Solutions

20. Recall that a nonempty subset N of an R-module M is a submodule if and only if, for every $x, y \in N$ and every $r \in R$, we have that $x + ry \in N$.

    So let $x, y \in N = \cup_{i=1}^{\infty} N_i$ and $r \in R$ be given. Then, for some $i, j \in \mathbb{N}$ with $i \leq j$, we have that $x \in N_i$ and $y \in N_j$. Hence, $x, y \in N_j$. Since N_j is given to be a submodule, we have that $x + ry \in N_j$. Therefore, $x + ry \in N$, completing the proof.

21. Suppose, to get a contradiction, that M is free over the set $\{a_1, \ldots, a_n\} \subset M$. First, we claim that $n = 1$. For otherwise we have that

    $$-a_2 \cdot a_1 + a_1 \cdot a_2 = 0 \cdot a_1 + 0 \cdot a_2.$$

    But then we have that $M = \langle a_1 \rangle$, which is impossible because $\langle 2, x \rangle$ is not principle.

22. The main computational tool to use in problems like this is the Chinese Remainder Theorem (see Exercises 10.3.16 and 10.3.17 in Dummit & Foote). In the particular case of quotient rings of $\mathbb{Q}[x]$, the Chinese Remainder Theorem states that if $a(x), b(x) \in \mathbb{Q}[x]$ have no nonconstant common divisor, then $\mathbb{Q}[x]/a(x)b(x) \cong \mathbb{Q}[x]/a(x) \oplus \mathbb{Q}[x]/b(x)$. Our goal is then to break-up and re-group the summands of V to get expressions in invariant factor and elementary divisor form, all the while using the Chinese Remainder Theorem to guarantee that we still have the same $Q[x]$-module.

    In this case, we get

    $$\begin{aligned} V &\cong \mathbb{Q}[x]/(x+1)^2 \oplus \mathbb{Q}[x]/(x-1)(x^2+1)^2 \oplus \mathbb{Q}[x]/(x+1)^2(x-1) \\ &\cong \mathbb{Q}[x]/(x+1)^2 \oplus \mathbb{Q}[x]/(x-1) \oplus \mathbb{Q}[x]/(x^2+1)^2 \oplus \mathbb{Q}[x]/(x+1)^2 \\ &\quad \oplus \mathbb{Q}[x]/(x-1) && \text{(elementary divisor form)} \\ &\cong \mathbb{Q}[x]/(x+1)^2(x-1) \oplus \mathbb{Q}[x]/(x^2+1)^2(x+1)^2(x-1) && \text{(invariant factor form)} \end{aligned}$$

23. Suppose that R is an integral domain, and let $x, y \in \operatorname{Tor}(M)$ and $r \in R$ be given. Then there exist $r_1, r_2 \in R \backslash \{0\}$ such that $r_1 x = r_2 y = 0$. Thus, since R is an integral domain (and so commutative), we have that $r_2 r_1 \neq 0$ and $r_2 r_1 (x + ry) = 0$. Thus, $x + ry \in \operatorname{Tor}(M)$, so $\operatorname{Tor}(M)$ is a submodule.

    To show that $\operatorname{Tor}(M/\operatorname{Tor}(M)) = 0$, let $z + \operatorname{Tor}(M) \in \operatorname{Tor}(M/\operatorname{Tor}(M))$ be given. We want to show that $z \in \operatorname{Tor}(M)$. Now, for some $r_3 \in R \backslash \{0\}$, we have that $r_3(z + \operatorname{Tor}(M)) \in \operatorname{Tor}(M)$. Since $\operatorname{Tor}(M)$ is a submodule (as we just proved), this implies that $r_3 z \in \operatorname{Tor}(M)$, i.e, for some $r_4 \in R \backslash \{0\}, r_4 r_3 z = 0$. Since R is an integral domain, $r_4 r_3 \neq 0$, so $z \in \operatorname{Tor}(M)$, as desired.

