1)/2}\frac{\varphi_n(q)}{\varphi_i(q)\varphi_{n-i}(q)} = \sum_{i=0}^{n} q^{i(i+1)/2}\binom{n}{i}_q \\
&= \operatorname{Card}\left(\bigcup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_o x_{\underline{\varepsilon}}\right),
\end{aligned}$$

as claimed. □

Corollary 3.2.1. *If $t \in \Delta_n$ has type 0, then the set of vertices of Y_n adjacent to t is $\cup_{\underline{\varepsilon}\in\mathcal{E}(n)} Q_t x_{\underline{\varepsilon}}$.*

Proof. First note that any type 0 vertex of Δ_n has a primitive representative by Proposition 1.5.2. Since the arguments used to prove the last proposition only used the fact that o has a primitive representative, the corollary follows from the last proposition. □

Before we use (3.5) to compute the spectral radius $\rho(Y_n)$ of Y_n, we consider

$$\frac{\operatorname{Card}(Q_{x_{\underline{\varepsilon}}}o)}{\operatorname{Card}(Q_o x_{\underline{\varepsilon}})} \qquad \text{and} \qquad \frac{\operatorname{Card}(Q_{gx_{\underline{\varepsilon}}}o)}{\operatorname{Card}(Q_o g x_{\underline{\varepsilon}})}$$

for $g \in Q_o$. Fix $g \in Q_o$, and note that $Q_{gx_{\underline{\varepsilon}}}o = gQ_{x_{\underline{\varepsilon}}}g^{-1}o$ by Lemma 2.2.1. Then $Q_{gx_{\underline{\varepsilon}}}o = gQ_{x_{\underline{\varepsilon}}}o$ and $\mathrm{Card}(Q_{gx_{\underline{\varepsilon}}}o) = \mathrm{Card}(gQ_{x_{\underline{\varepsilon}}}o) = \mathrm{Card}(Q_{x_{\underline{\varepsilon}}}o)$. Finally, $g \in Q_o$ implies $\mathrm{Card}(Q_o g x_{\underline{\varepsilon}}) = \mathrm{Card}(Q_o x_{\underline{\varepsilon}})$; i.e., for all $g \in Q_o$,

$$\frac{\mathrm{Card}(Q_{x_{\underline{\varepsilon}}}o)}{\mathrm{Card}(Q_o x_{\underline{\varepsilon}})} = \frac{\mathrm{Card}(Q_{gx_{\underline{\varepsilon}}}o)}{\mathrm{Card}(Q_o g x_{\underline{\varepsilon}})}. \tag{3.19}$$

Proposition 3.2.3. *The spectral radius of Y_n is*

$$\rho(Y_n) = 2^n q^{n(n+1)/4}.$$

Proof. By (3.5), Proposition 3.2.2, and (3.19),

$$\begin{aligned}
\rho(Y_n) &= \sum_{t' \sim o} \sqrt{\frac{\mathrm{Card}(Q_{t'}o)}{\mathrm{Card}(Q_o t')}} = \sum_{\underline{\varepsilon} \in \mathcal{E}(n)} \left(\sum_{t' \in Q_o x_{\underline{\varepsilon}}} \sqrt{\frac{\mathrm{Card}(Q_{t'}o)}{\mathrm{Card}(Q_o t')}} \right) \\
&= \sum_{\underline{\varepsilon} \in \mathcal{E}(n)} \left(\sum_{g x_{\underline{\varepsilon}}, g \in Q_o} \sqrt{\frac{\mathrm{Card}(Q_{gx_{\underline{\varepsilon}}}o)}{\mathrm{Card}(Q_o g x_{\underline{\varepsilon}})}} \right) \\
&= \sum_{\underline{\varepsilon} \in \mathcal{E}(n)} \mathrm{Card}(Q_o x_{\underline{\varepsilon}}) \sqrt{\frac{\mathrm{Card}(Q_{x_{\underline{\varepsilon}}}o)}{\mathrm{Card}(Q_o x_{\underline{\varepsilon}})}} \\
&= \sum_{\underline{\varepsilon} \in \mathcal{E}(n)} \sqrt{\mathrm{Card}(Q_o x_{\underline{\varepsilon}}) \mathrm{Card}(Q_{x_{\underline{\varepsilon}}}o)}.
\end{aligned}$$

It follows from (3.16) that

$$\rho(Y_n) = \sum_{\underline{\varepsilon} \in \mathcal{E}(n)} \sqrt{q^{n(n+1)/2}} = 2^n q^{n(n+1)/4}. \qquad \square$$

Theorem 3.2.1. *The isoperimetric constant of Y_n satisfies $i(Y_n) > 0$.*

Proof. By Propositions 3.1.1 and 1.6.4, Y_n is a connected, $(\prod_{m=1}^{n} (q^m + 1))$-regular graph. Thus, by [3, Theorem 3.1] and the last proposition, it suffices to show that $\rho(Y_n) = 2^n q^{n(n+1)/4} < \prod_{m=1}^{n} (q^m + 1)$. For all $n \geq 3$, note that $n^2 - 3n \geq 0$ implies $n(n+5) \leq 2n(n+1)$ and $n(n+5)/4 \leq n(n+1)/2$. Then

$$2^n q^{n(n+1)/4} \leq q^n q^{n(n+1)/4} = q^{n(n+5)/4} \leq q^{n(n+1)/2} < \prod_{m=1}^{n} (q^m + 1)$$

since $\prod_{m=1}^{n} (q^m + 1)$ has the form $q^{n(n+1)/2} + f(q)$, where $f(x)$ is a non-constant polynomial with non-negative integer coefficients. For $n = 2$, let $f(x) = x^3 + x^2 - 4x^{3/2} + x + 1$. Then for $x > 0$, $f'(x) = 3x^2 + 2x - 6x^{1/2} + 1$, and $f'(x) = 0$ when $x = 1$. Since $f'(4) > 0$ and $f(2) > 0$, $f(x)$ is increasing for all $x > 1$; i.e., $f(x) > 0$ for all $x \geq 2$. It follows that $\rho(Y_2) = 4q^{3/2} < q^3 + q^2 + q + 1 = (q+1)(q^2+1)$. □

Corollary 3.2.2. *The graph Y_n is non-amenable.*

Proof. This follows from our remarks at the beginning of this chapter. □

For any graph X, the *graph distance* between any two vertices $x, y \in X$, denoted $\mathrm{d}(x,y)$, is the length of a shortest walk in X connecting x and y; if no walk in X connects x and y, then $\mathrm{d}(x,y) := \infty$.[5] For a graph X, a vertex $x \in X$, and $i \in \mathbb{Z}^{\geq 0}$, let $B_i(x) = \{y \in X : \mathrm{d}(x,y) \leq i\}$.

