UC Davis

Jan 14, 2020

studying sp_4 webs

- a "light leaves" basis (arxiv 2009.13786)
- connection to tilting modules
- formulas for projectors (in preparation)

Elijah Bodish
(University of Oregon)

- $Sp_4 \circlearrowright \mathbb{C}^4$ preserving ω

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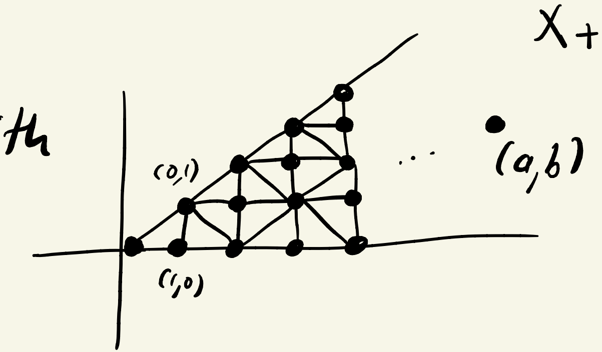
- $\mathrm{Rep}^{f.d.}(Sp_4)$ has irreducibles in bijection with
  $X_+ = \{\text{dominant integral weights}\}$



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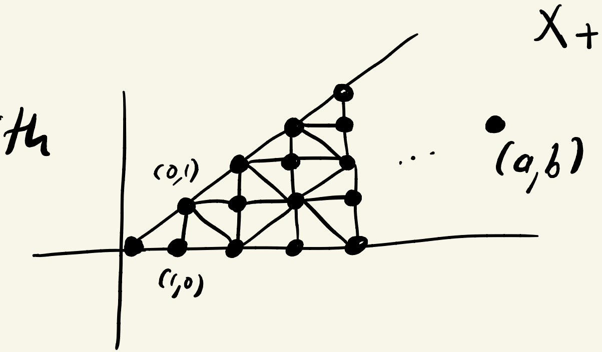

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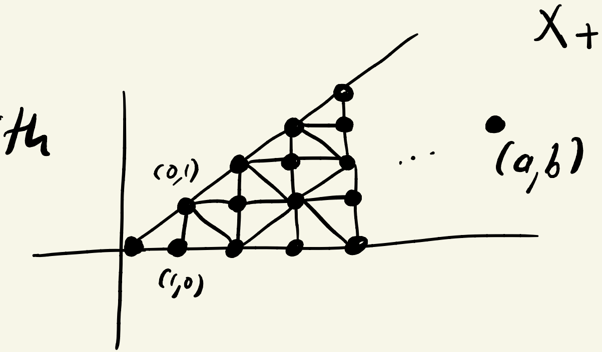

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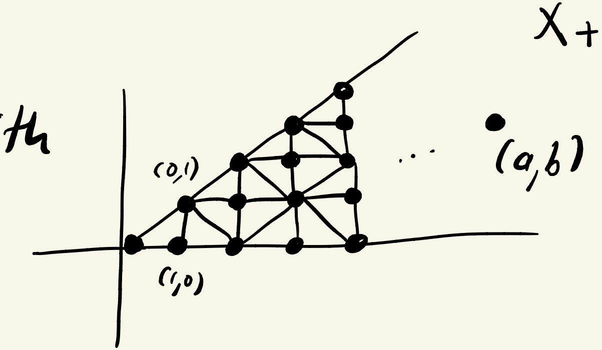

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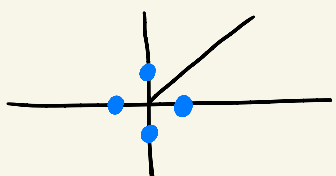
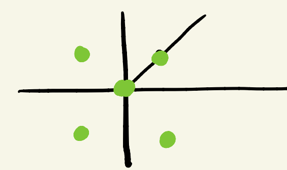
- $L(1,0) = \mathbb{C}^4$



wt $L(1,0)$

- $L(0,1) = \Lambda^2 \mathbb{C}^4 / \mathbb{C}\cdot\omega$



wt $L(0,1)$

__Notation__

Let $\underline{w} = (w_1 w_2 \dots w_n)$ be a word in alphabet $\{1, 2\}$

Set $L(1) := L(1,0)$

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and $L(\underline{w}) := L(w_1) \otimes L(w_2) \otimes \dots \otimes L(w_n)$

