Contents

Chapter 1

Getting to Know ΩB_n

1.1 Introduction

This paper aims to explore the structure of Mantaci and Reutenauer's [5] descent algebra ΩB_n which is a subalgebra of $\mathbb{Q}[B_n]$. First it is necessary to define the hyperoctahedral group B_n. After introducing more notation and terminology, we can define ΩB_n. In later chapters we'll show ΩB_n is an algebra and we'll eventually find a complete family of minimal orthogonal idempotents. We build up to this result by introducing the Free Lie Algebra on an alphabet A and examining ΩB_n's action on products of Lie polynomials. First we find an idempotent $\mathcal{I}_{(n)}$ that lives in ΩB_n and projects to $Lie_n[A]$. From it, we build the I_p which are another basis for ΩB_n, and from the I_p we build the E_λ, which are complete minimal and orthogonal idempotents.

Acknowledgements: The definitions and computations made throughout this paper were motivated foremost by the lectures of Nantel Bergeron [1]. Mantaci and Reutenauer developed the definition of ΩB_n in [5], which is analogous to Solomon's descent algebra ΣA_{n-1}. The multiplication table for ΣA_{n-1} is given in [4] along with definitions for idempotents analogous to the $\mathcal{I}_{(n)}, I_p$, and E_λ which are adapted to ΩB_n and discussed in Chapters 3 and 4 of this paper. The work with the Free Lie Algebra is developed in [3] and [2], and also discussed in [4]. Adaptations of definitions and computations to ΩB_n were made under the guidance of Nantel Bergeron for which I am very grateful.

1.2 Some Basic Definitions

The hyperoctahedral group B_n is the wreath product of $\mathfrak{S}_n$, the symmetric group on n letters, by C_2, the cyclic group of order 2. B_n is more commonly pictured as $n \times n$ permutation matrices with entries ± 1. For example:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Under this presentation, it is clear $|B_n| = 2^n n!$.

However, we will view an element $\sigma \in B_n$ as a permutation of $\{\pm 1, \pm 2, \dots, \pm n\}$ (noting $\sigma(x) = y$ if and only if $\sigma(-x) = -y$). We'll depict σ as follows. Suppose

$$\sigma: \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \text{-5} & 4 & 3 & \text{-1} & 2 \end{array}$$

Then we'll write $\sigma = \overline{5}43\overline{1}2$ (moving the negative signs on the left to bars on top, so that we really don't think of $\overline{5}$ as negative 5, but as an entirely different number incomparable to 5). Another way to think of $\sigma \in B_n$ that will be consistent with our later notation is as $\phi_\sigma(\tau_\sigma)$ where $\tau_\sigma = |\sigma_1||\sigma_2| \cdots |\sigma_n|$ and $\phi_\sigma : \{1, 2, \dots, n\} \to \{\pm 1\}$, $\phi_\sigma : i \mapsto \text{sgn}(\sigma_i)$. The ϕ tells us where to place the bars. For example, if $\tau = 54312$ and $\phi : i \mapsto (-1)^i$, then $\phi(\tau) = \overline{5}4\overline{3}1\overline{2}$. If we think of the -1 overhead as being the indication of a bar, we can also think of $\phi(\tau)$ as $\overset{\phi_1}{\tau_1}\overset{\phi_2}{\tau_2} \dots \overset{\phi_n}{\tau_n}$. Notice that $\tau_\sigma \in \mathfrak{S}_n$. Since ϕ takes values in $\{\pm 1\}$ this is another easy way to see B_n as a wreath product. (Although we can certainly use this notation to talk about $\phi(\sigma)$ where $\sigma \in B_n$.) Let the group algebra $\mathbb{Q}[B_n]$ be formal sums of elements in B_n with rational coefficients.

Now we'll define descents and descent classes. We'll call the linear span of these descent classes ΩB_n, our descent algebra. (see [5])

Definition $\sigma \in B_n$ has a *descent* in $i \in \{1, 2, \dots, n-1\}$ if $|\sigma(i)| > |\sigma(i+1)|$ and $\text{sgn}(\sigma(i)) = \text{sgn}(\sigma(i+1))$, or if $\text{sgn}(\sigma(i)) \neq \text{sgn}(\sigma(i+1))$ (although I like to think of this latter sort of descent as a "cut" because we don't compare values with opposite sign).

Example $\sigma = \overline{2}\overline{7}7453\overline{1}6$ has a descent in 2, 4, 5, and 6. (Note the descents in 2,5, and 6 come from sign changes.)

The next definition will be useful in describing our descent classes.

Definition A *signed composition* p of $n \in \mathbb{Z}$, denoted $p \models n$, is a sequence of nonzero integers $p = (p_1 p_2 \dots p_k)$ such that $\sum_{i=1}^{k} |p_i| = n$.

We call the p_i the parts of p, and we will denote k the number of parts of p by $k(p)$. Again, we'll put a bar on top of negative numbers (instead of to the left).

Ordinarily, we would place commas between the parts of p, but for now we will suppress the commas if p consists of single digits or variables.

Example $p = (13\overline{2}2\overline{1})$ is a signed composition of 9 with 5 parts.

We can put a partial ordering on the signed compositions by refinement with respect to signs. We say $p \leq q$ if p is a finer composition than q.

Example Let $q = (4\overline{2}2\overline{1})$. Referring to the example above, $p < q$. Note q is maximal with respect to this partial ordering because we cannot combine parts with different signs.

It is easy to show there are $2 \cdot 3^{n-1}$ signed compositions of n. Consider the map

$$\phi : \{p | p \models n\} \to 2^{\{\pm 1, \pm 2, \ldots, \pm n-1\}}$$

which sends

$$p = (p_1 p_2 \ldots p_k) \mapsto \{p_1, \mathrm{sgn}(p_2) \cdot (|p_1| + |p_2|), \ldots, \mathrm{sgn}(p_{k-1}) \cdot (p_1| + \ldots + |p_{k-1}|)\}.$$

For example, $\phi((13\overline{2}2\overline{1})) = \{1, 4, \overline{6}, 8\}$. ϕ has kernel of size 2 ($p = (n)$ and $p = (\overline{n})$) and the image of ϕ is of size 3^{n-1}, since in composing such a set, for each number $i \in \{1, 2, \ldots, n-1\}$ we have three mutually exclusive choices: don't include i in the set, include $+i$ in the set, or include $-i$ in the set.

Definition The *descent shape* of $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in B_n$ is the signed composition $p = (p_1 \ldots p_k)$ such that descents of σ occur in $|p_1|, |p_1| + |p_2|, \ldots,$ and $|p_1| + \ldots + |p_{k-1}|$, and $\mathrm{sgn}(p_j) = \mathrm{sgn}(\sigma_{|p_1|+|p_2|+\cdots+|p_j|})$. (Note that the whole "$p_j^{th}$ chunk" of σ has the same sign by the way descent was defined, i.e. $\mathrm{sgn}(p_1) = \mathrm{sgn}(\sigma_1) = \mathrm{sgn}(\sigma_2) = \cdots = \mathrm{sgn}(\sigma_{|p_1|})$, etc.) Furthermore, p is the maximal signed composition satisfying this. (So if σ has $k-1$ descents, then p has k parts.)

Example The descent shape of $\sigma = 513\overline{6}\overline{7}\overline{9}24\overline{8}$ is $p = (13\overline{2}2\overline{1})$.

A nice way to read off the descent shape of σ is to make a cut in σ everywhere there is a sign change and everywhere there is a descent, then read off p by the sizes of the cut chunks, and assigning signs accordingly. For example, for $\sigma = \overline{5}341\overline{2}$ we cut as $\overline{5}|34|1|\overline{2}$ and read off $p = (\overline{1}21\overline{1})$.

Now we can begin talking about elements in $\mathbb{Q}[B_n]$ called descent classes.

Definition The *descent class*

$$x_p = \sum_{\sigma:\ \text{descent shape of } \sigma \geq p} \sigma.$$

Example $x_{(\overline{21})} = \overline{123} + \overline{132} + \overline{231} + \overline{123}$

Definition $\Omega B_n =$ the linear span of $\{x_p\}_{p \vDash n}$.

In the next section we will show that ΩB_n is an algebra with basis $\{x_p\}_{p \vDash n}$ by finding coefficients $\alpha^r_{p,q}$ such that $x_p x_q = \sum_{r:r \vDash n} \alpha^r_{p,q} x_r$. Furthermore, because the x_p are indexed by signed compositions, the dimension $\dim \Omega B_n = 2 \cdot 3^{n-1}$.

1.3 Shuffles and Multiplication

The goal of this section is to show

$$x_p x_q = \sum_{r:r \vDash n} \alpha^r_{p,q} x_r$$

(where $\alpha^r_{p,q}$ are the number of matrices with column sum q and row sum p and content r. Note that, as defined below, having a given row and column sum are interrelated concepts in the way they respect signs) and hence that ΩB_n is a subalgebra of $\mathbb{Q}[B_n]$. (see [4])

It turns out that the way the descent classes multiply models how cards shuffle. Let's examine how cards shuffle more closely.

Denote the shuffle product by the symbol $\sqcup\!\sqcup$. It will be clear by the following example what we mean by $E_1 \sqcup\!\sqcup E_2$:

Example $12 \sqcup\!\sqcup 34 = 1234 + 1324 + 1342 + 3124 + 3142 + 3412$ Notice that 12 stays in that relative order as a subword of $12 \sqcup\!\sqcup 34$, as does 34. We can view a shuffle (of distinct integers) as a sum of permutations and hence as an element in $\mathbb{Q}[B_n]$.

Clearly the shuffle product is commutative and associative. How do shuffles compose? We could expand the shuffles into sums of permutations and then compose them in $\mathbb{Q}[B_n]$. We can make our computations shorter by noting that shuffling and then permuting is the same as permuting then shuffling, i.e. $(23145) \cdot (12 \sqcup\!\sqcup 34 \sqcup\!\sqcup 5) = 23 \sqcup\!\sqcup 14 \sqcup\!\sqcup 5$. Hence we have:

Example

$(1 \sqcup\!\sqcup 2345) \cdot (12 \sqcup\!\sqcup 34 \sqcup\!\sqcup 5)$

$$
\begin{aligned}
&= (12345 + 21345 + 23145 + 23415 + 23451) \cdot (12 \shuffle 34 \shuffle 5) \\
&= 12 \shuffle 34 \shuffle 5 + 21 \shuffle 34 \shuffle 5 + 23 \shuffle 14 \shuffle 5 + 23 \shuffle 41 \shuffle 5 + 23 \shuffle 45 \shuffle 1 \\
&= (12 + 21) \shuffle 34 \shuffle 5 + (14 + 41) \shuffle 23 \shuffle 5 + 1 \shuffle 23 \shuffle 45 \\
&= 1 \shuffle 2 \shuffle 34 \shuffle 5 + 1 \shuffle 23 \shuffle 4 \shuffle 5 + 1 \shuffle 23 \shuffle 45
\end{aligned}
$$

Note we can write this as a sum of terms that look like $E_1 \shuffle E_2 \shuffle \cdots \shuffle E_m$ that yield $12 \ldots n$ when concatenated as $E_1 E_2 \cdots E_m$ because these shuffles are all performed on a deck in that original order. Consider the term $41 \shuffle 23 \shuffle 5$ appearing above. Because 1 and 4 appear out of order here, they must appear in different segments of the first shuffle $(1 \shuffle 2345)$ and hence $14 \shuffle 23 \shuffle 5$ must also occur in the sum. When added together they yield $1 \shuffle 4 \shuffle 23 \shuffle 5 = 1 \shuffle 23 \shuffle 4 \shuffle 5$.

So we have a sum of permutations from $\underset{2345}{\overset{1}{\shuffle}}$ cut as $\_\_|\_\_|\_$. We can think of encoding this in a matix with 1 in the first row and 2345 in the second row, in that order, apportioned by two numbers in the first column, two numbers in the second column, and one number in the third. All matrices obtained in this way are

$$
\begin{array}{c} \\ 1 \\ 2345 \end{array}
\begin{array}{c} \begin{matrix} \_\_ & \_\_ & \_ \end{matrix} \\ \begin{pmatrix} 1 & & \\ 2 & 34 & 5 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ 1 \\ 2345 \end{array}
\begin{array}{c} \begin{matrix} \_\_ & \_\_ & \_ \end{matrix} \\ \begin{pmatrix} & 1 & \\ 23 & 4 & 5 \end{pmatrix} \end{array}
\qquad
\begin{array}{c} \\ 1 \\ 2345 \end{array}
\begin{array}{c} \begin{matrix} \_\_ & \_\_ & \_ \end{matrix} \\ \begin{pmatrix} & & 1 \\ 23 & 45 & \end{pmatrix} \end{array}
$$

and we can read off the shuffles across rows: $1 \shuffle 2 \shuffle 34 \shuffle 5 + 1 \shuffle 23 \shuffle 4 \shuffle 5 + 1 \shuffle 23 \shuffle 45$ which corresponds to our answer in the example above.

Since we know the numbers occur in the order $12 \cdots n$, only the cardinalities of each block matter. We can associate $E_1 \shuffle E_2 \shuffle \cdots \shuffle E_k$ with the composition $q = (|E_1||E_2| \cdots |E_k|)$ where $|E_i|$ is the length of the block E_i and associate $p = (|F_1| \cdots |F_h|)$ with $F_1 \shuffle \cdots \shuffle F_h$. In fact we can call the shuffles E_q and E_p. Then their product is a sum of shuffles obtained from the compositions filling a $k \times h$ matirx with row sum q and column sum p. We will denote the row sum $(\sum_{i=1}^h m_{1i}, \sum_{i=1}^h m_{2i}, \ldots, \sum_{i=1}^h m_{ki})$ of a matrix $M = (m_{ij})$ by the composition $r(M)$ and the column sum $(\sum_{i=1}^k m_{i1}, \ldots, \sum_{i=1}^k m_{ih})$ as $c(M)$. By $w(M)$ we will denote the composition obtained from reading the entries of M from left to right, row by row. Then in this notation, the argument above shows:

$$
E_q \cdot E_p = \sum_{M : \substack{r(M)=q \\ c(M)=p}} E_{w(M)}.
$$

In the above discussion, none of the σ's involved negative signs. Because barring is an involution, it is clear that the bars apportion in the matrix as follows. We bar an entry in the matrix if it occurs in a barred column or barred row but not both. So we must modify the definition of E_q to $q_i =$

$|E_i|$ and q_i is barred if and only if E_i is barred, and we must also modify the statements $r(M) = q, c(M) = p$ to mean we apportion bars on the entries of M accordingly. Hence $r(M) = q, c(M) = p$ are interrelated, as opposed to separate independent statements. This gives us a nice association between shuffles and signed compositions.