Corollary 3.2.3. *There is a constant $C > 1$ such that for all $x \in Y_n$ and for all $i \in \mathbb{Z}^{\geq 0}$, $\mathrm{Card}(B_i(x)) > C^i$.*

Proof. This follows from the last theorem and [3, Theorem 2.2]. □

Note that since Δ_n is a subcomplex of Ξ_{2n} by Proposition 1.3.1, the graph Y_n is a subgraph of X_{2n}, the one-complex of Ξ_{2n}. By [12, Theorem 1], the spectral radius of X_{2n} is

$$\rho(X_{2n}) = \sum_{i=1}^{2n-1} \binom{2n}{i} q^{i(2n-i)/2}.$$

Let $f(x) = x(x-1)/4$. Since $f(x) > 0$ for all $x > 1$, $n^2/2 > n(n+1)/4$ for all $n \geq 2$. Furthermore, $\binom{2n}{n} > 2^n$ for all $n \geq 2$: when $n = 2$, $\binom{2n}{n} = 6$ and $2^n = 4$. Proceeding inductively, suppose $n \geq 2$ and $\binom{2n}{n} > 2^n$. Then

$$\binom{2(n+1)}{n+1} = \frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n)!}{n!n!} \cdot \frac{(2n+1)(2n+2)}{(n+1)^2}$$
$$= \binom{2n}{n} \cdot 2 \cdot \frac{2n+1}{n+1} > 2^n \cdot 2 \cdot \frac{2n+1}{n+1} > 2^{n+1}$$

by the induction hypothesis and the fact that $(2n+1)/(n+1) > 1$ for all $n \geq 2$. It

[5] Suppose $x, y \in X$ such that there is a walk in X connecting x and y. Then the minimal length of a walk in X connecting x and y exists and is the length of a walk in X connecting x and y: consider the set S of walks in X connecting x and y. For each walk in S, put its length in the set S'. Then S' is a non-empty subset of $\mathbb{Z}^{\geq 0}$, which is well-ordered; hence, S' has a minimal element m, and, by the way we created S', is the length of a walk in X connecting x and y.

follows that for all $n \geq 2$,

$$\rho(Y_n) = 2^n q^{n(n+1)/4} < \binom{2n}{n} q^{n(n+1)/4} < \binom{2n}{n} q^{n^2/2}$$
$$\leq \sum_{i=1}^{2n-1} \binom{2n}{i} q^{i(2n-i)/2} = \rho(X_{2n}).$$

Remark. We noted at the beginning of this chapter that the one-complex X_n of Ξ_n is non-amenable for all $n \geq 2$, and Theorem 3.2.1 implies that Y_n is also non-amenable for all $n \geq 2$. The above computation shows that $\rho(Y_n) < \rho(X_{2n})$ for all $n \geq 2$, but can we say more? More precisely, let $C(\Delta_n) > 1$ be maximal such that for all $t \in Y_n$ and for all $i \in \mathbb{Z}^{\geq 0}$, $\mathrm{Card}(B_i(t)) > C(\Delta_n)^i$, and let $C(\Xi_{2n}) > 1$ be the analogous number for X_{2n} ($C(\Xi_{2n})$ exists by [55, Example (12.20)] and [3, Theorem 2.2]). Is there a relationship between $C(\Delta_n)$ and $C(\Xi_{2n})$?

Appendices

Appendix A

Symplectic Divisors Revisited

In this appendix, $n \geq 2$, V is a $2n$-dimensional K-vector space endowed with a non-degenerate, alternating bilinear form $\langle \cdot, \cdot \rangle$, and a lattice of V is a free $\mathcal{O}$-module of rank $2n$. We indicate how to obtain the analogues of the results in Section 2.2.2 when we replace G^S and $\mathrm{Sp}_n(\mathcal{O})$ with $\mathrm{GSp}_n(K)$ and $\mathrm{GSp}_n(\mathcal{O}) = \{g \in M_{2n}(\mathcal{O}) : g^t J_n g = \nu(g) J_n,\ \nu(g) \in \mathcal{O}^\times\}$, respectively. Recall that a lattice L of V is symplectic if it has the form $L = \mathcal{O}u_1 + \cdots + \mathcal{O}u_n + \mathcal{O}w_1 + \cdots + \mathcal{O}w_n$ for some symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V and that a lattice of V is symplectic if and only if it is primitive. Let $\Gamma = \mathrm{GSp}_n(\mathcal{O})$. Fix a symplectic lattice L_0 of V, and let $\mathcal{R} = \mathcal{R}(L_0) = \{gL_0 : g \in \mathrm{GSp}_n(K)\}$.

Proposition A.1. *A lattice L of V is in $\mathcal{R}$ if and only if $[L]$ is a special vertex of Δ_n.*

Proof. First note that since L_0 is symplectic, L_0 is primitive; hence, Propositions 1.5.2 and 1.5.3 imply $[L_0]$ is a special vertex of Δ_n. Let $L \in \mathcal{R}$. Then there is a $g \in \mathrm{GSp}_n(K)$ such that $L = gL_0$; i.e., such that $[L] = g[L_0]$. The fact that L is special follows from Corollary 1.6.3. The converse also follows from Corollary 1.6.3. □

For any $L \in \mathcal{R}$, Proposition 2.2.1 allows us to identify $\Gamma = \mathrm{GSp}_n(\mathcal{O})$ with $\{g \in \mathrm{GSp}_n(K) : gL = L\}$, where g acts on L as the matrix of a linear transformation with respect to a fixed basis of L.