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Problems

- decompose $L(\underline{\omega}) \cong \bigoplus_{(a,b) \in X_+} L(a,b)^{\oplus m_{\underline{\omega}}^{(a,b)}}$
- understand morphisms $L(\underline{\omega}) \longrightarrow L(\underline{u})$

Solution is inspired by ideas of

Libedinsky — Soergel Bimodules

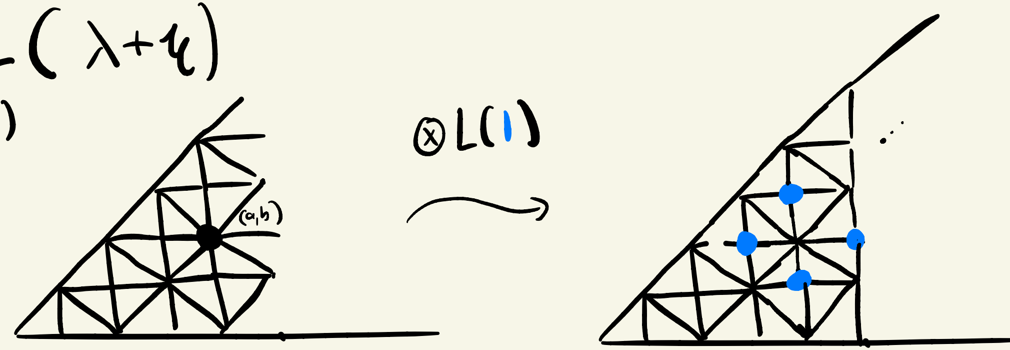
Elias, Williamson — diagrammatic Soergel Category

Elias — $\mathfrak{gl}_n$ webs

Probably the ideas go back much further

- $\lambda = (a,b) \in X_+$

- $L(\lambda) \otimes L(1) \cong \bigoplus_{\eta \in wt\, L(1)} L(\lambda + \eta)$



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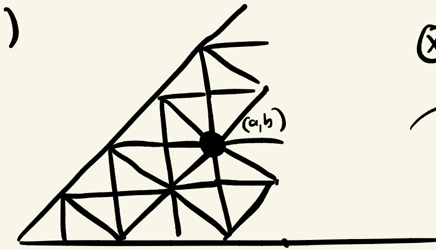

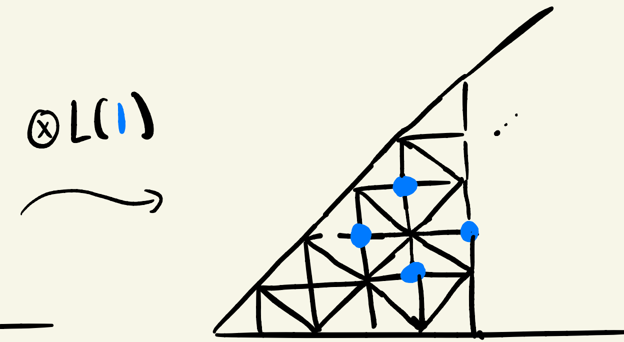

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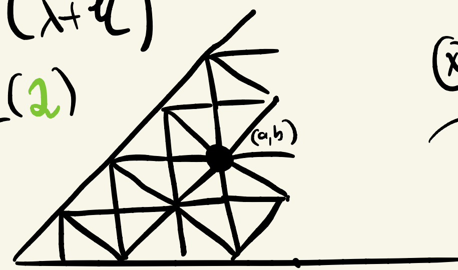