Example $(1 \sqcup\!\sqcup \overline{2345})\cdot(12 \sqcup\!\sqcup \overline{34} \sqcup\!\sqcup 5) = 1 \sqcup\!\sqcup \overline{2} \sqcup\!\sqcup 34 \sqcup\!\sqcup \overline{5} + \overline{1} \sqcup\!\sqcup \overline{23} \sqcup\!\sqcup 4 \sqcup\!\sqcup \overline{5} + 1 \sqcup\!\sqcup \overline{23} \sqcup\!\sqcup 45$
So the matrices we count now for this example are:

$$\begin{array}{c} \\ 1 \\ 4 \end{array}\begin{array}{ccc} 2 & \overline{2} & 1 \\ \multicolumn{3}{c}{\begin{pmatrix} 1 & & \\ \overline{1} & 2 & \overline{1} \end{pmatrix}} \end{array} \qquad \begin{array}{c} \\ 1 \\ 4 \end{array}\begin{array}{ccc} 2 & \overline{2} & 1 \\ \multicolumn{3}{c}{\begin{pmatrix} & \overline{1} & \\ \overline{2} & 1 & \overline{1} \end{pmatrix}} \end{array} \qquad \begin{array}{c} \\ 1 \\ 4 \end{array}\begin{array}{ccc} 2 & \overline{2} & 1 \\ \multicolumn{3}{c}{\begin{pmatrix} & & 1 \\ \overline{2} & 2 & \end{pmatrix}} \end{array}$$

Note that the signed compositions we read off above coincide with the shuffles in the answer. In terms of real-world card shuffles, we can think of the bars as meaning we turn the card face up (or down)–which is clearly an involution.

To see how these shuffles relate to ΩB_n, we introduce the anti-automorphism

$$* \ : \mathbb{Q}[B_n] \to \mathbb{Q}[B_n]$$
$$* \ : \sigma \mapsto \sigma^{-1}$$

extended algebraically. It is an anti-automorphism because $(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$.

What does $*$ do to ΩB_n? Let's examine it on a particular descent class x_p.

Let $n = 9, p = (13\overline{2}2\overline{1})$. Consider any σ that occurs as a summand of x_p. Because σ has shape p or less fine (coarser), it is clear which σ_i are barred and that $\sigma_2 < \sigma_3 < \sigma_4$, $|\sigma_5| < |\sigma_6|$, $\sigma_7 < \sigma_8$. Descents may or may not occur (because of the "less fine") in 1. To figure out what σ^{-1} is we want to rearrange the σ_i in increasing order. If

$$\sigma : \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 5 & 1 & 3 & 6 & \overline{7} & \overline{9} & 2 & 4 & \overline{8} \end{array}$$

then σ^{-1} is encoded by the same picture with the arrows pointing up $\uparrow$. We can rearrange entire columns, yielding:

$$\sigma^{-1} : \begin{array}{ccccccccc} 2 & 7 & 3 & 8 & 1 & 4 & 5 & 9 & 6 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & 2 & 3 & 4 & 5 & 6 & \overline{7} & \overline{8} & \overline{9} \end{array}$$

and read $\sigma^{-1} = 273814\overline{596}$ off the top row. In doing this rearranging, since $\sigma_2 < \sigma_3 < \sigma_4$ when we rearrange $\begin{array}{ccc} 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow \\ \sigma_2 & \sigma_3 & \sigma_4 \end{array}$ we have 234 in that relative order appearing in σ^{-1}. Likewise $\overline{56}$ and 78 will occur in that relative order. In this manner we see every possible σ occurring as a summand in x_p will have an inverse appearing in $1 \sqcup\!\sqcup 234 \sqcup\!\sqcup \overline{56} \sqcup\!\sqcup 78 \sqcup\!\sqcup \overline{9}$. So $x_p^* = E_p$ in the notation for shuffles used above.

Hence

$$\begin{aligned} x_p x_q &= ((x_p x_q)^*)^* = (x_q^* x_p^*)^* \\ &= (E_q \cdot E_p)^* = (\sum_{M:\, \substack{r(M)=q \\ c(M)=p}} E_{w(M)})^* \\ &= \sum_{M:\, \substack{r(M)=q \\ c(M)=p}} (E_{w(M)})^* \\ &= \sum_{M:\, \substack{r(M)=q \\ c(M)=p}} x_{w(M)}. \end{aligned}$$

Or we can write

$$x_p x_q = \sum \alpha_{p,q}^r x_r \qquad (1.1)$$

where $\alpha_{p,q}^r = |\{M : c(M) = p, r(M) = q, w(M) = r\}|$. In particular $\alpha_{p,q}^r \in \mathbb{Z}$.

Chapter 2

The Free Lie Algebra

2.1 Getting from $\mathbb{Q}[A^*]$ to $Lie[A]$

In this next section, we will introduce the free Lie algebra on an alphabet A and develop certain Lie polynomials that will be useful in decomposing $\mathbb{Q}[A^*]$.

Let A be an alphabet, $A = \{a_1 < a_2 < \cdots < a_f\}$. For our purposes, we stipulate that if $a_i \in A$ then $\overline{a}_i \in A$ as well. (We'll often let $A = \{\overline{1}, \overline{2}, \ldots, \overline{n}, 1, 2, \cdots, n\}$.) If $w = w_1 w_2 \cdots w_k$ where $w_i \in A$ then we say the length of w, denoted $|w|$, is equal to k. Let A^* be all words on A and A^n be all words on A of length n. That is, $A^n = \{w \in A^* : |w| = n\}$. Note, we use the symbol $|\ |$ to denote absolute value of numbers, length of words, and cardinality of sets depending on context. We denote by $\mathbb{Q}[A^*]$ all formal sums with finite support of words in A^* with coefficients in $\mathbb{Q}$.

The free Lie algebra on A turns out to be generated by brackets of elements of A. For instance, typical elements look like a or $[a, b]$ or $[c, [a, b]]$ etc. There are also relations on the brackets, namely

$$[f, g] = -[g, f] \qquad \text{and} \qquad [f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \tag{2.1}$$

where $f, g,$ and h are elements of the free Lie algebra. It is much easier to think of the free Lie algeba, $Lie[A]$ as the image of these brackets in $\mathbb{Q}[A^*]$ under the map which sends

$$[f, g] \mapsto fg - gf. \tag{2.2}$$

We'll speak of $[f, g]$ and $fg - gf$ interchangably.

By a simple inductive argument employing the identities 2.1 above, it is easy to show that any bracketing may be represented as a standard left bracket, that is, something of the form $[[\cdots[[w_1, w_2], w_3], \ldots], w_k]$.

Just as $\mathbb{Q}[A^*] = \oplus_{n\geq 0}\mathbb{Q}[A^n]$ is a graded algebra, $Lie[A] = \oplus_{n\geq 0} Lie_n[A]$ where $Lie_n[A]$ is generated by left bracketings of n letters, i.e. under our identification 2.2 above $Lie_n[A] \subset \mathbb{Q}[A^n]$.

Because we are thinking of $Lie[A]$ as sitting inside $\mathbb{Q}[A^*]$, it would be nice if we could find a map projecting $\mathbb{Q}[A^*]$ onto $Lie[A]$ or find a family of maps projecting $\mathbb{Q}[A^n]$ to $Lie_n[A]$. We could achieve this by defining an operator Θ_n that takes a word $w = w_1w_2\cdots w_n$ to its standard left bracket $[[\cdots[[w_1, w_2], w_3], \ldots], w_k]$. (see [3]) In order to do this consider the right action of B_n on the positions of letters in words of $\mathbb{Q}[A^n]$. If A includes $\overline{a}_i$ as well as a_i then the barring operation of B_n makes sense as well. For example, $a\overline{bc}\overline{d}eaa \cdot 2\overline{13}7465\overline{5} = b\overline{a}ca\overline{d}a\overline{e}$.

Definition Let $\Theta_n = \prod_{i=2}^{n}(1-\gamma_i) = \sum_{S:S\subset\{2,\ldots n\}}(-1)^{|S|}\prod_{i\in S}\gamma_i$ where $\gamma_i = i12\cdots i-l\ i+1\ i+2\cdots n$.

It will be clear by the following induction that Θ_n is the operator we want. Let $w = w_1w_2$. Notice that $w\cdot\Theta_2 = w\cdot(1-\gamma_2) = w_1w_2 - w_2w_1 = [w_1, w_2]$.

Let $w = w_1\cdots w_n$. By our inductive hypothesis, $w_1\cdots w_{n-1}\cdot\Theta_{n-1} = [[\cdots[w_1, w_2], \ldots, w_{n-2}], w_{n-1}]$. We can think of Θ_{n-1} as sitting in B_n by just letting it fix n. Indeed, since γ_i fixes $i+1, i+2, \ldots$ we can consider γ_i as an element of B_j for any $j > i$. Thus $\Theta_n = \Theta_{n-1}(1-\gamma_n)$.

$$\begin{aligned} w\cdot\Theta_n &= w\cdot\Theta_{n-1}(1-\gamma_n) \\ &= w_1\cdots w_{n-1}\cdot\Theta_{n-1}w_n(1-\gamma_n) \end{aligned}$$

(where the w_n just sort of hanging out there is concatenated on)

$$\begin{aligned} &= [[\cdots[w_1, w_2], \ldots, w_{n-2}], w_{n-1}]w_n(1-\gamma_n) \\ &= [\cdots[w_1, w_2], \ldots, w_{n-1}]w_n - w_n[\cdots[w_1, w_2], \ldots, w_{n-1}] \\ &= [[[\cdots[w_1, w_2], \ldots, w_{n-2}], w_{n-1}], w_n] = \text{the left bracketing of w} \end{aligned}$$

Hence $\mathbb{Q}[A^n]\Theta_n \subset Lie_n[A]$.

In fact, it can be shown that $\Theta_n \in \Omega B_n$. (see [2]) We want to show this because then we know Θ_n^* is a sum of shuffles, and we can use this fact in later computations. Let

$$y_q = \sum_{\sigma:\text{ descent shape of }\sigma\text{ is }p} \sigma.$$

Then clearly $x_p = \sum_{q:q\geq p} y_q$, so by the inclusion- exclusion principle,

$$y_q = \sum_{p:p\geq q}(-1)^{k(q)-k(p)}x_p.$$

Now consider $\prod_{i\in S}\gamma_i$ where $S = \{i_1 < i_2 < \ldots < i_k\}$. This product is the permutation $i_k i_{k-1}\cdots i_1 12\cdots\widehat{i_1}\cdots\widehat{i_k}\cdots n$, which has k descents and then an increasing sequence of $(n-k)$ numbers. Hence this product appears in

$y_{(\underbrace{11\cdots 1}_{k}(n-k))}$. And, in fact, $y_{(11\cdots 1(n-k))}$ is the sum of all permutations with k descents and an increasing sequence of $(n-k)$ numbers (all unbarred), so as we sum over all S of cardinality k, we get all of $y_{(11\cdots 1(n-k))}$.

$$\begin{aligned}
\Theta_n &= \sum_{S:S\subset\{2,\ldots n\}} (-1)^{|S|} \prod_{i\in S} \gamma_i \\
&= \sum_{k=0}^{n-1} \sum_{\substack{S:S\subset\{2,\ldots n\} \\ |S|=k}} (-1)^k \prod_{i\in S} \gamma_i \\
&= \sum_{k=0}^{n-1} (-1)^k y_{(11\cdots 1(n-k))} \\
&= \sum_{k=0}^{n-1} \sum_{p:p\geq(11\cdots 1(n-k))} (-1)^{k(p)-1} x_p.
\end{aligned}$$

2.2 Getting to Know $Lie[A]$ Better

The following proposition (from [1]) will show that $Lie_n[A] \subset \mathbb{Q}[A^n]\Theta_n$ and hence that $\mathbb{Q}[A^n]\Theta_n = Lie_n[A]$. It will also be useful in dealing with Lie polynomials later on. But first we need to introduce some more definitions.

We will make extensive use of the following scalar product $\langle , \rangle$.

Definition For words $u, v \in [A^*]$ let

$$\langle u, v\rangle = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise} \end{cases} \quad \text{and extend linearly to } \mathbb{Q}[A^*].$$

One important fact to notice about $\langle , \rangle$ is that

$$\text{for any } w \in \mathbb{Q}[A^*],\ w = \sum_{u\in A^*} \langle w, u\rangle u. \tag{2.3}$$

(Also note that this expression is the same if we sum over non- empty u, i.e. over $u \in A^* - 1$.) Another useful fact is that for $\tau \in \mathbb{Q}B_n$, $\langle u\tau, v\rangle = \langle u, v\tau^*\rangle$. This is easy to see, because $u\sigma = v$ if and only if $u = v\sigma^{-1}$.