Lemma A.1. *Let $L, M \in \mathcal{R}$. Then there is a basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of V with $\langle x_i, y_j \rangle = \pi^r \delta_{ij}$ for some $r \in \mathbb{Z}$ and $\langle x_i, x_j \rangle = 0 = \langle y_i, y_j \rangle$ and elements $\alpha_i, \beta_i \in K^\times$ with $\beta_1 \mathcal{O} \subseteq \cdots \subseteq \beta_n \mathcal{O} \subseteq \alpha_n \mathcal{O} \subseteq \cdots \subseteq \alpha_1 \mathcal{O}$ and $\beta_i \alpha_i = \pi^{r'-r}$ for some $r' \in \mathbb{Z}$ such* *that*

$$L = \mathcal{O}x_1 + \cdots + \mathcal{O}x_n + \mathcal{O}y_1 + \cdots + \mathcal{O}y_n,$$
$$M = \mathcal{O}\alpha_1 x_1 + \cdots + \mathcal{O}\alpha_n x_n + \mathcal{O}\beta_1 y_1 + \cdots + \mathcal{O}\beta_n y_n.$$

Remark. The ideals $\alpha_i\mathcal{O}$ and $\beta_i\mathcal{O}$ are the *symplectic divisors* of M in L and coincide with the standard elementary divisors of M in L. In other words, if we choose two lattices from $\mathcal{R}$ and consider their elementary divisors in the traditional sense, they satisfy the above-stated additional properties. For $L, M \in \mathcal{R}$, write $\{L : M\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$ if there are bases of L and M as in the lemma.

Proof. Replace Proposition 2.2.3 with Proposition A.1 in the proof of Lemma 2.2.5. □

Lemma A.2. *Let $L \in \mathcal{R}$. For $h_1, h_2 \in \mathrm{GSp}_n(K)$, the left cosets $h_1\Gamma$ and $h_2\Gamma$ are equal if and only if $h_1L = h_2L$.*

Proof. Replace Proposition 2.2.4 with Proposition 2.2.1 in the proof of Lemma 2.2.6. □

Lemma A.3. *Let $L, M, M' \in \mathcal{R}$ with L symplectic. Then $\{L : M\} = \{L : M'\}$ if and only if there is a $g \in \Gamma$ such that $gM = M'$.*

Proof. Replace Proposition 2.2.4, Lemma 2.2.5, S, G^S, and $\mathrm{Sp}_n(\mathcal{O})$ with Proposition 2.2.1, Lemma A.1, $K^\times$, $\mathrm{GSp}_n(K)$, and $\mathrm{GSp}_n(\mathcal{O})$, respectively, in the proof of Lemma 2.2.7. □

Proposition A.2. *Let $L \in \mathcal{R}$ be a symplectic lattice, and identify $\Gamma = \mathrm{GSp}_n(\mathcal{O})$ with the stabilizer of L in $\mathrm{GSp}_n(K)$. Let $g \in \mathrm{GSp}_n(K)$ with*

$$\Gamma g \Gamma = \Gamma \mathrm{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\Gamma.$$

Then $h\Gamma \mapsto hL$ gives a one-to-one correspondence between the cosets $h\Gamma$ in $\Gamma g\Gamma$ and the lattices $M \in \mathcal{R}$ with $\{L : M\} = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n\}$.

Proof. Replace Proposition 2.2.4, Lemma 2.2.7, and Lemma 2.2.6 with Proposition 2.2.1, Lemma A.3, and Lemma A.2, respectively, in the proof of Proposition 2.2.5. □

The following is the analogue of Proposition 2.2.6 when $\Gamma = \mathrm{GSp}_n(\mathcal{O})$.

Proposition A.3. *If $t \in \Delta_n$ is a type 0 vertex, then the number of vertices of Δ_n close to t is the number of left cosets of Γ in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. Replace Proposition 2.2.5 with Proposition A.2 in the proof of Proposition 2.2.6. □

Theorem A.1. *If $t \in \Delta_n$ is a special vertex, then the number of vertices of Δ_n close to t is the number of left cosets of Γ in*

$$\Gamma \mathrm{diag}(1, \underbrace{\pi, \ldots, \pi}_{n-1}, \pi^2, \pi, \ldots, \pi)\Gamma.$$

Proof. The proof is the same as that of Theorem 2.2.1 using the last proposition (Proposition A.3) rather than Proposition 2.2.6. □

Appendix B

Markov Chains

We show how Theorem 3.1.1 is a reformulation of a special case of [55, Theorem (12.10)]. We start by recalling the graph theory terminology and notation from Chapter 3 that we need. A *graph* X is a set of vertices together with a set $E(X)$ of edges, where an edge is an unordered pair $\{x, y\}$ of distinct vertices. Two vertices $x, y \in X$ are *adjacent*, denoted $x \sim y$, if $\{x, y\} \in E(X)$, and for $x \in X$, $\deg(x)$ denotes the number of vertices of X adjacent to x. A graph X is *locally finite* if $\deg(x) < \infty$ for all $x \in X$, and X is *regular* if $\deg(x) = \deg(y)$ for all $x, y \in X$. An isomorphism between two graphs X and Y is a bijection $\varphi : X \to Y$ such that $x, y \in X$ adjacent implies $\varphi(x), \varphi(y) \in Y$ adjacent, and an *automorphism* of a graph X is an isomorphism of X with itself.

Following [55, pp. 2 – 3, 8, 12 – 14], we now give the relevant Markov chain terminology. A *Markov chain* is given by a countable (finite or infinite) state space X and a transition matrix $P = (p(x, y))_{x,y \in X}$, where $p(x, y)$ is the probability of moving from x to y in one step. Let (X, P) be a Markov chain. For $j \geq 1$, let $p^{(j)}(x, y)$ be the (x, y)-entry of P^j. Then the *spectral radius* of P is

$$\rho(P) = \limsup_{j \to \infty} \; p^{(j)}(x, y)^{\frac{1}{j}}.$$

By [55, p. 3], $\rho(P)$ lies in $(0, 1]$ and is independent of the choice of $x, y \in X$. The *simple random walk* on a locally finite graph X is the Markov chain with state space X and transition probabilities

$$p(x, y) = \begin{cases} 1/\deg(x) & \text{if } y \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

For the simple random walk (X, P) on a locally finite graph X, let $\mathrm{AUT}(X)$ be the group of automorphisms of X and $\mathrm{AUT}(X, P)$ the group of $g \in \mathrm{AUT}(X)$ such that $p(gx, gy) = p(x, y)$ for all $x, y \in X$. The Markov chain (X, P) is *reversible* if there is a measure $m : X \to (0, \infty)$ such that $m(x)p(x, y) = m(y)p(y, x)$ for all $x, y \in X$.