    To give a ring R and a module M for which $\operatorname{Tor}(M)$ is not a submodule, we obviously need R to be not an integral domain. In fact, it suffices to take $R = M = \mathbb{Z}/6\mathbb{Z}$ and consider M as a module over itself acting by the usual multiplication. Then we have $2, 3 \in \operatorname{Tor}(M)$, but $2 + 3 \notin \operatorname{Tor}(M)$.

24. Let $d = (m, n)$. First, note that

$$\sum (a_i \otimes b_i) = \left(\sum a_i b_i\right)(1 \otimes 1),$$

so $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$ is a cyclic group generated by the element $1 \otimes 1$. Moreover, since $m(1 \otimes 1) = n(1 \otimes 1) = 0$, the order of $\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m$ divides d. Thus, to complete the proof, it remains only to show that $1 \otimes 1$ has order at least d.

Note that the map $\varphi\colon \mathbb{Z}_n \times \mathbb{Z}_m \to \mathbb{Z}_d$ defined by

$$\varphi(a \bmod n, b \bmod m) = ab \bmod d$$

is bilinear over $\mathbb{Z}$. It therefore follows from the universal property of tensor products that the map $\Phi\colon \mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \to \mathbb{Z}_d$ given by

$$\Phi((a \bmod n) \otimes (b \bmod m)) = ab \bmod d$$

is a well-defined $\mathbb{Z}$-module homomorphism. Since Φ maps $1 \otimes 1$ to an element of order d in $\mathbb{Z}_d$, $1 \otimes 1$ must have order at least d, as needed.

25. (Recall that an R-module M is a *torsion R-module* if $\mathrm{Tor}(M) = M$.) Suppose that G is a finite abelian group, and let $g \in G$ be given. We want to show that $g \in \mathrm{Tor}(G)$. Since G is finite, the submodule $\{ng : n \in \mathbb{Z}\}$ is finite. Thus, there exist distinct $m, n \in \mathbb{Z}$ such that $mg = ng$, i.e, $(m - n)g = 0$. Since $m - n \neq 0$, this shows that $g \in \mathrm{Tor}(G)$, as desired.

For an example of an infinite abelian group M that is a torsion $\mathbb{Z}$-module, put $M = \oplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, where the $\mathbb{Z}$-module structure on M is inherited from $\mathbb{Z}/2\mathbb{Z}$ in the usual way. Then we have $2x = 0$ for each $x \in M$.

26. $\mathrm{Hom}_R(\oplus A_i, B) \simeq \prod_i \mathrm{Hom}_R(A_i, B)$:

We want to identify a given homomorphism $\varphi\colon \oplus A_i \to B$ with a tuple $(\varphi_1, \varphi_2, \dots)$ of homomorphisms $\varphi_i\colon A_i \to B$. This may be achieved by setting

$$\varphi_i(a) = \varphi(0, \dots, 0, a, 0, \dots), \qquad a \in A_i. \tag{1}$$

where the a is the ith argument of φ. It is straightforward to verify that this is an group homomorphism. Moreover, if R is commutative, then this map is an R-module homomorphism.

To show that this mapping is moreover an isomorphism, we need to show that, given a tuple $(\varphi_1, \varphi_2, \dots)$, we can recover a unique homomorphism $\varphi\colon \oplus A_i \to B$ satisfying equation (1). That equation (1) defines a homomorphism $\varphi\colon \oplus A_i \to B$ follows from the observation that elements of $\oplus A_i$ are finite sums of elements of the form $(0, \dots, 0, a, 0, \dots)$. Therefore, if we use (1) to define the map φ on elements of the form $(0, \dots, 0, a, 0, \dots)$, then there exists a unique way to linearly extend φ to a map on $\oplus A_i$.

$\mathrm{Hom}_R(A, \prod B_j) \simeq \prod_j \mathrm{Hom}_R(A, B_j)$:

Given a homomorphism $\varphi\colon A \to \prod B_j$, define a tuple $(\varphi_1, \varphi_2, \dots)$ of maps $\varphi_j\colon A \to B_j$ by setting

$$\varphi(a) = (\varphi_1(a), \varphi_2(a), \dots).$$

The reader may verify that this establishes the desired isomorphism.