Let $\mathbb{Q}_{\sqcup\!\sqcup}[A^*]\overline{\otimes}\mathbb{Q}.[A^*]$ be the completion of the tensor product of $\mathbb{Q}[A^*]$ under the shuffle product with $\mathbb{Q}[A^*]$ under the concatenation product. So

$$\mathbb{Q}_{\sqcup\!\sqcup}[A^*]\overline{\otimes}\mathbb{Q}.[A^*] = \left\{ \sum_{u,v\in\mathbb{Q}[A^*]} f_{u,v}\, u\otimes v : \begin{array}{l} \text{for fixed u, } \sum f_{u,v} v \in \mathbb{Q}[A^*] \\ \text{for fixed v, } \sum f_{u,v} u \in \mathbb{Q}[A^*] \end{array} \right\}$$

Then we can define a map

$$\Delta : \mathbb{Q}[A^*] \to \mathbb{Q}_{\sqcup\!\sqcup}[A^*]\overline{\otimes}\mathbb{Q}.[A^*]$$

by sending a letter in the alphabet

$$\Delta : a \mapsto a \otimes 1 + 1 \otimes a$$

and extending algebraically.

Note

$$\Delta(w) = \sum_{u,v \in A^*} \langle w, u \sqcup\!\sqcup v \rangle u \otimes v \tag{2.4}$$

because $\Delta(w) = \Delta(w_1 w_2 \ldots w_n) = (w_1 \otimes 1 + 1 \otimes w_1)(w_2 \otimes 1 + 1 \otimes w_2) \cdots (w_n \otimes 1 + 1 \otimes w_n) = \sum_{S \subset \{1,\ldots,n\}} w_S \otimes w_{S^c}$ because in each summand we have a choice of whether to pick $w_i \otimes 1$ or $1 \otimes w_i$ from each factor–which means we get w_i on one side of the $\otimes$ but not the other. So for one summand, as i varies from 1 to n we get all of w assorted on either side of the $\otimes$. That is to say, the summand looks like $w_S \otimes w_{S^c}$, where

the *subword* $w_S = w_{i_1} w_{i_2} \cdots w_{i_k}$ for $S = \{i_1 < i_2 < \cdots < i_k\}$,

and S^c denotes the complement of S. So we can write the above sum as $\Delta(w) = \sum u \otimes v$ where u is some subword of w and v is the complementary subword. Hence w appears in $u \sqcup\!\sqcup v$, and if we think of each w_i distinctly, then w appears only once in this shuffle. Futhermore, w appears in $u \sqcup\!\sqcup v$ only when u and v are complementary subwords of w. Hence, $\Delta(w) = \sum_{u,v \in A^*} \langle w, u \sqcup\!\sqcup v \rangle u \otimes v$.

Proposition 1 *Given $p \in \mathbb{Q}[A^n]$, the following are equivalent:*

1. $p \in Lie_n[A]$
2. $\Delta(p) = p \otimes 1 + 1 \otimes p$
3. $\langle p, u \sqcup\!\sqcup v \rangle = 0$ *unless* $u = 1$ *or* $v = 1$ ($_{\text{sometimes denoted } \emptyset}^{\text{the empty word,}}$)
4. $p \cdot \Theta_n = np$

Proof (1) $\implies$ (2) Induct on n. The $n = 0$ case is trivial, and if $n = 1$ then $p = a \in A$ and by definition, $\Delta(a) = a \otimes 1 + 1 \otimes a$.

$p \in Lie_n[A]$ so $p = [r, s]$ where clearly $r \in Lie_j[A], s \in Lie_k[A]$ and $j, k < n$. Hence $\Delta(p) = \Delta(rs - sr) = \Delta(r)\Delta(s) - \Delta(s)\Delta(r) = (r \otimes 1 + 1 \otimes r)(s \otimes 1 + 1 \otimes s) - (s \otimes 1 + 1 \otimes s)(r \otimes 1 + 1 \otimes r)$ by the inductive hypothesis. By expanding these terms, we see $\Delta(p) = (rs - sr) \otimes 1 + 1 \otimes (rs - sr) = p \otimes 1 + 1 \otimes p$.

(2) $\implies$ (3) $\Delta(p) = p \otimes 1 + 1 \otimes p$. Recall $\Delta(p) = \sum_{u,v \in A^*} \langle p, u \sqcup\!\sqcup v \rangle u \otimes v$ by equation 2.4. Equating terms, clearly $\langle p, u \sqcup\!\sqcup v \rangle = 0$ unless $u = 1$ or $v = 1$.

(3) $\implies$ (4) $p\cdot\Theta_n = \sum_{u\in A^*}\langle p\cdot\Theta_n, u\rangle u = \sum_{u\in A^*}\langle p, u\cdot\Theta_n^*\rangle u = \sum_{u\in A^*}\langle p, u\cdot (\sum_{k=0}^{n-1}\sum_{p:p\geq(11\cdots 1(n-k))}(-1)^{k(p)-1}x_p^*\rangle u$. But, by assumption, $\langle p, u \sqcup\!\sqcup\, v\rangle = 0$ unless $u \sqcup\!\sqcup\, v$ is an empty shuffle. x_p^* is an empty shuffle exactly when $k(p) = 1$; in this case, when $p = (n)$, and hence $x_p^* = x_p = 12\cdots n = Id$ the identity of B_n. So $p\cdot\Theta_n = \sum_{u\in A^*}\langle p, u\cdot(\sum_{k=0}^{n-1} Id)\rangle u = \sum_{u\in A^*} n\langle p, u\cdot Id\rangle u = np$.

(4) $\implies$ (1) $p\cdot\Theta_n = np$. We already showed that $\mathbb{Q}[A^n]\Theta_n \subset Lie_n[A]$. Hence $np \in Lie_n[A]$ which implies $p \in Lie_n[A]$.

□

One consequence of this is that $\frac{1}{n}\Theta_n$ is an idempotent. Another is that, in particular, if $p \in Lie_n[A]$ then $\frac{1}{n}p \in Lie_n[A] \subset \mathbb{Q}[A^n]$ so $\frac{1}{n}p\cdot\Theta_n = n(\frac{1}{n}p) = p$ which implies $p \in \mathbb{Q}[A^n]\Theta_n$. Hence $Lie_n[A] \subset \mathbb{Q}[A^n]\Theta_n$, so

$$Lie_n[A] = \mathbb{Q}[A^n]\Theta_n.$$

In the next section we will define other idempotents that will be useful in finding a basis of idempotents for ΩB_n. We will use Proposition 1 to show they actually are in $Lie_n[A]$.

Chapter 3

Fun with Idempotents

3.1 The $\mathcal{I}_{(n)}$ Idempotents

Now we will define the $\mathcal{I}_{(n)}$ idempotents (see [4]), and use Proposition 1 to show they project into $Lie_n[A]$ as well.

Let

$$D = \sum_{u \in A^* - 1} u \otimes u \in \mathbb{Q}_{\sqcup\!\sqcup}[A^*] \overline{\otimes} \mathbb{Q}.[A^*].$$

Note $1 + D = \sum_{u \in A^*} u \otimes u$. Then

$$\begin{aligned} D^k &= \sum_{u_1, u_2, \dots, u_k \in A^* - 1} u_1 \sqcup\!\sqcup u_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k \otimes u_1 u_2 \cdots u_k \\ &= \sum_{u_1, \dots, u_k \in A^* - 1} \sum_{w \in A^*} \langle w, u_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k \rangle w \otimes u_1 \cdots u_k \\ &= \sum_{w \in A^*} w \otimes (\sum_{u_1, \dots, u_k \in A^* - 1} \langle w, u_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k \rangle u_1 \cdots u_k) \end{aligned}$$

But $u_1 \sqcup\!\sqcup u_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k = u_1 u_2 \cdots u_k {x_q}^*$ where $q_i = |u_i|$. For w to appear in this shuffle, we need $\sum_{i=1}^k |u_i| = |w|$, hence $q \models |w|$.

$$\begin{aligned} D^k &= \sum_{w \in A^*} w \otimes (\sum_{\substack{u_1, \dots, u_k \in A^* - 1 \\ q_i = |u_i|}} \langle w, u_1 \cdots u_k {x_q}^* \rangle u_1 \cdots u_k) \\ &= \sum_{w \in A^*} w \otimes (\sum_{\substack{u_1, \dots, u_k \in A^* - 1 \\ q_i = |u_i|}} \langle w x_q, u_1 \cdots u_k \rangle u_1 \cdots u_k) \end{aligned}$$

Now we know that if $u = u_1 \cdots u_k$, the right-hand side of the tensor becomes $\sum_{u \in A^* - 1} \langle w x_q, u \rangle u = w x_q$. But how many ways can $u_1 \cdots u_k = u$? As many ways as we can reapportion the letters of u into k parts, i.e. as many positive (unsigned) compositions there are of $|w|$. To note the fact that q has all positive parts, we may write $q \leq (|w|)$.

$$D^k = \sum_{w\in A^*} w \otimes (\sum_{\substack{u\in A^*-1 \\ q\models |w|,\ \text{positive}, k(q)=k}} \langle w x_q, u\rangle u) = \sum_{w\in A^*} w \otimes (\sum_{\substack{q\leq(|w|) \\ k(q)=k}} w x_q)$$

Let $\Phi =\ \log(1{+}D)$. Note this infinite formal sum is well-defined in $\mathbb{Q}_{ш}[A^*]\overline{\otimes}\mathbb{Q}.[A^*]$. Then

$$\begin{aligned}
\Phi &= \log(1+D) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} D^k \\
&= \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \sum_{w\in A^*} w \otimes (\sum_{\substack{q\leq(|w|) \\ k(q)=k}} w x_q) \\
&= \sum_{w\in A^*} w \otimes w \left(\sum_{q\leq(|w|)} \frac{(-1)^{k(q)-1}}{k(q)} x_q \right)
\end{aligned}$$

We will define

Definition Let $\mathcal{I}_{(n)} = \sum_{q\leq(n)} \frac{(-1)^{k(q)-1}}{k(q)} x_q$ and $\mathcal{I}_{[w]} = w\mathcal{I}_{(|w|)}$.

So $\Phi = \sum_{w\in A^*} w \otimes \mathcal{I}_{[w]}$. It is clear from the above definition that $\mathcal{I}_{[w]} \in \mathbb{Q}[A^{|w|}]$.

The following string of equations will show that $\mathcal{I}_{[w]} \in Lie_{|w|}[A]$.

$$\begin{aligned}
(Id\otimes\Delta)(1+D) &= \sum_{w\in A^*} w\otimes\Delta(w) \\
&= \sum_{w\in A^*} w \otimes \left(\sum_{u,v\in A^*} \langle w, u ш v\rangle u\otimes v \right) \\
&= \sum_{u,v\in A^*} \left(\sum_{w\in A^*} \langle w, u ш v\rangle w \right) \otimes u \otimes v \\
&= \sum_{u,v\in A^*} u ш v \otimes u \otimes v \\
&= (\sum_{u\in A^*} u\otimes u\otimes 1)\cdot(\sum_{v\in A^*} v\otimes 1\otimes v) \\
&= \exp(\sum_{u\in A^*} u\otimes \mathcal{I}_{[u]}\otimes 1)\cdot \exp(\sum_{v\in A^*} v\otimes 1\otimes \mathcal{I}_{[v]})
\end{aligned}$$

which is clear since $\exp(\sum_{w\in A^*} w\otimes w) = \sum_{w\in A^*} w\otimes \mathcal{I}_{[w]}$

$$
\begin{aligned}
&= \exp(\sum_{u\in A^*} u\otimes \mathcal{I}_{[u]}\otimes 1 + \sum_{v\in A^*} v\otimes 1\otimes \mathcal{I}_{[v]}) \\
&= \exp\left(\sum_{w\in A^*} w\otimes \mathcal{I}_{[w]}\otimes 1 + w\otimes 1\otimes \mathcal{I}_{[w]}\right) \\
&= \exp\left(\sum_{w\in A^*} w\otimes (\mathcal{I}_{[w]}\otimes 1 + 1\otimes \mathcal{I}_{[w]})\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{But also } (Id\otimes\Delta)(1+D) &= (Id\otimes\Delta)(\exp(\Phi)) \\
&= (Id\otimes\Delta)\left(\sum_{n\geq 0}\frac{\left(\sum_{w\in A^*} w\otimes \mathcal{I}_{[w]}\right)^n}{n!}\right) \\
&= \sum_{n\geq 0}\frac{\left(\sum_{w\in A^*} w\otimes \Delta\left(\mathcal{I}_{[w]}\right)\right)^n}{n!} \\
&= \exp\left(\sum_{w\in A^*} w\otimes \Delta(\mathcal{I}_{[w]})\right)
\end{aligned}
$$

Hence $\exp(\sum_{w\in A^*} w\otimes(\mathcal{I}_{[w]}\otimes 1 + 1\otimes\mathcal{I}_{[w]}) = \exp(\sum_{w\in A^*} w\otimes\Delta(\mathcal{I}_{[w]}))$ which implies $\sum_{w\in A^*} w\otimes(\mathcal{I}_{[w]}\otimes 1 + 1\otimes\mathcal{I}_{[w]}) = \sum_{w\in A^*} w\otimes\Delta(\mathcal{I}_{[w]})$ and thus

$$\mathcal{I}_{[w]}\otimes 1 + 1\otimes\mathcal{I}_{[w]} = \Delta(\mathcal{I}_{[w]}).$$

By Proposition 1, $\mathcal{I}_{[w]}\in Lie_{|w|}[A]$. If we let our alphabet be $A = \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$ and let $w_o = 12\cdots n$ which we can think of as the identity, then $\mathcal{I}_{[w_o]} = w_o\mathcal{I}_{(n)} = \mathcal{I}_{(n)}$ and so

$$\mathcal{I}_{(n)}\in Lie_n[A].$$

Now we will show that $\mathcal{I}_{(n)}$ actually projects to $Lie_n[A]$. Suppose $p\in Lie_n[A]$. Then

$$
\begin{aligned}
p\mathcal{I}_{(n)} &= \sum_{u\in A^*}\langle p\mathcal{I}_{(n)}, u\rangle u \\
&= \sum_{u\in A^*}\langle p\sum_{q\leq(n)}\frac{(-1)^{k(q)-1}}{k(q)}x_q, u\rangle u \\
&= \sum_{u\in A^*}\sum_{q\leq(n)}\frac{(-1)^{k(q)-1}}{k(q)}\langle p, ux_q^*\rangle u = \sum_{u\in A^*}\langle p, ux_{(n)}^*\rangle u
\end{aligned}
$$

since ${ux_q}^*$ is a shuffle unless $q = (n)$(or $q = (\overline{n})$), and $\langle p, \text{shuffle}\rangle = 0$ for a non-empty shuffle.

$$
\begin{aligned}
&= \sum_{u\in A^*}\langle px_{(n)}, u\rangle u = \sum_{u\in A^*}\langle p, u\rangle u \\
&= p
\end{aligned}
$$

In particular, $\mathcal{I}_{(n)} \in Lie_n[A]$ so $\mathcal{I}_{(n)}\mathcal{I}_{(n)} = \mathcal{I}_{(n)}$ which means $\mathcal{I}_{(n)}$ is an idempotent. The above equations show $Lie_n[A]\mathcal{I}_{(n)} = Lie_n[A]$. Since we already had a realization of $Lie_n[A] \subset \mathbb{Q}[A^n]$, this shows that $Lie_n[A] \subset \mathbb{Q}[A^n]\mathcal{I}_{(n)}$. Furthermore, recall for $w \in \mathbb{Q}[A^n]$, $w\mathcal{I}_{(n)} = \mathcal{I}_{[w]} \in Lie_n[A]$ so $\mathbb{Q}[A^n]\mathcal{I}_{(n)} \subset Lie_n[A]$. Thus we have found an idempotent such that

$$\mathbb{Q}[A^n]\mathcal{I}_{(n)} = Lie_n[A].$$

Now we will break down $Lie[A]$ further.