Following [54, Definition 10.1], a group G is *amenable* if there is a finitely additive measure μ on the family of all subsets of G such that $\mu(G) = 1$ and μ is left-invariant ($\mu(gA) = \mu(A)$ for $g \in G$ and $A \subseteq G$). Recall that if G is a group acting on a set S and $a \in S$, we write G_a for $\{g \in G : ga = a\}$, the stabilizer of a in G. The following is [55, Theorem (12.10)] (with Γ replaced with Q).

Theorem B.1. *If (X, P) is a reversible Markov chain and $Q \leq \mathrm{AUT}(X, P)$ is amenable, acts transitively, and has a left Haar measure $\mu(\cdot)$, then*

$$\rho(P) = \sum_{y \in X} p(x, y)\sqrt{\frac{\mu(Q_y)/m(y)}{\mu(Q_x)/m(x)}}.$$

Remark. The function $m(\cdot)$ given above is the measure $m : X \to (0, \infty)$ as in the definition of a reversible Markov chain.

Proof. See the proof of [55, Theorem (12.10)]. ☐

From now on, (X, P) is the simple random walk on a locally finite, regular graph X. Then $p(x, y) = p(y, x)$ for all $x, y \in X$: if $x, y \in X$ with x not adjacent to y, then y is not adjacent to x and $p(x, y) = 0 = p(y, x)$. On the other hand, if $x, y \in X$ are adjacent, then X regular implies $p(x, y) = 1/\deg(x) = 1/\deg(y) = p(y, x)$. It follows that the Markov chain (X, P) is reversible with measure $m : X \to (0, \infty)$ given by $m(x) = 1$ for all $x \in X$.

Suppose there is a group Q that

1. is solvable,
2. acts transitively on X as a group of automorphisms, and
3. has a left Haar measure $\mu(\cdot)$

(see Theorem 3.1.1). Then by [54, p. 149], Q solvable implies Q amenable. We claim that $Q \leq \mathrm{AUT}(X, P)$. Suppose $g \in Q$, and let $x, y \in X$. First suppose $x \sim y$. Then g an automorphism of X implies $gx \sim gy$. Since X is regular, $p(gx, gy) = 1/\deg(gx) = 1/\deg(x) = p(x, y)$. Now suppose x and y are not adjacent. Then gx and gy are not adjacent (otherwise, $gx \sim gy$, and $g^{-1} \in Q$ an automorphism implies $x = g^{-1}(gx)$ and

$y = g^{-1}(gy)$ adjacent, a contradiction); hence, $p(gx, gy) = 0 = p(x, y)$, establishing the claim. It follows that the group Q satisfies the conditions of Theorem B.1. Since (X, P) also satisfies the conditions of Theorem B.1,

$$\rho(P) = \sum_{y \in X} p(x, y) \sqrt{\frac{\mu(Q_y)/m(y)}{\mu(Q_x)/m(x)}} = \sum_{y \in X} p(x, y) \sqrt{\frac{\mu(Q_y)}{\mu(Q_x)}}.$$

Substituting in the values of $p(x, y)$ in the above formula gives

$$\rho(P) = \sum_{\substack{y \in X, \\ y \sim x}} \frac{1}{\deg(x)} \sqrt{\frac{\mu(Q_y)}{\mu(Q_x)}} = \frac{1}{\deg(x)} \sum_{y \sim x} \sqrt{\frac{\mu(Q_y)}{\mu(Q_x)}}.$$

For $x, y \in X$, write $Q_x y$ for $\{gy : g \in Q_x\}$, the orbit of y under the action of Q_x. The following is Lemma (1.29) of [55] (with Γ replaced by Q).

Lemma B.1. *Let* $Q \leq \mathrm{AUT}(X)$ *have a left Haar measure* $\mu(\cdot)$. *Then*

$$\frac{\mu(Q_x)}{\mu(Q_y)} = \frac{\mathrm{Card}(Q_x y)}{\mathrm{Card}(Q_y x)}.$$

Proof. See the proof of [55, Lemma (1.29)]. □

Since $\mathrm{AUT}(X, P) \leq \mathrm{AUT}(X)$, this implies

$$\rho(P) = \frac{1}{\deg(x)} \sum_{y \sim x} \sqrt{\frac{\mu(Q_y)}{\mu(Q_x)}} = \frac{1}{\deg(x)} \sum_{y \sim x} \sqrt{\frac{\mathrm{Card}(Q_y x)}{\mathrm{Card}(Q_x y)}},$$

and writing t' for y and t_0 for x gives

$$\rho(P) = \frac{1}{\deg(t_0)} \sum_{t' \sim t_0} \sqrt{\frac{\mathrm{Card}(Q_{t'} t_0)}{\mathrm{Card}(Q_{t_0} t')}}.$$

It therefore remains to compare $\rho(P)$ and $\rho(X)$ (compare the above formula with that in Theorem 3.1.1). But X regular implies $\rho(X) = \deg(x)\rho(P)$ for any $x \in X$, giving the formula in Theorem 3.1.1.

Appendix C

Solvable Groups

Let $n \in \mathbb{Z}^{\geq 2}$, K a field, and P_n the group of upper triangular matrices in $\mathrm{GL}_n(K)$. We show that P_n is solvable. First recall that a group G is *solvable* if there are subgroups $H_0 = G, H_1, \ldots, H_m = \{1\}$ of G such that

$$\{1\} = H_m \trianglelefteq H_{m-1} \trianglelefteq \cdots \trianglelefteq H_0 = G$$

and H_r/H_{r+1} is abelian for all $0 \leq r \leq m-1$.