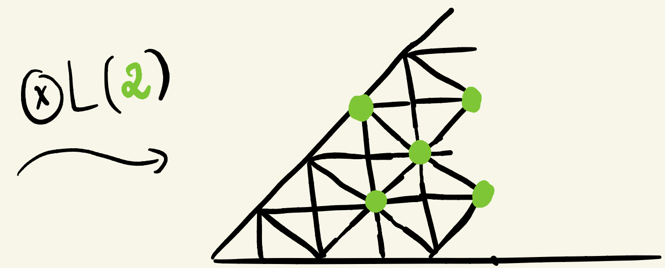

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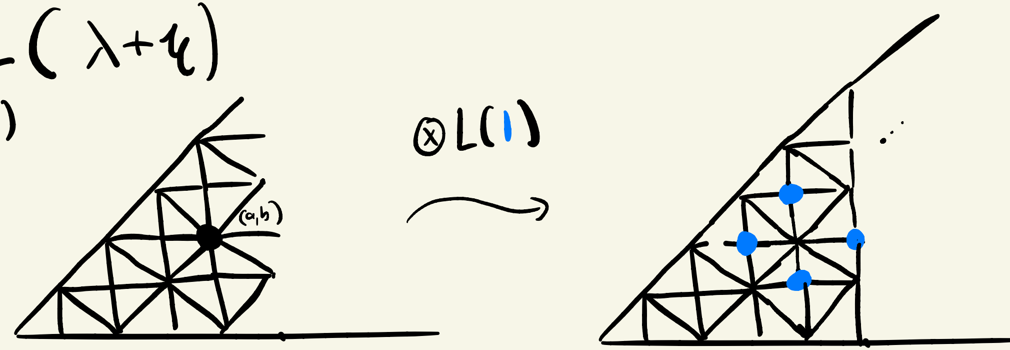

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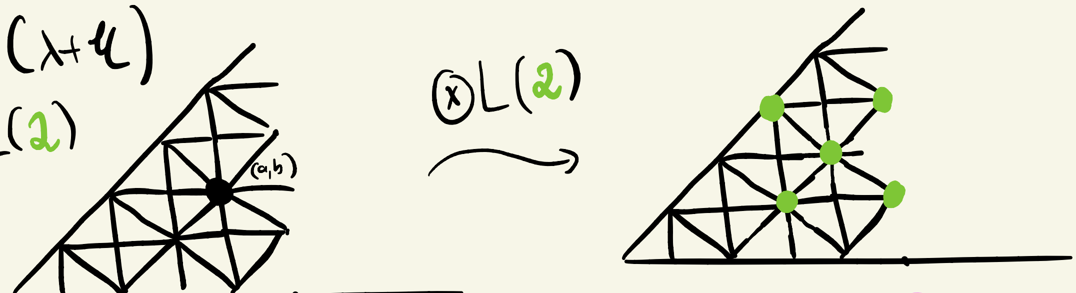
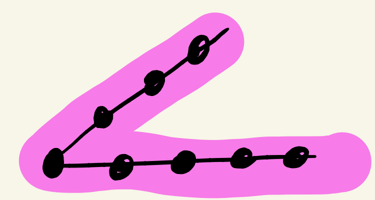
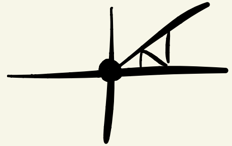


- Must reinterpret when λ is "on the wall". $\lambda \in$ 

example : $L(111)$

- Step 0 $\mathbb{1} = L(0,0)$



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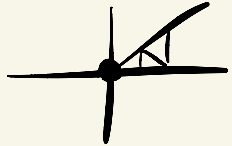

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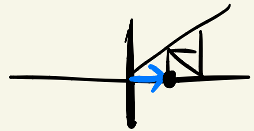