3.2 $Lie^+[A]$ and $Lie^-[A]$

Definition
Let $Lie^+[A] = \{p \in Lie[A] : p = \overline{p}\}$, $Lie^-[A] = \{p \in Lie[A] : p = -\overline{p}\}$.

Claim $Lie[A] = Lie^+[A] \oplus Lie^-[A]$

The proof of the claim is clear if we consider the map

$$\Psi : Lie[A] \to Lie^+[A] \oplus Lie^-[A]$$
$$p \mapsto (\frac{p+\overline{p}}{2}, \frac{p-\overline{p}}{2})$$

□

It is clear that Ψ respects the grading of $Lie[A]$, that is to say, $Lie_n[A] = Lie_n^+[A] \oplus Lie_n^-[A]$ where $Lie_n^+[A] = \{p \in Lie_n[A] : p = \overline{p}\}$, and $Lie_n^-[A] = \{p \in Lie_n[A] : p = -\overline{p}\}$.

Let

$$I^+_{(n)} = \frac{\mathcal{I}_{(n)} + \overline{\mathcal{I}_{(n)}}}{2} \qquad I^-_{(n)} = \frac{\mathcal{I}_{(n)} - \overline{\mathcal{I}_{(n)}}}{2}.$$

Then $I^+_{(n)} + I^-_{(n)} = \mathcal{I}_{(n)}$. Clearly $I^+_{(n)} \in Lie_n^+[A]$ and $I^-_{(n)} \in Lie_n^-[A]$ and each of these operators fixes those respective spaces. Since barring is an involution, $\overline{\mathcal{I}_{(n)}\mathcal{I}_{(n)}} = \mathcal{I}_{(n)}$ and $\mathcal{I}_{(n)}\overline{\mathcal{I}_{(n)}} = \overline{\mathcal{I}_{(n)}}\mathcal{I}_{(n)} = \overline{\mathcal{I}_{(n)}}$. Hence $I^+_{(n)}$ and $I^-_{(n)}$ are both idempotents, and $I^+_{(n)}I^-_{(n)} = 0$.

Let $n = |w|$. If $u = w\mathcal{I}_{(n)}$ then $w\overline{\mathcal{I}_{(n)}} = \overline{u}$. So $I^+_{(n)}$ projects into $Lie_n^+[A]$, $I^-_{(n)}$ projects into $Lie_n^-[A]$. These idempotents are orthogonal and their sum projects to all of $Lie_n[A]$. Hence it must be that

$$\mathbb{Q}[A^n]I^+_{(n)} = Lie_n^+[A] \quad \text{and} \quad \mathbb{Q}[A^n]I^-_{(n)} = Lie_n^-[A].$$

For convenience, for $\epsilon \in \{\pm 1\}$ let

$$I^\epsilon_{(n)} = \begin{cases} I^+_{(n)} & \text{if } \epsilon = +1 \\ I^-_{(n)} & \text{if } \epsilon = -1. \end{cases}$$

So $I^{\epsilon}_{(n)} = \frac{\mathcal{I}_{(n)}+\epsilon\overline{\mathcal{I}_{(n)}}}{2}$.

3.3 Defining I_p and Finding an Expression in Terms of x_q

Definition Given $p \models n$ where $k(p) = k$, let $\epsilon_i = \text{sgn}(p_i)$. Then we define

$$I_p = \sum_{\substack{S_1+S_2+\cdots+S_k=\{1,2,\ldots,n\} \\ |S_i|=p_i}} I^{\epsilon_1}_{[S_1]} I^{\epsilon_2}_{[S_2]} \cdots I^{\epsilon_k}_{[S_k]}$$

Above we used the notation:

$$S_1 + S_2 + \cdots + S_k = \{1, 2, \ldots, n\}$$

to mean $\{1, 2, \ldots, n\}$ is the disjoint union of the S_i's. That is, $\bigcup_{i=1}^k S_i = \{1, \ldots, n\}$ and $\forall i, j$ $S_i \neq \emptyset$ and $S_i \cap S_j = \emptyset$. Also, we always order S_j as $S_j = \{i_1 < i_2 < \cdots < i_r\}$, so that by $I_{[S_j]} = S_j I_{(|S_j|)}$ we mean $i_1 i_2 \cdots i_r I_{(r)}$.

It turns out that I_q is a linear combination of x_r, and that the x_r is a linear combination of I_q's, from which we see that $\{I_q\}_{q \models n}$ is also a basis of ΩB_n. In order to compute such an expression it is necessary to introduce special notation to keep track of signs and bars. We will introduce the notation throughout the body of the next claim, and will use it in subsequent computations.

Claim

$$I_q = \frac{1}{2^k} \sum_{\phi:\{1,\ldots k\}\to\{\pm 1\}} \sum_{r : r \leq \phi(q)} \frac{-1^{k(r)-k}}{k(r,q)} \Upsilon(\phi(q), q) x_r. \tag{3.1}$$

Notation: We will follow Garsia and Reutenauer's notation [4] and call

$$k(r, q) = \prod_{i=1}^{k} k(r_{(i)})$$

where $r = (r_{(1)} r_{(2)} \cdots r_{(k)})$ concatenated together, and $r_{(i)} \models |q_i|$. ($k(r, q)$ says we cut r as q dictates and then count the sizes of the pieces).

For $w \in \mathbb{Q}[A^*]$ and $\phi_i \in \{\pm 1\}$, we say

$$\overset{\phi_i}{w} = \begin{cases} w & \text{if } \phi_i = 1 \\ \overline{w} & \text{if } \phi_i = -1. \end{cases}$$

Note that $\overline{\overline{w}} = w$, so that if $\phi_i = -1$ then $\overset{\phi_i}{\overline{w}} = w$. We will also make use of the symbols ϵ_i which will generally denote the sign of some part of a composition, where as ϕ usually denotes a map that distributes bars.

We also introduce the following notation:

$$\Upsilon_i(\phi, q) = \begin{cases} -1 & \text{if } \phi_i < 0 \text{ and } q_i < 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\text{and } \Upsilon(\phi, q) = \prod_{i=1}^{k(q)} \Upsilon_i(\phi, q)$$

Proof We will now compute I_q in terms of x_r's using the fact that $\mathcal{I}_{(n)} = \sum_{r:r\leq(n)} \frac{-1^{k(r)-1}}{k(r)} x_r$. Let $q \models n$ where $k(q) = k$ and $\epsilon_i = \text{sgn}(q_i)$.

$$\begin{aligned} I_q &= \sum_{\substack{S_1+\cdots+S_k=\{1,\dots,n\} \\ |S_i|=|q_i|}} I^{\epsilon_1}_{[S_1]} I^{\epsilon_2}_{[S_2]} \cdots I^{\epsilon_k}_{[S_k]} \\ &= \sum_{\substack{S_1+\cdots+S_k=\{1,\dots,n\} \\ |S_i|=|q_i|}} S_1 \frac{(\mathcal{I}_{(|q_1|)} + \epsilon_1 \overline{\mathcal{I}}_{(|q_1|)})}{2} \cdots S_k \frac{(\mathcal{I}_{(|q_k|)} + \epsilon_k \overline{\mathcal{I}}_{(|q_k|)})}{2} \end{aligned}$$

$$= \sum_{\substack{S_1+\cdots+S_k=[n], |S_i|=|q_i| \\ r_{(i)}:r_{(i)}\leq(|q_i|)}} \frac{1}{2^k} \prod_{i=1}^{k} \frac{-1^{k(r_{(i)})-1}}{k(r_{(i)})} S_1(x_{r_{(1)}} + \epsilon_1 \overline{x}_{r_{(1)}}) \cdots S_k(x_{r_{(k)}} + \epsilon_k \overline{x}_{r_{(k)}})$$

Above, we used the notation $[n] = \{1, 2, \dots, n\}$ for brevity—well, actually, to make the equation fit on the page. We will make further use of this notation where necessary. Later, we will pull the S_i's out to the left of the expression, understanding that $x_{r_{(i)}}$ acts only on that piece of $S_1 S_2 \cdots S_k$. Note that $\overline{x_p} = x_{\overline{p}}$ so that we can get rid of the bars over the x_r's in the above expression. For example $\overline{x}_{(13\overline{2}21)} = x_{(\overline{13}2\overline{21})}$. Now we can simplify the product of terms that look like $x_{r_{(i)}} + \epsilon_i x_{\overline{r_{(i)}}}$ by using the ϕ on top notation we introduced above. Let ϕ be a map

$$\phi : \{1, 2, \dots k\} \to \{\pm 1\}$$

and let ϕ_i denote $\phi(i)$. From each of the k factors in the former product, we have a choice of whether to choose $x_{r_{(i)}}$ or $x_{\overline{r_{(i)}}}$; each series of choices corresponds to some ϕ, of which there are a total of 2^k. Hence $\prod_{i=1}^{k} (x_{r_{(i)}} + x_{\overline{r_{(i)}}}) = \sum_\phi x_{\overset{\phi_1}{r_{(1)}}} x_{\overset{\phi_2}{r_{(2)}}} \cdots x_{\overset{\phi_k}{r_{(k)}}}$.

Of course, we still have to deal with the factor of ϵ_i that is multiplied by $x_{\overline{r_{(i)}}}$. This is negative only if $\epsilon_i = -1$ (i.e. $q_i < 0$) and we actually chose the barred term out of the product, i.e. if $\phi_i = -1$ as well. To be more specific, what is really important is whether $\overset{\phi_i}{r_{(i)}}$ is barred (since later on we will have ϕ acting on signed compositions, not just positive ones). This information is encoded in the symbol defined in the beginning of this section $\Upsilon_i(\phi, q)$, because this is negative when both ϕ_i and q_i are. The product of these signs is $\Upsilon(\phi, q)$.

Now something really nice happens. The S_i's disappear, so to speak. $\sum S_1 \cdots S_k x_{r_{(1)}} \cdots x_{r_{(k)}} = 12 \cdots n x_{(r_{(1)} \cdots r_{(k)})} = x_r$. Why is this? Well, $x_{r_{(i)}}$ mixes up S_i but doesn't touch the other S_j. So we end up getting something that has the descent structure of $x_{r_{(i)}}$ on the S_i piece, and may or may not have descents leading in or out of this i^{th} chunk. As we sum over all such $S_1 + \cdots + S_k = \{1, \ldots, n\}$, we certainly get all rearrangements of $12 \cdots n$ that have this descent shape, i.e. the descent shape of $r = (r_{(1)} \cdots r_{(k)})$. The composition we call r is made by concatenating the $r_{(i)} \leq (|q_i|)$, so we get that r is a finer composition than $(|q_1| \cdots |q_k|)$.

By an even further abuse of the ϕ notation, we can call the composition we obtain by barring $\overset{\phi_1}{r}_{(1)} \overset{\phi_2}{r}_{(2)} \cdots \overset{\phi_k}{r}_{(k)}$ in that manner $\phi(r)$. Also, instead of considering functions of $\phi(r)$ where $r \leq q$, we can consider functions of r where $r \leq \phi(q)$. Now we are ready to further simplify the equations above using this new notation.

$$\begin{aligned} I_q &= \frac{1}{2^k} \sum_{\substack{S_1 + \cdots + S_k = [n] \\ |S_i| = |q_i|}} \sum_{\substack{r_{(i)} : r_{(i)} \leq (|q_i|) \\ \phi : \{1, \ldots k\} \to \{\pm 1\}}} \frac{-1^{(\sum k(r_{(i)}) - 1)}}{\prod k(r_{(i)})} \Upsilon(\phi, q) S_1 \cdots S_k x_{\overset{\phi_1}{r}_{(1)} \cdots \overset{\phi_k}{r}_{(k)}} \\ &= \frac{1}{2^k} \sum_{\substack{r = r_{(1)} \cdots r_{(k)} \\ r_{(i)} : r_{(i)} \leq (|q_i|)}} \frac{-1^{k(r) - k}}{k(r, q)} \sum_{\phi} \Upsilon(\phi, q) x_{\phi(r)} \\ &= \frac{1}{2^k} \sum_{\phi : \{1, \ldots k\} \to \{\pm 1\}} \sum_{r : r \leq \phi(q)} \frac{-1^{k(r) - k}}{k(r, q)} \Upsilon(\phi(q), q) x_r \end{aligned}$$

Note in the above expression, because we've changed the order of summation, we don't care about the sign of ϕ_i itself but about the sign of $\phi(q)_i$, where $\phi(q)$ is the composition obtained from q adding and deleting bars according to ϕ. So we care about the sign of $\Upsilon_i(\phi(q), q)$, which is negative only when both $\phi(q)_i$ and q_i are.

