The enumeration of regions in the Shi arrangement with a given separating wall

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The roots $\alpha_1$, $\alpha_2$, and $\theta$ and their reflecting hyperplanes.
For any positive integers $n$ and $m$, the extended Shi arrangement is

$$\{ H_{\alpha_{ij},k} | k \in \mathbb{Z}, -m < k \leq m \text{ and } 1 \leq i \leq j \leq n-1 \}$$
Regions in the dominant chamber

\[ H_{\alpha_1,0} H_{\alpha_1,1} H_{\alpha_1,2} \]

\[ H_{\alpha_2,2} \]

\[ H_{\alpha_2,1} \]

\[ H_{\alpha_2,0} \]

\[ H_{\theta,1} H_{\theta,2} \]

\[ C_{nm} = \frac{1}{nm + 1} \binom{n(m + 1)}{n} \]

\[ C_{32} = 12 \]

12 regions
A hyperplane $H$ separates two regions if they lie on opposite sides of it. A hyperplane $H$ is a separating wall for a region $R$ if $H$ is a supporting hyperplane of $R$ and $H$ separates $R$ from $R_0$.

There are two regions which have $H_{\theta,m}$ as a separating wall.
Problem

Given a hyperplane $H_{\alpha_{ij},m}$, how many dominant regions have it as a separating wall? In other words, the rest of the hyperplanes in the arrangement cut $H_{\alpha_{ij},m}$ into regions. How many?

There are three regions which have $H_{\alpha_{1},m}$ as a separating wall.
An \( n \)-core is an integer partition \( \lambda \) such that \( n \nmid h_{ij} \) for all boxes \((i, j)\) in \( \lambda \). Some 3-cores. Boxes contain their hook numbers.

Not a 3-core.
Abacus description of $n$-cores

The hooklengths from the first column of a partition $\lambda$, plus all negative integers, are a set of $\beta$-numbers for $\lambda$. Construct an $n$-abacus for a partition by putting its $\beta$-numbers on an $n$-runner abacus.

A partition $\lambda$ is an $n$-core if and only if its abacus is flush.
Regions on $H_{\theta,m}$

Shaded regions have hook $4 = n(m - 1) + 1$ in box $(1,1)$. 

$n = 3$ and $m = 2$
Regions on $H_{\theta,m}$

$n = 3$ and $m = 3$

Shaded regions have hook $7 = n(m - 1) + 1$ in box $(1,1)$. 
Number of regions on $H_{\theta,m}$

$\lambda$ is an $n$-core and box $(1,1)$ has hook length $n(m - 1) + 1$. What does its abacus look like? Runner 0 has 0 beads, runner 1 has $m$ beads, and all other runners have from 0 to $m - 1$ beads. There are $m^{n-2}$ such abacuses.
Base case done

What about the rest of the $H_{\alpha,m}$?

\[
\begin{array}{cccc}
\alpha_{14} & \alpha_{13} & \alpha_{12} & \alpha_{11} \\
\alpha_{24} & \alpha_{23} & \alpha_{22} \\
\alpha_{34} & \alpha_{33} \\
\alpha_{44} \\
\end{array}
\]
Generating functions

\[ f_{\alpha m}^n(p, q) = \sum_{R \in \mathfrak{h}_{\alpha m}^n} p^{r(R)} q^{c(R)}. \]

\[ r(R) = |\{(j, k) : R \text{ and } R_0 \text{ are separated by } H_{\alpha_{1j}, k} \text{ where } 1 \leq k \leq m \text{ and } 1 \leq j \leq n - 1\}|. \]

\[ c(R) = |\{(i, k) : R \text{ and } R_0 \text{ are separated by } H_{\alpha_{in-1}, k} \text{ where } 1 \leq k \leq m \text{ and } 1 \leq i \leq n - 1\}|. \]

\[ \mathfrak{h}_{\alpha m}^n = \{m\text{-Shi regions which have } H_{\alpha, m} \text{ as a separating wall.}\} \]
Regions on $H_{\alpha_1,m}$

\[ f^3_{\alpha_{12}}(p, q) = p^4 q^2 (1 + q + q^2) \]
Region coordinates

For each region $R$ keep track of all hyperplanes crossed in a triangular array. Let $r_{ij}$ be the number of translates of $H_{\alpha_{ij}}$ which separate the region $R$ from $R_0$.