- Step 1 $L(0,0) \otimes L(1) \simeq L(1)$

<u>example</u> : $L(\textcolor{blue}{111})$

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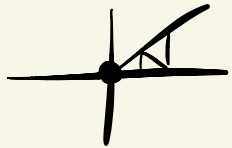
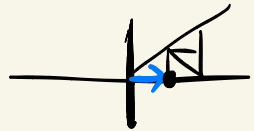
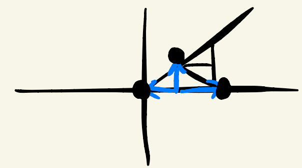


- <u>Step 1</u> $L(0,0) \otimes L(\textcolor{blue}{1}) \cong L(\textcolor{blue}{1})$





- <u>Step 2</u> $L(\textcolor{blue}{1}) \otimes L(\textcolor{blue}{1}) \cong L(2,0) \oplus L(0,1) \oplus L(0,0)$

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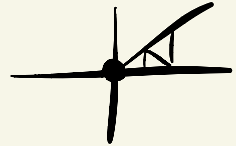

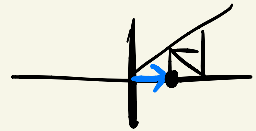

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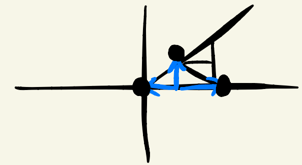
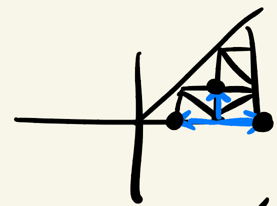


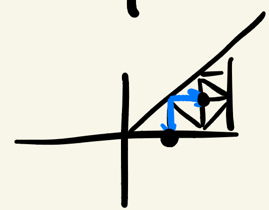
- Step 2 $L(1) \otimes L(1) \cong L(2,0) \oplus L(0,1) \oplus L(0,0)$

$L(2,0) \otimes L(1)$



- Step 3 $\left(L(1)\otimes L(1)\right) \otimes L(1) \cong \begin{pmatrix} L(2,0) \\ L(0,1) \\ L(0,0) \end{pmatrix} \otimes L(1)$

$L(0,1) \otimes L(1)$



$$\cong L(3,0) \oplus L(1,1) \oplus L(1,0) \oplus L(1,1) \oplus L(1,0) \oplus L(1,0)$$

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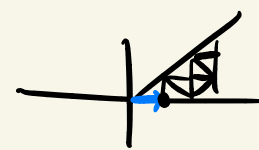
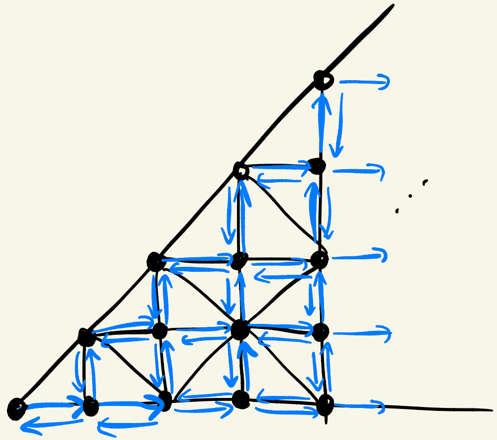
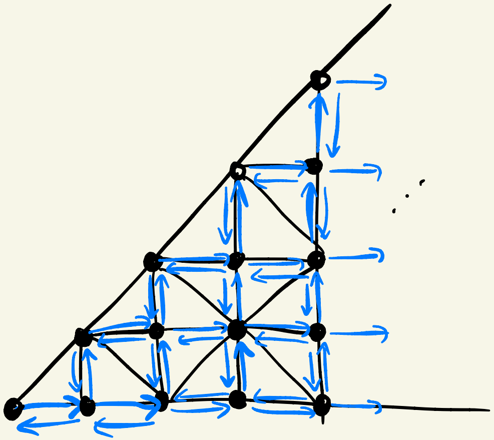




$$L(\underbrace{111\ldots1}_{n}) \cong \bigoplus L(a,b)^{m^{(a,b)}_{111\ldots1}}$$