□

3.4 The Action of B_n on a Product of Signed Lie Polynomials

Now we would like to find out what $I_p x_q$ and $I_p I_q$ are. In order to do that, we first study the more general action of x_p.

For $Q \in Lie[A]$ we say Q is a homogeneous Lie polynomial if for some j, $Q \in Lie_j[A]$ and we call $j = \deg(Q)$ the degree of Q. For example, the degree of Θ_n is n. Given a collection of homogeneous Lie polynomials $Q_1, Q_2, \ldots, Q_m$ and $S = \{i_1 < i_2 < \cdots < i_r\} \subset \{1, 2, \ldots, m\}$ we will call $Q_S = Q_{i_1} Q_{i_2} \cdots Q_{i_r}$. By $\deg(Q_S)$ we mean $(\deg(Q_{i_1}) \deg(Q_{i_2}) \cdots \deg(Q_{i_r}))$ which is a composition of the sum of degrees of the Q_{i_j}. Furthermore, if $Q_i \in Lie^{\epsilon_i}[A]$, we can say $\operatorname{sgn}(Q_i) = \epsilon_i$. If this happens, we can call Q_i a signed Lie polynomial.

Lemma Let Q_i be homogeneous signed Lie polynomials for which $\epsilon_i = \operatorname{sgn}(Q_i)$ and $\sum_{i=1}^{m} \deg(Q_i) = n$. Let $p \models n$ where $k(p) = k$ and $\phi_i = \operatorname{sgn}(p_i)$. Then

$$Q_1 Q_2 \cdots Q_m x_p = \sum_{\substack{S_1+S_2+\cdots+S_k=\{1,2,\ldots,m\} \\ \deg(Q_{S_i}) \models |p_i|}} \Upsilon(p,\ \operatorname{sgn}(Q_S)) Q_{S_1} Q_{S_2} \cdots Q_{S_k}$$

Proof

$$\begin{aligned} Q_1 Q_2 \cdots Q_m x_p &= \sum_{u \in A^*} \langle Q_1 Q_2 \cdots Q_m x_p, u \rangle u \\ &= \sum_{u \in A^*} \langle Q_1 Q_2 \cdots Q_m, u x_p^* \rangle u \\ &= \sum_{\substack{u_1,\ldots,u_k \in A* \\ |u_i|=|p_i|, u_1 \cdots u_k = u}} \langle Q_1 \cdots Q_m, \overset{\phi_1}{u_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overset{\phi_k}{u_k} \rangle u \end{aligned}$$

Now remember our friend Δ. We can define an analogue of it $\Delta^{k-1} : \mathbb{Q}[A^*] \to \overline{\otimes_k} \mathbb{Q}[A^*]$, that sends for $a \in A$, $\Delta^{k-1} : a \mapsto \underbrace{a \otimes 1 \otimes \cdots \otimes 1}_{k} + 1 \otimes a \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a$. Also analogously to equation 2.4,

$$\Delta^{k-1}(f) = \sum_{u_1,\ldots,u_k \in A^*} \langle f, u_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup u_k \rangle u_1 \otimes \cdots \otimes u_k.$$

So $\langle Q_1 \cdots Q_m, \overset{\phi_1}{u_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overset{\phi_k}{u_k} \rangle$ is actually the coefficient of $\overset{\phi_1}{u_1} \otimes \cdots \otimes \overset{\phi_k}{u_k}$ in $\Delta^{k-1}(Q_1 \cdots Q_m)$.

Following the proof of Proposition 1, we also have that $Q \in Lie[A]$ if and only if $\Delta^{k-1}(Q) = Q \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes Q$. Since Δ^{k-1} extends algebraically, $\Delta^{k-1}(Q_1 \cdots Q_m) = \sum_{S_1+\cdots+S_k=[m]} Q_{S_1} \otimes \cdots \otimes Q_{S_k}$. This has a coefficient on $\overset{\phi_1}{u_1} \otimes \cdots \otimes \overset{\phi_k}{u_k}$ of $\prod \langle Q_{S_i}, \overset{\phi_i}{u_i} \rangle = \prod \langle \overset{\phi_i}{Q}_{S_i}, u_i \rangle$ when $\overset{\phi_i}{u_i}$ occurs in Q_{S_i}, so in particular, $\deg(Q_{S_i}) \models |u_i|$. Hence the coefficient

$$\begin{aligned} \langle Q_1 \cdots Q_m, \overset{\phi_1}{u_1} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \overset{\phi_k}{u_k} \rangle &= \sum_{\substack{S_1+\cdots+S_k=[m] \\ \deg(Q_{S_i}) \models |u_i|}} \langle \overset{\phi_1}{Q}_{S_1}, u_1 \rangle \cdots \langle \overset{\phi_k}{Q}_{S_k}, u_k \rangle \\ &= \sum_{\substack{S_1+\cdots+S_k=[m] \\ \deg(Q_{S_i}) \models |u_i|}} \langle \overset{\phi_1}{Q}_{S_1} \cdots \overset{\phi_k}{Q}_{S_k}, u_1 \cdots u_k \rangle \end{aligned}$$

So, to continue:

$$\begin{aligned}
Q_1 Q_2 \cdots Q_m x_p &= \sum_{u \in A^*} \sum_{\substack{S_1 + \cdots + S_k = \{1, \dots, m\} \\ \deg(Q_{S_i}) \models |p_i|}} \langle \overset{\phi_1}{Q}_{S_1} \overset{\phi_2}{Q}_{S_2} \cdots \overset{\phi_k}{Q}_{S_k}, u \rangle u \\
&= \sum_{\substack{S_1 + \cdots + S_k = \{1, \dots, m\} \\ \deg(Q_{S_i}) \models |p_i|}} \overset{\phi_1}{Q}_{S_1} \overset{\phi_2}{Q}_{S_2} \cdots \overset{\phi_k}{Q}_{S_k} \\
&= \sum_{\substack{S_1 + S_2 + \cdots + S_k = \{1, 2, \dots, m\} \\ \deg(Q_{S_i}) \models |p_i|}} \Upsilon(p,\ \operatorname{sgn}(Q_S)) Q_{S_1} Q_{S_2} \cdots Q_{S_k}
\end{aligned}$$

because $\overline{Q}_i = \epsilon_i Q_i$. Notice Q_{S_i} is signed (its sign is $\prod_{j \in S_i} \epsilon_j$). Hence, each time we remove a bar from atop a Q_{S_i}, it contributes a negative sign exactly when $\operatorname{sgn}(Q_{S_i}) < 0$. But it's only actually barred when $\phi_i = \operatorname{sgn}(p_i) < 0$. As i varies from 1 to k, this information is encoded in $\Upsilon(p,\ \operatorname{sgn}(Q_S))$.

$\square$

Using this lemma, we will be able to compute $I_p x_q$ and $I_p I_q$. In order to do this, we require some identities involving $\Upsilon(\phi, q)$. Proving these identities involves some lengthy computations. The reader may want to skip to the next section.

3.5 Identities Involving $\Upsilon(\phi, q)$

Claim Let $\phi, \psi : \{1, 2, \dots k(p)\} \to \{\pm 1\}$. Let $p \models n$. Then $\psi_i \neq \phi_i$ if and only if

$$\Upsilon_i(\phi(p), p) \Upsilon_i(\psi\phi(p), \phi(p)) = -\Upsilon_i(\psi(p), p) \Upsilon_i(\phi\psi(p), \psi(p)) \tag{3.2}$$

Proof Suppose $\phi_i \neq \psi_i$. Without loss of generality let $\phi(p)_i < 0$, hence $\psi(p)_i > 0$. Since ϕ and ψ differ on i, $\psi(\phi(p))_i = -p_i$. Then what is $\Upsilon_i(\phi(p), p) \cdot \Upsilon_i(\psi\phi(p), \phi(p))$? It is negative; we can think of it as $\Upsilon_i(-1, p)\Upsilon_i(\psi\phi(p), -1) = \Upsilon_i(-1, p)\Upsilon_i(-p, -1) = -1$. So the product on the left-hand side of equation 3.2 is negative, because p_i is either positive or negative. On the other hand, $\Upsilon_i(\psi(p), p)\Upsilon_i(\phi\psi(p), \psi(p)) = \Upsilon_i(+1, p)\Upsilon_i(\phi\psi(p), +1) = +1$ which is always positive. So each time ϕ and ψ differ, the signs of the expressions in equation 3.2 are different.

Suppose ϕ and ψ *agree* at the i^{th} place. So $\psi\phi(p)_i = p_i$. Then $\Upsilon_i(\phi(p), p)\Upsilon_i(\psi\phi(p), \phi(p)) = \Upsilon_i(\phi(p), p)\Upsilon_i(p, \phi(p)) = +1$ because the above terms are the same—and $(\pm 1)^2 = +1$. Likewise, $\Upsilon_i(\psi(p), p)\Upsilon_i(\phi\psi(p), \psi(p)) = +1$ when ϕ and ψ agree at the i^{th} place. So the signs of the expressions in equation 3.2 are the same.

Hence $\Upsilon_i(\phi(p), p)\Upsilon_i(\psi\phi(p), \phi(p)) = -\Upsilon_i(\psi(p), p)\Upsilon_i(\phi\psi(p), \psi(p))$ *exactly* when ϕ and ψ differ on i.

$\square$

Claim

$$\frac{1}{2^{k(p)}}\sum_{\phi,\psi}\Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p))x_{\psi\phi(p)} = x_p. \tag{3.3}$$

Proof Notice that ϕ and ψ commute and so $x_{\psi\phi(p)} = x_{\phi\psi(p)}$. First we will show that $\Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p)) = -\Upsilon(\psi(p),p)\Upsilon(\phi\psi(p),\psi(p))$ when $\phi_i \neq \psi_i$ for an odd number of i. This will simplify the above sum greatly.

Case 1: Suppose ϕ and ψ differ on an *odd* number of places. We can cancel all the terms out of the products where they agree (recall Υ is a product of Υ_i's). Then we are looking at two products of an odd number $(2m+1)$ of plus and minus ones. Suppose the first cancelled product contains j negative signs. Then by equation 3.2 the second cancelled product will contain $2m+1-j$ negative signs. But these two numbers are of opposite parity, so the entire products are of opposite sign, i.e. $\Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p)) = -\Upsilon(\psi(p),p)\Upsilon(\phi\psi(p),\psi(p))$. Hence the coefficient of $x_{\phi\psi(p)}$ is negative that of $x_{\psi\phi(p)}$, and equation 3.3 is reduced to $\frac{1}{2^{k(p)}}\sum_\phi \sum_{\substack{\psi \text{ that differ from } \phi \\ \text{on an even number of places}}} \Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p))x_{\psi\phi(p)}$.

Case 2: Suppose ϕ and ψ differ on an *even* number of places. Now instead of cancelling the $x_{\psi\phi(p)}$ term with the $x_{\phi\psi(p)}$ term, we construct ϕ' and ψ' to do the job. Assume $\phi \neq \psi$. Let j be the first place that ϕ and ψ differ on. Choose ϕ' to differ from ϕ only on j. So $\phi'(p)_j = -\phi(p)_j$ and for all $i \neq j, \phi'(p)_i = \phi(p)_i$. Let ψ' differ from ψ only in j as well. Note that ϕ' and ψ' differ from each other in an even number of places, so they occur in our slightly simplified sum above. Also, $\phi'(\psi'(p)) = \phi\psi(p)$, so $x_{\phi'(\psi'(p))} = x_{\phi\psi(p)}$, but

$$\Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p)) = -\Upsilon(\phi'(p),p)\Upsilon(\psi'\phi'(p),\phi'(p)).$$

Why is this? Well, the products are exactly the same except for the j^{th} term by the way ϕ' and ψ' were constructed. So all we have to do is compare $\Upsilon_j(\phi(p),p)\Upsilon_j(\psi\phi(p),\phi(p))$ to $\Upsilon_j(\phi'(p),p)\Upsilon_j(\psi'\phi'(p),\phi'(p))$. Now, since $\phi\psi(p)_j = -p_j$, $\Upsilon_j(\phi(p),p)\Upsilon_j(\psi\phi(p),\phi(p)) = \Upsilon_j(\phi(p),p)\Upsilon_j(-p,\phi(p)) = \text{sgn}(\phi(p)_j)$.