Let $g = (g_{ij}) \in P_n$. If $i < j$, say that g_{ij} is in diagonal r if $j - i = r$. Define subsets $H_1, \ldots, H_n$ of G as follows:

- H_1 consists of the matrices in P_n whose diagonal entries are all 1, and
- for all $2 \leq r \leq n$, H_r consists of the matrices in H_1 whose entries in diagonals $1, \ldots, r-1$ are all 0.

Note that for $1 \leq m \leq n-1$, the entries in diagonal m of $g = (g_{ij}) \in P_n$ are $g_{i,i+m}$, where $1 \leq i \leq n-m$. In particular, for a given $2 \leq r \leq n$, $g = (g_{ij}) \in H_r$ implies $g_{ij} = 0$ if $i > j$ ($g \in P_n$), $g_{ii} = 1$ for all $1 \leq i \leq n$ ($g \in H_1$), and

$$g_{ij} = 0 \text{ if } 1 \leq i \leq n-1 \text{ and } i+1 \leq j \leq \min\{i+r-1, n\}. \tag{C.1}$$

Finally,

$$\{I_n\} = H_n \subseteq H_{n-1} \subseteq \cdots \subseteq H_1 \subseteq H_0 := P_n.$$

We now show that

$$\{I_n\} = H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_1 \trianglelefteq H_0 := P_n.$$

Lemma C.1. *For all* $1 \leq r \leq n-1$, $H_r \leq P_n$. *In particular, each* H_r *is a group and* $\{I_n\} = H_n \trianglelefteq H_{n-1}$.

Proof. Fix $1 \leq r \leq n-1$, and let $g = (g_{ij}), h = (h_{ij}) \in H_r$. Then P_n a group implies that whenever $i > j$, the (i,j)-entry of gh is 0. Since $g_{ii} = 1 = h_{ii}$ for all $1 \leq i \leq n$, the (i,i)-entry of gh is

$$\sum_{m=1}^{n} g_{im}h_{mi} = \sum_{m=i}^{n} g_{im}h_{mi} = g_{ii}h_{ii} = 1.$$

We now check that all the entries in diagonals $1, \ldots, r-1$ of gh are 0. Suppose $i < j$ with $j - i \in \{1, \ldots, r-1\}$. Then the (i,j)-entry of gh is

$$\sum_{m=1}^{n} g_{im}h_{mj} = \sum_{m=i}^{n} g_{im}h_{mj} = \sum_{m=i}^{j} g_{im}h_{mj} = h_{ij} + \sum_{m=i+1}^{j} g_{im}h_{mj} = 0$$

since $g_{ii} = 1$, $h_{ij} = 0$ whenever $1 \leq j-i \leq r-1$, and $g_{im} = 0$ whenever $1 \leq m-i \leq r-1$.

Write $g^{-1} = (g'_{ij})$. Since P_n is a group, $g'_{ij} = 0$ whenever $i > j$. Moreover, $gg^{-1} = I_n$ implies

$$1 = \sum_{m=1}^{n} g_{im}g'_{mi} = \sum_{m=i}^{n} g_{im}g'_{mi} = g_{ii}g'_{ii}$$

and $g'_{ii} = g_{ii}^{-1} = 1$. Finally, we obtain g^{-1} from g by applying a finite sequence of elementary row operations to g; i.e., by multiplying g on the left by finitely many elementary matrices $E_1, \ldots, E_\ell$ so that $g^{-1} = E_\ell \cdots E_1$. Write E_{ij} for the matrix with all entries equal to zero except for a 1 in the (i,j)-entry, and note that $(I_n + \alpha E_{ij})g$ is the matrix obtained from g by adding α times the jth row of g to row i. Then since $g_{ii} = 1$ for all $1 \leq i \leq n$ and $g_{ij} = 0$ whenever $i > j$, all the E_m have the form $I_n + \alpha E_{ij}$ for some $\alpha \in K$ with $1 \leq i < j \leq n$. In particular, since all the entries in diagonals 1 through $r-1$ of g are zero, $i \in \{1, \ldots, n-r\}$, and for each $1 \leq i \leq n-r$, the possible values of j are $i+r, \ldots, n$ (cf. (C.1)); hence, all the $E_m \in H_r$, and $g^{-1} = E_\ell \cdots E_1 \in H_r$ by the last paragraph. It follows that $H_r \leq P_n$ for all $1 \leq r \leq n-1$. □

Thus, since H, G groups implies $H \cap G \leq H$ and $H \cap G \leq G$,

$$\{I_n\} = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = P_n.$$

Lemma C.2. *Let* $0 \le r \le n-1$*. If* $h = (h_{ij}) \in H_r$ *and* $h^{-1} = (h'_{ij})$*, then for any* $1 \le i \le n-r$*,* $h'_{i,i+r} + h_{i,i+r} = 0$*.*

Proof. First note that by the last lemma, $h^{-1} \in H_r$. Let $1 \le i \le n-r$. Then $r+1 \le i+r \le n$, and $hh^{-1} = I_n$, $h, h^{-1} \in P_n$,

$$h_{ii} = 1 = h'_{i+r,i+r}, \text{ and } h_{im} = 0 \text{ for all } i+1 \le m \le i+(r-1), \tag{C.2}$$

imply

$$\begin{aligned} 0 &= \sum_{m=1}^{n} h_{im}h'_{m,i+r} = \sum_{m=i}^{i+r} h_{im}h'_{m,i+r} = h'_{i,i+r} + \sum_{m=i+1}^{i+(r-1)} h_{im}h'_{m,i+r} + h_{i,i+r} \\ &= h'_{i,i+r} + h_{i,i+r}, \end{aligned}$$

as claimed. □

Proposition C.1. *For all* $1 \le r \le n-1$*,* $H_r \trianglelefteq H_{r-1}$*.*

Proof. Let $g = (g_{ij}) \in H_r$, $h = (h_{ij}) \in H_{r-1}$, and write $h^{-1} = (h'_{ij}) \in H_{r-1}$. Suppose $r = 1$. Since $gh^{-1} \in P_n$, the (m,i)-entry of gh^{-1} is

$$\sum_{\ell=1}^{n} g_{m\ell}h'_{\ell i} = \sum_{\ell=m}^{i} g_{m\ell}h'_{\ell i},$$

and the (i,i)-entry of hgh^{-1} is

$$\sum_{m=1}^{n} h_{im}\left(\sum_{\ell=m}^{i} g_{m\ell}h'_{\ell i}\right) = \sum_{m=i}^{n} h_{im}\left(\sum_{\ell=m}^{i} g_{m\ell}h'_{\ell i}\right) = h_{ii}g_{ii}h'_{ii} = 1$$

