Problem One

① <u>Proposition</u>. Let $m, n \in \mathbb{Z}_+$ with $m \mid n$, and let G be a group of order n. Then G need not have a subgroup of order m.

<u>pf</u>. Let $G = A_4$ and $m = 6 \mid n = 12$. We show that G has no subgroup of order m.

Suppose that H is a subgroup of A_4 with order 6. Then $[A_4 : H] = 2$. Let $\sigma = (ijk) \in A_4$ be a 3-cycle, and note that $\sigma \cdot \sigma H = H$ since either $\sigma \cdot \sigma H = \sigma H = H$ if $\sigma \in H$ or $\sigma \cdot \sigma H = H$ by $[A_4 : H] = 2$ if $\sigma \notin H$. It follows that $\sigma^2 \in H$ in either case so that H must then contain every 3-cycle in A_3. But then $|H| \geq 8$ — a contradiction. ∎

Problem Two

② <u>Proposition</u>. Define $\varphi : \mathbb{R}[x] \to \mathbb{C}$ by $\varphi(f) = f(\sqrt{-1})$. Then $\operatorname{Ker}\varphi = \langle x^2+1 \rangle$.

<u>pf</u>. Suppose that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathbb{R}[x]$ has $\sqrt{-1}$ as a root. Then the complex conjugate $\overline{\sqrt{-1}} = -\sqrt{-1}$ must also be a root since

$$f(-\sqrt{-1}) = a_0 + a_1 \overline{\sqrt{-1}} + \cdots + a_n \left(\overline{\sqrt{-1}}\right)^n = \overline{a_0 + a_1\sqrt{-1} + \cdots + a_n \sqrt{-1}^{\,n}} = 0.$$

Thus, if $f \in \operatorname{Ker}\varphi$, both $(x+\sqrt{-1}), (x-\sqrt{-1}) \mid f$ so that since the gcd $((x+\sqrt{-1}), (x-\sqrt{-1})) = 1$, $(x^2+1) \mid f$ as well. Therefore, since clearly $x^2+1 \in \operatorname{Ker}\varphi$, we must have $\operatorname{Ker}\varphi = \langle x^2+1 \rangle$. ∎

Problem Three

③ Proposition. Let G be a group and p be a prime number.
(a) If $|G| = p$, then G is abelian.
(b) If $|G| = p^2$, then G is abelian.
(c) If $|G| = p^3$, then G need not be abelian.

pf. (a), (b) } Cf. Problem 8 in the Groups Section.

(c) Consider $p = 2$ and $G = D_4 = \langle x, y \mid x^2 = y^4 = (xy)^2 = 1 \rangle$. Since clearly $xy \neq yx$, G is not abelian. ■

Problem Four

④ Proposition. Suppose $\{1\} \to K \to G \to H \to \{1\}$ is an exact sequence of groups with K and H both abelian. Then G need not be abelian.

pf. Consider the sequence $\{1\} \longrightarrow \mathbb{Z}_2 \xrightarrow{\varphi} S_3 \xrightarrow{\psi} \mathbb{Z}_3 \longrightarrow \{1\}$ where $\mathbb{Z}_n$ denotes the cyclic group of order n, S_3 the symmetric group of order 6, φ the inclusion map, and ψ the canonical map $\psi: S_3 \longrightarrow \mathbb{Z}_3 = S_3/\mathbb{Z}_2$. Then clearly $\varphi(\mathbb{Z}_2) = \mathbb{Z}_2 = \ker \psi$ so that the sequence is exact, but S_3 is not an abelian group. ■

Problem Five

pg 2/2 ⑤ Proposition. $\mathbb{Z}[x]$ is not a Principle Ideal Domain.
pf. $\langle 2, x \rangle$ cannot be rewritten as a principle ideal. ■