And $\Upsilon_j(\phi'(p),p)\Upsilon_j(\psi'\phi'(p),\phi'(p)) = \Upsilon_j(-\phi(p),p)\Upsilon_j(\psi\phi(p),-\phi(p)) = \Upsilon_j(-\phi(p),p)\Upsilon_j(-p,-\phi(p)) = \text{sgn}(-\phi(p)_j) = -\text{sgn}(\phi(p)_j)$. Since we constructed ϕ' and ψ' in a well-defined manner (i.e. we would use ϕ and ψ to cancel them out as well), every term in the above sum cancels except when $\phi = \psi$. This obviously happens $2^{k(p)}$ times, once for each ϕ. To sum up:

$$\frac{1}{2^{k(p)}}\sum_{\phi,\psi}\Upsilon(\phi(p),p)\Upsilon(\psi\phi(p),\phi(p))x_{\psi\phi(p)}$$

$$
\begin{aligned}
&= \frac{1}{2^{k(p)}} \sum_{\phi:\{1,\dots k(p)\}\to\{\pm 1\}} \Upsilon(\phi(p),p)\Upsilon(\phi\phi(p),\phi(p))x_{\phi\phi(p)} \\
&= \frac{1}{2^{k(p)}} \sum_{\phi:\{1,\dots k(p)\}\to\{\pm 1\}} \Upsilon(\phi(p),p)\Upsilon(p,\phi(p))x_p \\
&= \frac{1}{2^{k(p)}} \sum_{\phi:\{1,\dots k(p)\}\to\{\pm 1\}} (\pm 1)^2 x_p \\
&= x_p.
\end{aligned}
$$

□

Claim Let $p, q \models n, k(p) = k(q) = k$. Pick $\sigma \in \mathfrak{S}_k$. Let $p \cdot \sigma$ be the composition obtained from permuting the parts of p according to σ. Then

$$
\sum_{\phi} \Upsilon(\phi(q),q)\Upsilon(\phi(q),p\cdot\sigma) = \begin{cases} 2^k & \text{if } \forall i,\ \mathrm{sgn}(p_{\sigma_i}) = \mathrm{sgn}(q_i) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}
$$

Proof Suppose there exists a j for which $\mathrm{sgn}(p_{\sigma_j}) = -\mathrm{sgn}(q_j)$. As we sum over all ϕ, half of the time $\phi(q)_j$ is negative and hence $\Upsilon_j(\phi(q),q)\Upsilon_j(\phi(q),p\cdot\sigma) = \Upsilon_j(-1,q)\Upsilon_j(-1,-q) = -1$. The other half the time, $\phi(q)_j$ is positive which means $\Upsilon_j(\phi(q),q)\Upsilon_j(\phi(q),p\cdot\sigma) = +1$. Thus

$$
\begin{aligned}
&\sum_{\phi} \Upsilon(\phi(q),q)\Upsilon(\phi(q),p\cdot\sigma) \\
&\quad= \sum_{\phi:\phi(q)_j>0} \Upsilon_j(\phi(q),q)\Upsilon_j(\phi(q),p\cdot\sigma) \prod_{i:i\neq j} \Upsilon_i(\phi(q),q)\Upsilon_i(\phi(q),p\cdot\sigma) \\
&\qquad + \sum_{\phi:\phi(q)_j<0} \Upsilon_j(\phi(q),q)\Upsilon_j(\phi(q),p\cdot\sigma) \prod_{i:i\neq j} \Upsilon_i(\phi(q),q)\Upsilon_i(\phi(q),p\cdot\sigma) \\
&\quad= \sum_{\phi:\phi(q)_j>0} (+1) \prod_{i:i\neq j} \Upsilon_i(\phi(q),q)\Upsilon_i(\phi(q),p\cdot\sigma) \\
&\qquad + \sum_{\phi:\phi(q)_j<0} (-1) \prod_{i:i\neq j} \Upsilon_i(\phi(q),q)\Upsilon_i(\phi(q),p\cdot\sigma) \\
&\quad= 0
\end{aligned}
$$

because for each ϕ there exists a unique ϕ' agreeing with ϕ on all $i \neq j$ and differing exactly on j.

Now suppose $\mathrm{sgn}(p_{\sigma_i}) = \mathrm{sgn}(q_i)$ for all i. Then $\Upsilon_i(\phi(q),p\cdot\sigma) = \Upsilon_i(\phi(q),q)$ for all i. Hence

$$
\sum_{\phi} \Upsilon(\phi(q),q)\Upsilon(\phi(q),p\cdot\sigma) = \sum_{\phi} \Upsilon(\phi(q),q)\Upsilon(\phi(q),q) = \sum_{\phi} (\pm 1)^2 = 2^k.
$$

□

Equations 3.3 and 3.4 are really quite remarkable. Who would have thought all those awful signs would drop out so nicely?

3.6 Expressing x_p in Terms of I_q

We will put the work of section 3.5 to use in the following claim. Recall that, in keeping with the notation of Garsia and Reutenauer [4], by $k(q,p)$ we mean the product of how many parts of q (which is finer than p) match up to the parts of p. That is, q is a concatenation of $q_{(i)} \models p_i$ and $k(q,p) = \prod_{i=1}^{k(p)} k(q_{(i)})$. By $k!(q,p)$ we mean $k!(q,p) = \prod_{i=1}^{k(p)} k(q_{(i)})!$.

Claim

$$x_p = \sum_{q:q\leq p} \sum_{\phi:\{1,\ldots,k(q)\}\to\{\pm 1\}} \frac{1}{k!(q,p)} \Upsilon(\phi(q),q) I_{\phi(q)}$$

Proof

$$\sum_{q:q\leq p} \sum_{\phi:\{1,\ldots,k(q)\}\to\{\pm 1\}} \frac{1}{k!(q,p)} \Upsilon(\phi(q),q) I_{\phi(q)}$$

$$= \sum_{\substack{q:q\leq p \\ \phi}} \frac{1}{k!(q,p)} \Upsilon(\phi(q),q) \sum_{\substack{\psi:[k(q)]\to\{\pm 1\} \\ r:r\leq\psi(\phi(q))}} \frac{1}{2^{k(q)}} \frac{-1^{k(r)-k(q)}}{k(r,q)} \Upsilon(\psi\phi(q),\phi(q)) x_r$$

the above expression plugs in equation 3.1 and takes into account that $k(q) = k(\phi(q))$. Below, we switch the order of summation:

$$= \frac{1}{2^{k(q)}} \sum_{\substack{q:q\leq p \\ \phi}} \frac{1}{k!(q,p)} \Upsilon(\phi(q),q) \sum_{\substack{r:r\leq\phi(q) \\ \psi}} \frac{-1^{k(r)-k(q)}}{k(r,q)} \Upsilon(\psi\phi(q),\phi(q)) x_{\psi(r)}$$

$$= \frac{1}{2^{k(q)}} \sum_{\substack{q:q\leq p \\ r:r\leq q}} \sum_{\phi,\psi} \frac{1}{k!(q,p)} \frac{-1^{k(r)-k(q)}}{k(r,q)} \Upsilon(\phi(q),q) \Upsilon(\psi\phi(q),\phi(q)) x_{\psi\phi(r)}$$

$$= \frac{1}{2^{k(q)}} \sum_{\substack{r:r\models n \\ q:r\leq q\leq p}} \frac{1}{k!(q,p)} \frac{-1^{k(r)-k(q)}}{k(r,q)} \sum_{\phi,\psi} \Upsilon(\phi(q),q) \Upsilon(\psi\phi(q),\phi(q)) x_{\psi\phi(r)}$$

$$= \frac{1}{2^{k(q)}} \sum_{\substack{r:r\models n \\ q:r\leq q\leq p}} \frac{1}{k!(q,p)} \frac{-1^{k(r)-k(q)}}{k(r,q)} x_r$$

by equation 3.3.

$$= x_p$$

Note, the last step of the proof comes from [4]. According to Garsia and Reutenauer $\sum_{q:r\leq q\leq p} \frac{1}{k!(q,p)} \frac{-1^{k(r)-k(q)}}{k(r,q)} = 0$ unless $r = p$, in which case the sum is clearly 1. Also note that the above claim along with equation 3.1 show that $\{I_p\}_{p\models n}$ is a basis of ΩB_n.

□

3.7 Computing $I_p I_q$

Claim $I_p x_q = 0$ if $k(p) < k(q)$. If $k(p) = k(q)$, then

$$I_p x_q = \sum_{\substack{S_1+\cdots+S_k=\{1,\ldots,n\} \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ |p_{\sigma_i}|=|q_i|}} \Upsilon(q, p\cdot\sigma) I^{\epsilon_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon_{\sigma_k}}_{[S_{\sigma_k}]} \tag{3.5}$$

Proof Let $m = k(p)$, $\epsilon_i = \mathrm{sgn}(p_i)$ and $k = k(q), \phi_i = \mathrm{sgn}(q_i)$, and let $Q_i = I^{\epsilon_i}_{[S_i]}$. By the lemma of Section 3.4, then

$$\begin{aligned}
I_p x_q &= \sum_{\substack{S_1+S_2+\cdots+S_m=\{1,2,\ldots,n\} \\ |S_i|=|p_i|}} I^{\epsilon_1}_{[S_1]} I^{\epsilon_2}_{[S_2]} \cdots I^{\epsilon_m}_{[S_m]} x_q \\
&= \sum_{\substack{S_1+S_2+\cdots+S_m=\{1,2,\ldots,n\} \\ |S_i|=p_i}} Q_1 Q_2 \cdots Q_m x_q \\
&= \sum_{\substack{S_1+\cdots+S_m=\{1,\ldots,n\} \\ |S_i|=p_i}} \sum_{\substack{T_1+\cdots+T_k=\{1,\ldots,m\} \\ \deg(Q_{T_i})\models|q_i|}} \Upsilon(q,\ \mathrm{sgn}(Q_T)) Q_{T_1} Q_{T_2} \cdots Q_{T_k} \\
&= 0 \text{ if } m < k.
\end{aligned}$$

When $m < k$ there is no way to find T_i such that $T_1 + \cdots + T_k = \{1, 2, \ldots, m\}$, so the sum is empty and hence 0. But what if $m = k$? Then the only way to find $T_i \neq \emptyset$ such that $T_1 + T_2 + \cdots + T_k = \{1, 2, \ldots, k\}$ is to have $|T_i| = 1$. So each of these set decompositions corresponds to some permutation of $\{1, 2, \ldots, k\}$, i.e. to some $\sigma \in \mathfrak{S}_k$. Hence

$$I_p x_q = \sum_{\substack{S_1+\cdots+S_k=\{1,\ldots,n\} \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ \deg(Q_{\sigma_i})=|q_i|}} \Upsilon(q,\ \mathrm{sgn}(Q_\sigma)) Q_{\sigma_1} Q_{\sigma_2} \cdots Q_{\sigma_k}$$

resubstitute for the Q_i's and recall $\deg(I^{\epsilon_i}_{[S_i]}) = |S_i|$

$$= \sum_{\substack{S_1+\cdots+S_k=\{1,\ldots,n\} \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ |S_{\sigma_i}|=|q_i|}} \Upsilon(q,\ \mathrm{sgn}(I^{\epsilon_\sigma}_{[S_\sigma]})) I^{\epsilon_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon_{\sigma_k}}_{[S_{\sigma_k}]}$$

Recall $\mathrm{sgn}(I^{\epsilon_\sigma}_{[S_\sigma]}) = \epsilon_\sigma = \mathrm{sgn}(p\cdot\sigma)$.

$$= \sum_{\substack{S_1+\cdots+S_k=\{1,\ldots,n\} \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ |p_{\sigma_i}|=|q_i|}} \Upsilon(q, p\cdot\sigma) I^{\epsilon_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon_{\sigma_k}}_{[S_{\sigma_k}]} \quad \text{if } k(q) = k(p).$$

□

Definition Given a signed composition $p \models n$, define the symbol $\lambda(p)$ to be the composition obtained from p by rearranging p's positive parts in decreasing order then p's negative parts in decreasing order of absolute value. We can think of $\lambda(p) = (\lambda^+, \lambda^-)$ as a signed partition of n.

Example $\lambda(13\bar{2}2\bar{1}) = (321\overline{21})$

Notice there is a permutation $\sigma \in \mathfrak{S}_{k(p)}$ that acts on p to obtain $\lambda(p)$. We can talk about *Stab* p being those permutations that fix p. Note that $|Stab\ p| = |Stab\ \lambda(p)|$ because they are in the same orbit of this action.

Now we will finally compute what $I_p I_q$ is for certain p, q.

Lemma If $\lambda(p) = \lambda(q) = \lambda$ then

$$I_p I_q = |Stab\ \lambda| I_q. \tag{3.6}$$

Proof Let $k = k(q) = k(p)$. The first line of the next string of equations is a consequence of equation 3.1, and the next two lines come from equation 3.5.

$$\begin{aligned}
I_p I_q &= I_p \left(\sum_{\phi:[k]\to\{\pm 1\}} \frac{1}{2^k} \Upsilon(\phi(q), q) x_{\phi(q)} + \text{coefficients} \cdot x_{\substack{\text{compositions} \\ \text{with } > k \text{ parts}}} \right) \\
&= \frac{1}{2^k} \sum_{\phi} \Upsilon(\phi(q), q) I_p x_{\phi(q)} \\
&= \frac{1}{2^k} \sum_{\phi} \Upsilon(\phi(q), q) \sum_{\substack{S_1+\cdots+S_k=[n] \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ |p_{\sigma_i}|=|q_i|}} \Upsilon(\phi(q), p\cdot\sigma) I^{\epsilon^p_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon^p_{\sigma_k}}_{[S_{\sigma_k}]} \\
&= \frac{1}{2^k} \sum_{\substack{S_1+\cdots+S_k=[n] \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S} \\ |p_{\sigma_i}|=|q_i|}} I^{\epsilon^p_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon^p_{\sigma_k}}_{[S_{\sigma_k}]} \sum_{\phi} \Upsilon(\phi(q), q) \Upsilon(\phi(q), p\cdot\sigma)
\end{aligned}$$

Recalling the remarkable equation 3.4 :

$$= \sum_{\substack{S_1+\cdots+S_k=\{1,\ldots,n\} \\ |S_i|=|p_i|}} \sum_{\substack{\sigma\in\mathfrak{S}_k \\ p_{\sigma_i}=q_i}} I^{\epsilon^p_{\sigma_1}}_{[S_{\sigma_1}]} \cdots I^{\epsilon^p_{\sigma_k}}_{[S_{\sigma_k}]}$$

We emphasized that the ϵ_i denote the sign of p_i by superscripting with a p, because we want to make the substitution $\epsilon^p_{\sigma_i} = \epsilon^q_i$ (which comes from equation 3.4: $\text{sgn}(p_{\sigma_i}) = \text{sgn}(q_i)$ for all i).