($g_{ii} = 1$ and $hh^{-1} = I_n$ implies $1 = \sum_{m=1}^{n} h_{im}h'_{mi} = h_{ii}h'_{ii}$); i.e., $H_1 \trianglelefteq H_0 = P_n$ (hgh^{-1} is upper triangular since P_n is a group). Now suppose $2 \le r \le n-1$, and note that to show that $H_r \trianglelefteq H_{r-1}$, it suffices to show that all the entries of hgh^{-1} in diagonal $r-1$ are zero (we already know $H_r \le H_{r-1}$). Thus, let $1 \le i \le n-(r-1)$. Since P_n is a group, $h, g, h' \in P_n$, $g_{ii} = 1 = g_{i+(r-1),i+(r-1)}$, and $g_{i,\ell} = 0$ for all $i+1 \le \ell \le i+(r-1)$, (C.2) (with $r-1$ instead of r since $h \in H_{r-1}$), and the last lemma (with $r-1$ instead of r since $h \in H_{r-1}$) imply that the $(i, i+(r-1))$-entry of hgh^{-1} is

$$\begin{aligned}
\sum_{m=1}^{n} h_{im}(gh^{-1})_{m,i+(r-1)} &= \sum_{m=i}^{i+(r-1)} h_{im}(gh^{-1})_{m,i+(r-1)} \\
&= \sum_{m=i}^{i+(r-1)} h_{im}\left(\sum_{\ell=1}^{n} g_{m\ell}h'_{\ell,i+(r-1)}\right) \\
&= \sum_{\ell=1}^{n} g_{i\ell}h'_{\ell,i+(r-1)} + h_{i,i+(r-1)}\sum_{\ell=1}^{n} g_{i+(r-1),\ell}h'_{\ell,i+(r-1)} \\
&= \sum_{\ell=i}^{i+(r-1)} g_{i\ell}h'_{\ell,i+(r-1)} + h_{i,i+(r-1)} \\
&= h'_{i,i+(r-1)} + h_{i,i+(r-1)} = 0;
\end{aligned}$$

i.e., $hgh^{-1} \in H_r$ and $H_r \trianglelefteq H_{r-1}$. □

To show that P_n is solvable, it therefore remains to show that for all $0 \leq r \leq n-1$, H_r/H_{r+1} is abelian. We first find representatives for H_r/H_{r+1} for all $0 \leq r \leq n-1$.

Lemma C.3. *A set of representatives for H_0/H_1 is*

$$\{\mathrm{diag}(\alpha_1, \ldots, \alpha_n) : \alpha_1, \ldots, \alpha_n \in K^\times\},$$

and for all $1 \leq r \leq n-1$, a set of representatives for H_r/H_{r+1} is

$$\{g = (g_{ij}) \in H_r : g_{ij} = 0 \text{ if } i < j \text{ with } j - i \neq r,\ g_{ij} \in K \text{ if } i < j \text{ with } j - i = r\}.$$

Proof. As in Lemma C.1, let E_{ij} be the matrix with all entries 0 except for a 1 in the (i, j)-entry, and note that for $\alpha \in K$ and $h \in M_n(K)$, $h(I_n + \alpha E_{ij})$ is the matrix obtained from h by adding α times the ith column of h to the jth column of h. It follows that if $h = (h_{ij}) \in H_1$, then we can write h as a product of elementary matrices of the form $I_n + h_{ij}E_{ij}$, where $1 \leq i < j \leq n$. In particular, if $h = (h_{ij}) \in H_{r+1}$ for some $1 \leq r \leq n-1$, then h is a product of elementary matrices of the form $I_n + h_{ij}E_{ij}$, where $1 \leq i \leq n-r$ and $i + r \leq j \leq n$.

Let $1 \leq r \leq n-1$. Since H_{r+1} consists of those matrices $g \in H_r$ such that all entries in diagonal r of g are 0, for any $g \in H_r$, there is a matrix $h \in H_{r+1}$ such that all entries in diagonals $1, \ldots, r-1$ of gh are 0 ($H_{r+1} \leq H_r$ by Lemma C.1) and all entries in diagonals $r+1, \ldots, n-1$ are 0 (by our comments in the first paragraph of this proof). Since each of the entries in diagonal r of gh can be any element of K, we are done in this case. Similarly, if $g \in H_0$, there is a matrix $h \in H_1$ such that all entries in diagonals $1, \ldots, n-1$ of gh are 0 (by our comments in the first paragraph of this proof since all the diagonal entries of g are in $K^\times$). ☐

Lemma C.4. *For all $0 \le r \le n-1$, let $S(r)$ be the set of representatives for H_r/H_{r+1} given in the last lemma. Then for all $g, h \in S(r)$, $gH_{r+1} = hH_{r+1}$ if and only if $g = h$.*