\[
\begin{array}{cccc}
  r_{14} & r_{13} & r_{12} & r_{11} \\
  r_{24} & r_{23} & r_{22} \\
  r_{34} & r_{33} \\
  r_{44} \\
\end{array}
\]

Coordinates-array for a Shi region, $n = 5$. 
Shi tableaux and regions

\[ H_{\alpha_1,0}, H_{\alpha_1,1}, H_{\alpha_1,2} \]

\[ H_{\theta,0}, H_{\theta,1}, H_{\theta,2} \]
A region \( R \in S_{n,m} \) has \( H_{\alpha_{ij},m} \) as a separating wall if and only if \( r_{ij} = m \) and for all \( t \) such that \( i \leq t < j \), \( r_{it} + r_{t+1,j} = m - 1 \). The conditions on the coordinates translate into a recursion on the generating function.
Base case

\[
f_{\theta m}^n(p, q) = p^m q^m (p^{m-1} + p^{m-2} q + \ldots + p q^{m-2} + q^{m-1})^{n-2}
\]
\[
= \[m\]_{p, q}^{n-2} p^m q^m
\]

\[
f_{\theta m}^n(p, q) = \sum_{\lambda: \lambda \text{ "has" } H_{\theta, m} \text{ as separating wall}} p^\ell(\lambda) q^{\lambda_1}
\]
Recursion applied to generating function

The generating function recursion keeps track of the number of ways to attach a new first part on a core partition (new first column on Shi tableau) which corresponds to a boundary region from one dimension less.

\[
f_{\alpha m}^n(p, q) = \left( p^m(1 + q + q^2 + \cdots + q^{(n-1)m})f_{\alpha m}^{n-1}(p, q) \right) \leq q^{(n-1)m} = (p^m[(n - 1)m + 1]qf_{\alpha m}^{n-1}(p, q)) \leq q^{(n-1)m}
\]
Symmetry

$T$ is the Shi tableau for $R$ and $T'$ (conjugate) is the Shi tableau for $R'$.

$$
\begin{array}{cccc}
   r_{14} & r_{13} & r_{12} & r_{11} \\
   r_{24} & r_{23} & r_{22} \\
   r_{34} & r_{33} \\
   r_{44}
\end{array}
\quad
\begin{array}{cccc}
   r_{14} & r_{24} & r_{34} & r_{44} \\
   r_{13} & r_{23} & r_{33} \\
   r_{12} & r_{22} \\
   r_{11}
\end{array}
$$

$R \in \mathfrak{h}_{\alpha_{ij}m}^n$ if and only if $R \in \mathfrak{h}_{\alpha_{n-j,n-i}m}^n$.

In terms of generating functions, this becomes the following:

$$f_{\alpha_{ij}m}^n(p, q) = f_{\alpha_{n-j,n-i}m}^n(q, p).$$
Example

\( n = 7 \) and \( m = 2 \)

\[
f^7_{\alpha_{242}}(p, q)
\]

There are 781 regions in the dimension 7 2-Shi arrangement which have \( H_{\alpha_{242}} \) as a separating wall.
Example

\( n = 4 \)

\[
\begin{array}{|c|c|c|}
\hline
\alpha_{13} & \alpha_{12} & \alpha_{11} \\
\hline
\alpha_{23} & \alpha_{22} & \\
\hline
\alpha_{33} & & \\
\hline
\end{array}
\]

\[
f_{\alpha_{242}}^{7}(p, q) = (p^{2}[13]_q f_{\alpha_{242}}^{6}(p, q)) \leq q^{12}
\]

\[
= \left( p^{2}[13]_q \left( p^{2}[11]_q f_{\alpha_{242}}^{5}(p, q) \right) \leq q^{10} \right) \leq q^{12}
\]

\[
= \left( p^{2}[13]_q \left( p^{2}[11]_q \left( q^{2}[9]_p f_{\alpha_{132}}^{4}(q, p) \right) \leq p^{8} \right) \leq q^{10} \right) \leq q^{12}
\]

\[
= \left( p^{2}[13]_q \left( p^{2}[11]_q \left( q^{2}[9]_p \left( p^{2} q^{2}[2]_p q, \right) \leq p^{8} \right) \leq q^{10} \right) \right) \leq q^{12}
\]

\[
f_{\alpha_{242}}^{7}(1, 1) = 781
\]