How many such $\sigma \in \mathfrak{S}_k$ are there such that $p \cdot \sigma = q$? Since q lies in the orbit of p under the action of $\mathfrak{S}_k$, it is clear there are $|Stab\ p| = |Stab\ \lambda|$ such σ. Additionally, summing over the S_i is the same as if we relabel and sum over S_{σ_i}. And the condition $|S_i| = |p_i|$ is equivalent to $|S_{\sigma_i}| = |p_{\sigma_i}|$ which happens to be equal to $|q_i|$. Now we can complete our computation.

$$\begin{aligned} I_pI_q &= \sum_{\substack{\sigma\in\mathfrak{S}_k \\ p_{\sigma_i}=q_i}} \sum_{\substack{S_{\sigma_1}+\cdots+S_{\sigma_k}=\{1,\dots,n\} \\ |S_{\sigma_i}|=|q_i|}} I^{\epsilon^p_{\sigma_1}}_{[S_{\sigma_1}]}\cdots I^{\epsilon^p_{\sigma_k}}_{[S_{\sigma_k}]} \\ &= |Stab\lambda| \sum_{\substack{S_{\sigma_1}+\cdots+S_{\sigma_k}=\{1,\dots,n\} \\ |S_{\sigma_i}|=|q_i|}} I^{\epsilon^p_{\sigma_1}}_{[S_{\sigma_1}]}\cdots I^{\epsilon^p_{\sigma_k}}_{[S_{\sigma_k}]} \end{aligned}$$

relabelling S_{σ_i} by T_i and substituting $\epsilon^p_{\sigma_i}=\epsilon^q_i$ yields:

$$\begin{aligned} &= |Stab\lambda| \sum_{\substack{T_1+\cdots+T_k=\{1,\dots,n\} \\ |T_i|=|q_i|}} I^{\epsilon^q_1}_{[T_1]}\cdots I^{\epsilon^q_k}_{[T_{\sigma_k}]} \\ &= |Stab\lambda| I_q \end{aligned}$$

□

Hence we have generated more idempotents, namely $\frac{1}{|Stabp|}I_p$. Remark: The argument that keeps track of the signs in Section 3.5 shows that $I_pI_q=0$ if there exists *no* such $\sigma\in\mathfrak{S}_k$ such that for all i, $p_{\sigma_i}=q_i$, but there is a σ such that $|p_{\sigma_i}|=|q_i|$.

The next idempotents we will construct from the above ones form a complete family of minimal orthogonal idempotents. They are really the crux of the paper, because they are elements of ΩB_n and they decompose $\mathbb{Q}[B_n]$ as completely as possible in a nice way.

Chapter 4

A Complete Family of Minimal Orthogonal Idempotents

Definition Let λ by a signed partition of n. Then we define

$$E_\lambda = \frac{1}{k!} \sum_{p:\lambda(p)=\lambda} I_p.$$

The goal of this chapter is to prove the following theorem. (see [4])

Theorem 1 $\{E_\lambda\}_{\lambda \vdash n}$ *are a complete family of minimal orthogonal idempotents.*

Claim E_λ is an idempotent.

Proof It is quite easy to show the E_λ's are idempotents using equation 3.6.

$$\begin{aligned} E_\lambda E_\lambda &= \frac{1}{k!} \sum_{p:\lambda(p)=\lambda} I_p \frac{1}{k!} \sum_{q:\lambda(q)=\lambda} I_q = \frac{1}{k!k!} \sum_{\substack{p:\lambda(p)=\lambda \\ q:\lambda(q)=\lambda}} I_p I_q \\ &= \frac{1}{k!k!} \sum_{\substack{p:\lambda(p)=\lambda \\ q:\lambda(q)=\lambda}} |Stab\lambda| I_q \end{aligned}$$

How many p are there such that $\lambda(p) = \lambda$? As many as there are in the orbit of λ under the action of $\mathfrak{S}_k$, which we denote $|Orb\lambda|$.

$$
\begin{aligned}
&= \frac{1}{k!k!}\sum_{q:\lambda(q)=\lambda}|Orb\lambda||Stab\lambda|I_q = \frac{1}{k!k!}\sum_{q:\lambda(q)=\lambda}|\mathfrak{S}_k|I_q \\
&= \frac{1}{k!}\sum_{q:\lambda(q)=\lambda} I_q \\
&= E_\lambda
\end{aligned}
$$

□

While we're at it, notice if $\lambda(q) = \lambda$ then $E_{\lambda(q)}I_q = \frac{1}{k!}\sum_{p:\lambda(p)=\lambda} I_p I_q = \frac{1}{k!}\sum_{p:\lambda(p)=\lambda}|Stab\lambda|I_q = \frac{1}{k!}|Orb\lambda||Stab\lambda|I_q = I_q$, so

$$E_{\lambda(q)}I_q = I_q \tag{4.1}$$

4.1 $\sum_{\lambda:\lambda\vdash n} E_\lambda = Id$

In order to show that the E_λ are a complete family of idempotents, we need the following lemma.

Lemma

$$\sum_{\lambda:\lambda\vdash n} E_\lambda = Id$$

Proof We need to return to our work with $1 + D = \sum_{w\in A^*} w\otimes w = \exp(\Phi)$ from Section 3.1.

$$
\begin{aligned}
&\sum_{w\in A^*} w\otimes w \\
&\quad= \exp\left(\sum_{w\in A^*} w\otimes \mathcal{I}_{[w]}\right) \\
&\quad= \sum_{k\geq 0}\frac{\left(\sum_{w\in A^*} w\otimes\mathcal{I}_{[w]}\right)^k}{k!} \\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots u_k\in A^*} u_1 ш u_2 ш \cdots ш u_k \otimes \mathcal{I}_{[u_1]}\mathcal{I}_{[u_2]}\cdots\mathcal{I}_{[u_k]} \\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{u_1,\ldots u_k\in A^*}\sum_{w\in A^*}\langle w, u_1 ш \cdots ш u_k\rangle w\otimes\mathcal{I}_{[u_1]}\cdots\mathcal{I}_{[u_k]} \\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes\left(\sum_{u_1,\ldots u_k\in A^*}\langle w, u_1 ш \cdots ш u_k\rangle\mathcal{I}_{[u_1]}\cdots\mathcal{I}_{[u_k]}\right)
\end{aligned}
$$

Let $n = |w|$. Then w occurs in $u_1 ш \cdots ш u_k$ only if $|u_1|+\cdots|u_k| = n$ and each u_i is a subword of w. In other words $w_{S_i} = u_i$ and $S_1 + S_2 + \cdots + S_k = \{1,\ldots n\}$, where $|S_i| = |u_i|$. As we sum over all such set decompositions and all such possible subwords, we generate exactly those shuffles that w occurs in. So we can rewrite the sum

$$\sum_{w\in A^*} w\otimes w = \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes\left(\sum_{u_1,\dots u_k\in A^*}\sum_{\substack{S_1+S_2+\cdots+S_k=\{1,\dots n\}\\ w_{S_i}=u_i}} \mathcal{I}_{[u_1]}\cdots\mathcal{I}_{[u_k]}\right)$$

Since $w_{S_i} = u_i$, we can view $\mathcal{I}_{[u_1]}\cdots\mathcal{I}_{[u_k]}$ as w acted upon by $\mathcal{I}_{[S_1]}\cdots\mathcal{I}_{[S_k]}$. So the expression becomes

$$\sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes\left(\sum_{u_1,\dots u_k\in A^*}\sum_{\substack{S_1+S_2+\cdots+S_k=\{1,\dots n\}\\ |S_i|=|u_i|}} w\mathcal{I}_{[S_1]}\cdots\mathcal{I}_{[S_k]}\right).$$

Now our sum doesn't really involve any u_i's anymore, because as we sum over S_i's they tell us that the u_i's must be of the form w_{S_i}'s. The only thing we care about are the varying possible lengths of the u_i's. But all the different lengths of the $u_1,\dots,u_k$ correspond to all the different positive compositions of $n = |w|$ with k parts. Continuing:

$$\begin{aligned}
&\sum_{w\in A^*} w\otimes w\\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes w \sum_{\substack{p:p\models n,\ \text{positive}, k(p)=k\\ S_1+\cdots+S_k=\{1,\dots n\}:|S_i|=p_i}} \mathcal{I}_{[S_1]}\cdots\mathcal{I}_{[S_k]}\\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes w \sum_{\substack{p:p\leq(n), k(p)=k\\ S_1+\cdots+S_k=[n]:|S_i|=p_i}} (I^+_{[S_1]}+I^-_{[S_1]})\cdots(I^+_{[S_k]}+I^-_{[S_k]})\\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes w \sum_{\substack{p:p\leq(n), k(p)=k\\ S_1+\cdots+S_k=\{1,\dots n\}:|S_i|=p_i}}\ \sum_{\epsilon:\{1,\dots,k\}\to\{\pm 1\}} I^{\epsilon_1}_{[S_1]}\cdots I^{\epsilon_k}_{[S_k]}\\
&\quad= \sum_{k\geq 0}\frac{1}{k!}\sum_{w\in A^*} w\otimes w \sum_{\substack{p:p\models n,\text{signed}\\ k(p)=k}}\ \sum_{\substack{S_1+\cdots+S_k=\{1,\dots n\}\\ |S_i|=|p_i|}} I^{\epsilon_1}_{[S_1]}\cdots I^{\epsilon_k}_{[S_k]}
\end{aligned}$$

where ϵ_i is now $\mathrm{sgn}(p_i)$ since we encorporated the signs into the positive compositions

$$
\begin{aligned}
&= \sum_{w\in A^*} w\otimes w \sum_{k\geq 0}\frac{1}{k!}\sum_{\substack{p:p\models n\\ k(p)=k}} I_p \\
&= \sum_{w\in A^*} w\otimes w \sum_{k\geq 0}\sum_{\substack{\lambda:\lambda\vdash n\\ k(\lambda)=k}}\frac{1}{k!}\sum_{p:\lambda(p)=\lambda} I_p \\
&= \sum_{w\in A^*} w\otimes w \sum_{k\geq 0}\sum_{\substack{\lambda:\lambda\vdash n\\ k(\lambda)=k}} E_\lambda \\
&= \sum_{w\in A^*} w\otimes w \left(\sum_{\lambda:\lambda\vdash n} E_\lambda\right)
\end{aligned}
$$

All of the above implies that $w = w\sum_{\lambda:\lambda\vdash|w|} E_\lambda$ and hence

$$\sum_{\lambda:\lambda\vdash n} E_\lambda = Id,$$

the identity of ΩB_n.

$\square$

4.2 Orthogonality

The $E_\lambda \in \Omega B_n$ act on $\mathbb{Q}[A^*]$. Since the E_λ are complete idempotents, they project onto subspaces of $\mathbb{Q}[A^*]$, all of which sum to $\mathbb{Q}[A^*]$ itself. In fact, we will show that that sum is a direct one by examining the spaces that the E_λ project to more closely.

Definition Given $Q_i \in Lie_{k_i}[A]$, if $\deg(Q_i) = |q_i|$ and $\operatorname{sgn}(Q_i) = \operatorname{sgn}(q_i)$, we say the Lie polynomial $Q_1Q_2\cdots Q_m$ is of *type* $q = (q_1q_2\cdots q_m)$. We call

$$\frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m} Q_{\sigma_1}\cdots Q_{\sigma_m}$$

the *symmetrized product* of $Q_1\cdots Q_m$ and we say it has *shape* λ if $\lambda(q)=\lambda$.

The Poincare-Birkhoff-Witt Theorem implies that the enveloping algebra on $Lie[A]$ is $\mathbb{Q}[A^*]$ and hence that these symmetrized products span $\mathbb{Q}[A^*]$. (see [3]) One important result of this is that knowing how ΩB_n acts on products of signed Lie polynomials tells us how it acts on all of $\mathbb{Q}[A^*]$.

Definition Given a signed partition $\lambda\vdash n$, let HS_λ be the subspace of $\mathbb{Q}[A^*]$ spanned by symmetrized products of shape λ.

Recall we know $Lie[A] = Lie^+[A]\oplus Lie^-[A]$. Hence we can find a basis of

$Lie[A]$ that looks like $\{P_1^+, P_2^+, \dots\} \cup \{P_1^-, P_2^-, \dots\}$, where $P_i^{\epsilon_i}$ are signed Lie polynomials of sign ϵ_i. It follows from the Poincare-Birkhoff-Witt Theorem that {symmetrized products of $P_{i_1}^{\epsilon_1} P_{i_2}^{\epsilon_2} \cdots P_{i_n}^{\epsilon_n}$} where $i_1 \geq i_2 \geq \cdots \geq i_n$, and if $i_j = i_{j+1}$ then $\epsilon_j > \epsilon_j$, is a basis of $\mathbb{Q}[A]$. (see [3], [4]) So another result of the Poincare-Birkhoff-Witt Theorem is $\mathbb{Q}[A^*] = \oplus HS_\lambda$, or more specifically that

$$\mathbb{Q}[A^n] = \oplus_{\lambda \vdash n} HS_\lambda.$$

Claim $E_\lambda \in HS_\lambda$

Proof Let $\lambda \vdash n, k = k(\lambda)$. Recall that if $\lambda(p) = \lambda$ then under the action of $\mathfrak{S}_k$ we can get from λ to p.

$$\begin{aligned}
E_\lambda &= \frac{1}{k!} \sum_{p:\lambda(p)=\lambda} I_p \\
&= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{|Stab\lambda|} I_{\lambda \cdot \sigma} \\
&= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \frac{1}{|Stab\lambda|} \sum_{\substack{S_1 + \cdots + S_k = \{1, \dots, n\} \\ |S_i| = |\lambda_{\sigma_i}|}} I_{[S_1]}^{\epsilon_{\sigma_1}} \cdots I_{[S_k]}^{\epsilon_{\sigma_k}} \\
&= \frac{1}{k!} \frac{1}{|Stab\lambda|} \sum_{\sigma \in \mathfrak{S}_k} \sum_{\substack{S_{\sigma_1} + \cdots + S_{\sigma_k} = \{1, \dots, n\} \\ |S_{\sigma_i}| = |\lambda_{\sigma_i}|}} I_{[S_{\sigma_1}]}^{\epsilon_{\sigma_1}} \cdots I_{[S_{\sigma_k}]}^{\epsilon_{\sigma_k}} \\
&= \frac{1}{|Stab\lambda|} \sum_{\substack{S_1 + \cdots + S_k = \{1, \dots, n\} \\ |S_i| = |\lambda_i|}} \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} I_{[S_{\sigma_1}]}^{\epsilon_{\sigma_1}} \cdots I_{[S_{\sigma_k}]}^{\epsilon_{\sigma_k}} \\
&= \frac{1}{|Stab\lambda|} \sum_{\substack{S_1 + \cdots + S_k = \{1, \dots, n\} \\ |S_i| = |\lambda_i|}} \text{something symmetrized} \in HS_\lambda
\end{aligned}$$

□

Hence $E_\lambda \in HS_\lambda$. Clearly the right action by E_λ symmetrizes, i.e. for $w \in \mathbb{Q}[A^n], wE_\lambda \in HS_\lambda$. So E_λ projects into HS_λ, i.e. $\mathbb{Q}[A^n]E_\lambda \subset HS_\lambda$. But since we know they are complete, $\sum_{\lambda \vdash n} E_\lambda = 12 \cdots n = Id$, we know

$$\mathbb{Q}[A^n] = \mathbb{Q}[A^n] \sum_{\lambda \vdash n} E_\lambda = \sum_{\lambda \vdash n} \mathbb{Q}[A^n] E_\lambda = \oplus_{\lambda \vdash n} HS_\lambda.$$

This tells us that E_λ projects to all of HS_λ. In fact, this gives us that the E_λ are orthogonal. Since they are projection maps, E_λ decomposes as

$$E_\lambda = \sum_{\mu \vdash n} E_\lambda E_\mu$$

with $E_\lambda E_\mu \in HS_\mu$. Since our sum above is a direct sum, this decomposition is unique. We already know $E_\lambda E_\lambda = E_\lambda$, so for $\mu \neq \lambda$, $E_\lambda E_\mu = 0$. In other words, (using the Kroenecker delta) we have proven the following lemma.