Proof. First suppose $r = 0$, and let $g = \mathrm{diag}(g_1, \ldots, g_n), h = \mathrm{diag}(h_1, \ldots, h_n) \in S(0)$. Then $g^{-1}h = \mathrm{diag}(g_1^{-1}h_1, \ldots, g_n^{-1}h_n) \in H_1$ if and only if $g_i^{-1}h_i = 1$ for all $1 \le i \le n$; i.e., if and only if $g_i = h_i$ for all i. Now let $1 \le r \le n-1$ and $g = (g_{ij}), h = (h_{ij}) \in S(r)$. Write $g^{-1} = (g'_{ij}) \in H_r$. Then $g^{-1}h \in H_r$ (by Lemma C.1) implies all entries in diagonals $1, \ldots, r-1$ of $g^{-1}h$ are 0 and all diagonal entries in $g^{-1}h$ are 1. Suppose $gH_{r+1} = hH_{r+1}$; i.e., suppose $g^{-1}h \in H_{r+1}$. Then all entries in diagonal r of $g^{-1}h$ are 0; hence, for any $1 \le i \le n-r$, the $(i, i+r)$-entry of $g^{-1}h$ is

$$\begin{aligned} 0 &= \sum_{m=1}^{n} g'_{im}h_{m,i+r} = \sum_{m=i}^{i+r} g'_{im}h_{m,i+r} = h_{i,i+r} + \sum_{m=i+1}^{i+r-1} g'_{im}h_{m,i+r} + g'_{i,i+r} \\ &= h_{i,i+r} + g'_{i,i+r} \end{aligned}$$

since $g'_{ii} = 1 = h_{i+r,i+r}$ (H_1 is a group) and $h_{m,i+r} = 0$ for all m such that $1 \le (i+r)-m \le r-1$. But Lemma C.2 implies $g'_{i,i+r} + g_{i,i+r} = 0$; i.e., $0 = h_{i,i+r} + g'_{i,i+r} = h_{i,i+r} - g_{i,i+r}$. ☐

Proposition C.2. *For all $0 \le r \le n-1$, H_r/H_{r+1} is abelian. More precisely,*

1. *$H_0/H_1 \cong (K^\times)^n$, where the group operation of $(K^\times)^n$ is coordinate-wise multiplication, and*

2. *for all $1 \le r \le n-1$, $H_r/H_{r+1} \cong K^{n-r}$, where the group operation of K^{n-r} is coordinate-wise addition.*

Proof. First note that the groups $(K^\times)^n$ and K^{n-r} $(1 \le r \le n-1)$ under coordinate-wise multiplication and coordinate-wise addition, respectively, are abelian. As in the last lemma, for all $0 \le r \le n-1$, let $S(r)$ be the set of representatives for H_r/H_{r+1} given in Lemma C.3. Then by the last lemma, each $S(r)$ is a complete set of representatives for H_r/H_{r+1}. For part 1, if $g = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \in S(0)$, let $\varphi_0(g) = (\alpha_1, \ldots, \alpha_n)$. Then if $h = \mathrm{diag}(\beta_1, \ldots, \beta_n) \in S(0)$, $\varphi_0(gh) = \varphi_0(\mathrm{diag}(\alpha_1\beta_1, \ldots, \alpha_n\beta_n)) = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n) = (\alpha_1, \ldots, \alpha_n) \cdot (\beta_1, \ldots, \beta_n) = \varphi_0(g) \cdot \varphi_0(h)$, and φ_0 is a homomorphism. Since φ_0 is surjective and has kernel $\{I_n\}$, it is an isomorphism.

For part 2, fix $1 \leq r \leq n-1$. For $g = (g_{ij}) \in S(r)$, let $\varphi_r(g) = (g_{1,1+r}, \ldots, g_{n-r,n})$. Then if $h = (h_{ij}) \in S(r)$ and $1 \leq i \leq n-r$, the $(i, i+r)$-entry of gh is

$$\begin{aligned} \sum_{m=1}^{n} g_{im}h_{m,i+r} &= \sum_{m=i}^{i+r} g_{im}h_{m,i+r} = h_{i,i+r} + \sum_{m=i+1}^{i+(r-1)} g_{im}h_{m,i+r} + g_{i,i+r} \\ &= h_{i,i+r} + g_{i,i+r} \end{aligned}$$

since $g_{ii} = 1 = h_{i+r,i+r}$ $(g, h \in H_r \leq H_1)$ and $g_{im} = 0$ for all m such that $1 \leq m - i \leq r - 1$ $(g \in H_r)$. Thus, $\varphi_r(gh) = (h_{1,1+r} + g_{1,1+r}, \ldots, h_{n-r,n} + g_{n-r,n}) = (h_{1,1+r}, \ldots, h_{n-r,n}) + (g_{1,1+r}, \ldots, g_{n-r,n}) = \varphi_r(g) + \varphi_r(h)$, and φ_r is a homomorphism. But φ_r is also surjective and has kernel $\{I_n\}$; hence, φ_r is an isomorphism. □

Index of Notation

Bibliography

[1] Peter Abramenko and Gabriele Nebe. Lattice chain models for affine buildings of classical type. *Mathematische Annalen*, 322:537 – 562, 2002.

[2] Anatolij N. Andrianov. *Quadratic Forms and Hecke Operators*. Springer-Verlag, 1987.

[3] N. L. Biggs, Bojan Mohar, and John Shawe-Taylor. The spectral radius of infinite graphs. *The Bulletin of the London Mathematical Society*, 20:116 – 120, 1988.

[4] Béla Bollobás. *Graph Theory: An Introductory Course*. Springer, 1979.

[5] Nicolas Bourbaki. *Groupes et Algébres de Lie Chaptires IV – VI*. Hermann, 1968.

[6] Kenneth S. Brown. *Buildings*. Springer, 1989.

[7] François Bruhat. Représentations des groupes de Lie semi-simples complexes. *Comptes Rendus de l'Académie des Sciences, Paris*, 238:437 – 439, 1954.

[8] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local, I. Données radicielles valuées. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 41:5 – 251, 1972.

[9] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local, II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 60:197 – 376, 1984.

[10] Francis Buekenhout. An introduction to incidence geometry. In Francis Buekenhout, editor, *Handbook of Incidence Geometry*, pages 1 – 25. Elsevier Science B. V., 1995.

[11] Donald I. Cartwright. Harmonic functions on buildings of type $\tilde{A}_n$. In Wolfgang Woess Massimo Picardello, editor, *Symposia Mathematica Volume XXXIX*, pages 104 – 138. Cambridge University Press, 1999.

[12] Donald I. Cartwright, Patrick Solé, and Andrzej Żuk. Ramanujan geometries of type $\tilde{A}_n$. *Discrete Mathematics*, 269:35 – 43, 2003.

[13] C. Chevalley. Sur certains groupes simples. *The Tohôku Mathematical Journal*, 7(2):14 – 66, 1955.