Lemma

$$E_\lambda E_\mu = \delta_{\lambda,\mu} E_\lambda.$$

□

4.3 Minimality

Furthermore, these E_λ are a *minimal* family of complete orthogonal idempotents.

Lemma The E_λ are minimal with respect to being complete and orthogonal.

Suppose they were not. Then we could find non-zero E' and E'' such that for some $\lambda, E_\lambda = E' + E'', E'E'' = 0$, and $E'E_\mu = E''E_\mu = 0$ for $\mu \neq \lambda$. Then the right ideal decomposes as $E_\lambda \Omega B_n = E'\Omega B_n + E''\Omega B_n$. But don't forget, we can also think of this as a vector space over $\mathbb{Q}$, so we can write the sum as $V = V_1 \oplus V_2$. Since E', E'' are non-zero, V_1, V_2 are also non-zero. Recall that one basis for ΩB_n is $\{I_p\}_{p \models n}$. So $\{E_\lambda I_p\}_{p \models n}$ contains a basis for the ideal $E_\lambda \Omega B_n$. Recall we showed in equation 4.1 that $E_{\lambda(p)} I_p = I_p$. Hence

$$E_\lambda I_p = E_\lambda E_{\lambda(p)} I_p = \begin{cases} I_p & \text{if } \lambda(p) = \lambda \\ 0 & \text{if } \lambda(p) \neq \lambda. \end{cases}$$

This gives us that a basis for our ideal/vector space V is $\{I_p\}_{\lambda(p)=\lambda}$. For $f \in V_i$, we know we can write $f = \sum_{\lambda(p)=\lambda} c_p I_p$ in terms of our basis for V. Pick an $f \in V_i$ such that $\sum_{\lambda(p)=\lambda} c_p = c \neq 0$. We know such an f exists since such linear combinations are in V. Without loss of generality, $f \in V_1$. Using this f, we will show V_1 contains all of $\{I_p\}_{\lambda(p)=\lambda}$ and hence is equal to V. Pick any q such that $\lambda(q) = \lambda$. Then

$$fI_q = \sum_{\lambda(p)=\lambda} c_p I_p I_q = \sum_{\lambda(p)=\lambda} c_p |Stab\lambda| I_q = |Stab\lambda| I_q \sum_{\lambda(p)=\lambda} c_p = c|Stab\lambda| I_q.$$

Since V_1 is an ideal, $fI_q \in V_1$ which implies $I_q \in V_1$. Thus $V_1 = V$ and hence $V_2 = 0$, which is a contradiction. Because we cannot break down the E_λ into smaller orthogonal pieces, we say they are minimal.

□

Thus we have completed the proof of Theorem 1, too.

4.4 Conclusion

4.4.1 A Commutative Subalgebra

Finding the $\{E_\lambda\}_{\lambda\vdash n}$ is nice in and of itself, because they break down $\mathbb{Q}[A^n]$ (with $A = \{a_1, a_2, \dots, a_f, \overline{a}_1, \overline{a}_2, \dots, \overline{a}_f\}$) into the smallest possible really "nice" pieces. Because they are orthogonal, multiplication in $\text{span}\{E_\lambda\}_{\lambda\vdash n}$ is quite easy to compute. Clearly the span of $\{E_\lambda\}_{\lambda\vdash n}$ forms a commutative subalgebra of ΩB_n. But we don't have a very intuitive picture of what the E_λ's are. We would like another basis of $\text{span}\{E_\lambda\}_{\lambda\vdash n}$ for which we do have a more intuitive feel. We might want to ask what is $\text{span}\{E_\lambda\}_{\lambda\vdash n}$ in terms of shuffles (or anti-shuffles)? One way to answer this question is to find an alternate basis for $\text{span}\{E_\lambda\}_{\lambda\vdash n}$ that is a nicer expression in terms of x_p's. Or we could do this for some subalgebra of $\text{span}\{E_\lambda\}_{\lambda\vdash n}$.

For instance, analogous results have been found for Solomon's descent algebra, $\Sigma A_{n-1} = \text{span}\{x_p\}_{p\leq(n)} \subset \mathbb{Q}[\mathfrak{S}_n]$ (since no bars are involved) in [1], [3], and [4]. Consider $\text{span}\{\mathcal{Y}_\langle\}_{\langle=\infty}^{\backslash} \subset \Sigma\mathcal{A}_{\backslash-\infty}$ where

$$\mathcal{Y}_\langle = \sum_{\substack{\|(\surd)=\langle \\ \surd\leq(\backslash)}} \dagger_\surd.$$

(Recall the definition of y_p from Section 2.1.) These $\mathcal{Y}_\langle$ are expressible in terms of the analogous E_λ^Σ so multiplication in $\text{span}\{\mathcal{Y}_\langle\}_{\langle=\infty}^{\backslash}$ is commutative and easy to compute.

$$\sum_{h=1}^{n} \binom{z-h+n}{n} \mathcal{Y}_\langle = \sum_{\langle=\infty}^{\backslash} \Big(\sum_{\substack{\lambda\vdash\backslash,\ \text{positive} \\ \|(\lambda)=\langle}} \mathcal{E}_\lambda^\Sigma \Big) \ddagger^\langle = \sum_{\langle=\infty}^{\backslash} \Big(\sum_{\substack{\lambda\vdash\backslash \\ \|(\lambda)=\langle}} \mathcal{E}_\lambda \Big) \ddagger^\langle$$

The first equality comes from [1], [3], [4]. Why is the second equality true? In [4], they defined for $\lambda \vdash n$ a positive partition,

$$E_\lambda^\Sigma = \frac{1}{k!} \sum_{\lambda(p)=\lambda} \mathcal{I}_p \text{ where } \mathcal{I}_p = \sum_{\substack{S_1+\cdots+S_k=[n] \\ |S_i|=p_i}} \mathcal{I}_{[S_1]} \cdots \mathcal{I}_{[S_k]}.$$

Recall that $\mathcal{I}_{(n)} = I^+_{(n)} + I^-_{(n)}$, so $\mathcal{I}_p = \sum_{\phi:[k(p)]\to\{\pm1\}} \sum_{\substack{S_1+\cdots+S_k=[n] \\ |S_i|=p_i}} I^{\phi_1}_{[S_1]} \cdots I^{\phi_k}_{[S_k]} = \sum_\phi I_{\phi(p)}$. Hence, $\sum_{\substack{k(\lambda)=k \\ \lambda \text{ positive}}} E_\lambda^\Sigma = \frac{1}{k!} \sum_{\substack{k(\lambda)=k \\ \lambda \text{ positive}}} \sum_{\lambda(p)=\lambda} \sum_\phi I_{\phi(p)} = \frac{1}{k!} \sum_{\substack{k(\lambda)=k \\ \lambda \text{ signed}}} \sum_{\lambda(p)=\lambda} I_p = \sum_{\substack{k(\lambda)=k \\ \lambda \text{ signed}}} E_\lambda$.

So, in fact, $\text{span}\{\mathcal{Y}_\langle\} \subset \text{span}\{\mathcal{E}_\lambda\}$ and we do have a nice commutative subalgebra of ΩB_n. The $\mathcal{Y}_\langle$ model particular shuffles of a deck of n cards. One may want to know what happens when such a shuffle is iterated m times. This is rather difficult to compute directly, but if we change basis to express the problem in terms of E_λ, finding an answer is quite easy. Of course, $\text{span}\{\mathcal{Y}_\langle\}^\backslash_{\langle=\infty} \subset \Sigma\mathcal{A}_{\backslash-\infty} \subset \Omega\mathcal{B}_\backslash$, but we would hope to find a larger subalgebra of $\text{span}\{E_\lambda\}_{\lambda\vdash n}$ than this.

4.4.2 Conjecture

Although I was unable to prove it as yet, I propose that $\text{span}\{\mathcal{Y}^\epsilon_\langle\}^\backslash_{\langle=\infty}$ form a commutative subalgebra of ΩB_n as well, where

$$\mathcal{Y}^+_\langle = \sum_{\substack{\surd:\Vert(\surd)=\langle \\ \surd^\infty>'}} \dagger_\surd, \qquad \mathcal{Y}^-_\langle = \sum_{\substack{\surd:\Vert(\surd)=\langle \\ \surd^\infty<'}} \dagger_\surd.$$

If one wanted to develop the study of ΩB_n further, this is one direction s/he could take—it certainly warrants further research. Another avenue might be to develop such idempotents for the other generalizations of Solomon's algebra that are discussed in [5].

4.4.3 A Word on the Radical

In [4] it is shown that for A an algebra of $m \times m$ matrices, the radical

$$\sqrt{A} = \{a \in A : \text{ trace } ax = 0 \text{ for all } x \in A\}. \tag{4.2}$$

We can identify elements of ΩB_n with their images under the left regular representation of $\mathbb{Q}[B_n]$. Following steps taken in [4], (since $\tau\sigma = \sigma$ if and only if $\tau = Id$, for $\tau, \sigma \in B_n$) for $f \in \mathbb{Q}[B_n]$,

$$\text{trace } f = \sum_{\sigma \in B_n} \langle f\sigma, \sigma\rangle = 2^n n! \langle f,\ Id\rangle$$

if we think of $f \in \mathbb{Q}[B_n]$ as words in an alphabet $A = \{1, \ldots, n, \overline{1}, \ldots, \overline{n}\}$. Note that Id occurs exactly once in each x_p, so trace $x_p = 2^n n!$. One nice result of this is trace $(x_p - x_q) = 0$.

What is trace $(x_q - x_{\lambda(q)})x_r$? $(x_q - x_{\lambda(q)})x_r = \sum_{M:\substack{r(M)=r \\ c(M)=q}} x_{w(M)} - \sum_{N:\substack{r(N)=r \\ c(N)=q}} x_{w(N)}$. But $\lambda(q) = q \cdot \sigma$ for some $\sigma \in \mathfrak{S}_{k(q)}$. If $N = (n_{ij})$, let $N^\sigma = (n_{i\sigma_j})$. Then if N is such that $r(N) = r, c(N) = \lambda(q)$, we see $r(N^\sigma) = r, c(N^\sigma) = q$. So

$$\text{trace } (x_q - x_{\lambda(q)})x_r = \sum_{M:\substack{r(M)=r \\ c(M)=q}} \text{trace } (x_{w(M)} - x_{w(M^{\sigma^{-1}})}) = 0.$$

Now we can prove the following claim.

Claim

$$\sqrt{\Omega B_n} = \text{ span}\{x_q - x_{\lambda(q)}\}_{q \models n}$$

Proof Recall that by equation 4.1 it must follow that $E_\lambda \notin \sqrt{\Omega B_n}$. Also by equation 4.2, $x_q - x_{\lambda(q)} \in \sqrt{\Omega B_n}$. The E_λ are clearly independent, and so are the $x_q - x_{\lambda(q)}$. By a dimension argument our claim clearly follows.

□

Furthermore, we have also shown that

$$\text{span}\{E_\lambda\}_{\lambda \vdash n} \cong \Omega B_n / \sqrt{\Omega B_n}$$

so that span$\{E_\lambda\}$ is semi-simple and ΩB_n is not.

Recall that a major goal of this paper was to find the E_λ's and exhibit their properties. Not only have we used them to decompose $\mathbb{Q}[A^*]$, but we have also found some nice combinatorial results concerning their span. Our work with the radical has shown another remarkable fact about the E_λ. All this serves to justify (to the author at any rate) that it was a worthy undertaking to find them in the course of this paper.

Bibliography

[1] Bergeron, Nantel. Lectures in Algebraic Combinatorics, Math 208: Descent Algebras of Coxeter Groups. Harvard University, Cambridge, MA 02138, Sept.-Dec. 1992.

[2] Bergeron, F., Bergeron, N., and Garsia, A. M., *Idempotents for the Free Lie Algebra and q-Enumeration* IMA Vol. Math. Appl., 19, in "Collection: Invariant theory and tableaux" (Minneapolis, MN, 1988) 166-190.

[3] Garsia, Adriano M. *Combinatorics of the Free Lie Algebra and the Symmetric Group* in Collection: Analysis, et cetera, 1990, 309-382.

[4] Garsia, A. M. , Reutenauer, C. *A decomposition of Solomon's Descent Algebra* Advances in Mathematics 77, (1989) No 2, 189-262.

[5] Mantaci, R., Reutenauer, C. *A Generalization of Soloman's Algebra for Hyperoctahedral Groups and other Wreath Products.* Technical report from Institut Blaise Pascal, LITP 92.28 March 1992.