[14] C. Chevalley. *Classification des Groupes de Lie Algébriques.* Séminaire École Normale Supérieure, Paris, 1956 – 58. Mimeographed notes, 2 volumes.

[15] John B. Conway. *A Course in Functional Analysis.* Springer, second edition, 1990.

[16] Giuliana Davidoff, Peter Sarnak, and Alain Valette. *Elementary Number Theory, Group Theorey, and Ramanujan Graphs.* London Mathematical Society Student Texts. Cambridge University Press, 2003.

[17] David S. Dummit and Richard M. Foote. *Abstract Algebra.* John Wiley & Sons, Inc., second edition, 1999.

[18] Gábor Elek and Gábor Tardos. On roughly transitive amenable graphs and harmonic Dirichlet functions. *Proceedings of the American Mathematical Society*, 128(8):2479 – 2485, 2000.

[19] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications.* John Wiley & Sons, Inc., second edition, 1999.

[20] Paul Garrett. *Buildings and Classical Groups.* Chapman & Hall, 1997.

[21] Theo Grundhöfer. Basics on buildings. In Katrin Tent, editor, *Tits Buildings and the Model Theory of Groups*, pages 1 – 21. Cambridge University Press, 2002.

[22] Nagayoshi Iwahori and Hideya Matsumoto. On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 25:5 – 48, 1965.

[23] Kenkichi Iwasawa. *Local Class Field Theory.* Oxford University Press, 1986.

[24] Bruce Kleiner and Bernhard Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique*, 324(6):639 – 643, 1997.

[25] Alexander Lubotzky. Cayley graphs: Eigenvalue, expanders and random walks. In P. Rowlinson, editor, *Survey in Combinatorics*, pages 155 – 289. Cambridge University Press, 1995.

[26] Bojan Mohar. The spectrum of an infinite graph. *Linear Algebra and its Applications*, 48:245 – 256, 1982.

[27] R. V. Moody and K. L. Teo. Tits' systems with crystallographic Weyl groups. *Journal of Algebra*, 21:178 – 190, 1972.

[28] G. D. Mostow. *Strong Rigidity of Locally Symmetric Spaces.* Annals of Mathematics Studies, No. 78. Princeton University Press, 1973.

[29] James R. Munkres. *Topology.* Prentice Hall, second edition, 2000.

[30] M. Ram Murty. Ramanujan graphs. *Journal of the Ramanujan Mathematical Society*, 18(1):33 – 52, 2003.

[31] James Parkinson. *Buildings and Hecke Algebras.* PhD thesis, University of Sydney, 2005.

[32] James Parkinson. Buildings and Hecke algebras. *Journal of Algebra*, 297(1):1 – 49, 2006.

[33] James Parkinson. Isotropic random walks on affine buildings. *Annales de l'Institut Fourier*, to appear.

[34] James Parkinson. Spherical harmonic analysis on affine buildings. *Mathematische Zeitschrift*, to appear.

[35] Mark Ronan. *Lectures on Buildings.* Academic Press, Inc., 1989.

[36] Mark Ronan. Classification and construction of buildings. In William M. Kantor, Robert A. Liebler, Stanley E. Payne, and Ernest E. Schult, editors, *Finite Geometries, Buildings, and Related Topics*, pages 1 – 15. Oxford University Press, 1990.

[37] Mark Ronan. Buildings: Main ideas and applications. I. Main ideas. *The London Mathematical Society*, 24:1 – 51, 1992.

[38] Mark Ronan. Buildings: Main ideas and applications. II. Arithmetic groups, buildings and symmetric spaces. *The London Mathematical Society*, 24:97 – 126, 1992.

[39] Mark Ronan and Jacques Tits. Building buildings. *Mathematische Annalen*, 278(1 – 4):291 – 306, 1987.

[40] Joseph J. Rotman. *Advanced Modern Algebra*. Prentice Hall, 2002.

[41] Anne Schwartz and Thomas R. Shemanske. Maximal orders in central simple algebras and Bruhat-Tits buildings. *Journal of Number Theory*, 56(1):115 – 138, 1996.

[42] Jean-Pierre Serre. *Trees*. Springer-Verlag, 1980.

[43] Thomas R. Shemanske. The arithmetic and combinatorics of building for Sp_n. *Transactions of the American Mathematical Society*, to appear.

[44] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology*. Dover Publications, Inc., 1970.

[45] Gudlaugur Thorbergsson. Isoparametric foliations and their buildings. *Annals of Mathematics, Second Series*, 133(2):429 – 446, 1991.

[46] Jacques Tits. Théorème de Bruhat et sous-groupes paraboliques. *Comptes Rendus de l'Académie des Sciences, Paris*, 254:2910 – 2912, 1962.

[47] Jacques Tits. Géométries polyédriques et groupes simples. In *Atti della II Riunione del Groupement de Mathématiciens d'Expression Latine*, pages 66 – 88, Florence (1961), Edizioni Cremonese, Rome, 1963.

[48] Jacques Tits. Structures et groupes de Weyl. *Séminaire Bourbaki 1964 – 65*, exposé no. 288, February 1965.

[49] Jacques Tits. *Buildings of Spherical Type and Finite BN-pairs*, volume 386 of *Lecture Notes in Mathematics*. Springer-Verlag, 1974.

[50] Jacques Tits. On buildings and their applications. In *Proceedings of the International Congress of Mathematicians*, pages 209 – 220. Canadian Mathematical Congress, 1975.

[51] Jacques Tits. Reductive groups over local fields. *Proceedings of Symposia in Pure Mathematics*, 33:29 – 69, 1979.

[52] Jacques Tits. Buildings and Buekenhout geometries. In Michael J. Collins, editor, *Finite Simple Groups. II*, pages 309 – 320. Academic Press, Inc., 1980.

[53] Jacques Tits. Immeubles de type affine. In L. A. Rosati, editor, *Buildings and the Geometry of Diagrams*, pages 159 – 190. Springer, 1986.

[54] Stan Wagon. *The Banach-Tarski Paradox*. Cambridge University Press, 1985.

[55] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*. Cambridge University Press, 2000.

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