$m^{(a,b)}_{111\ldots1}$ = paths of length n from $(0,0)$ to (a,b)



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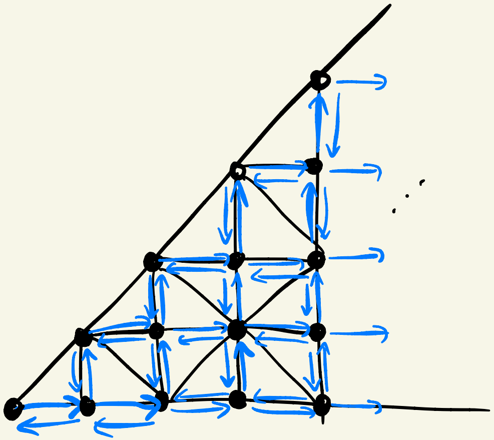

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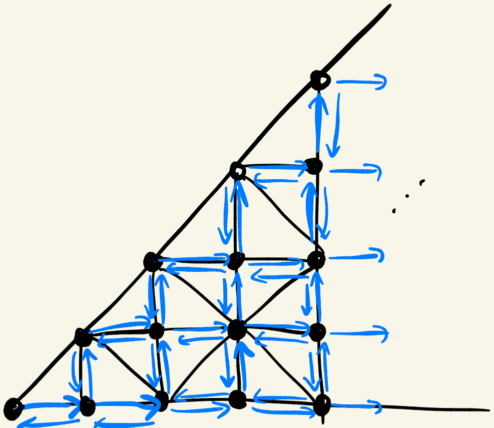

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$(\mu_0, \mu_1, \ldots, \mu_n)$ s.t.
- $\mu_0 = (0,0)$
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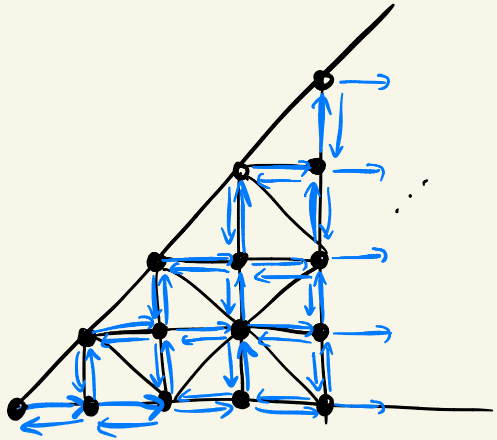

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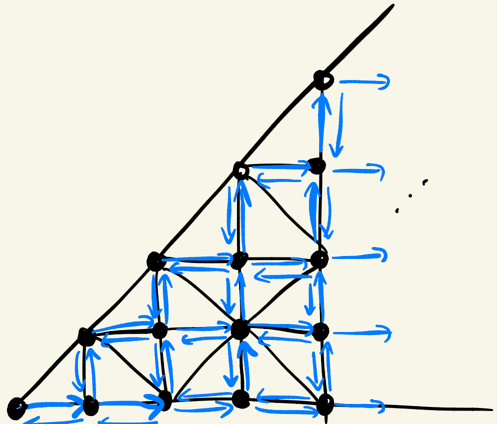

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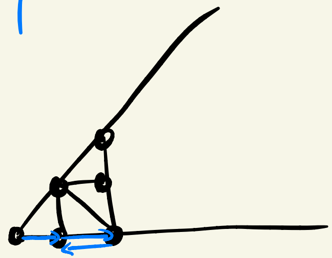
- $m^{(a,b)}_{\underline{w}} = \# E(\underline{w}, (a,b))$ i.e. $L(\underline{w}) \cong \bigoplus_{(a,b)} L(a,b)^{\# E(\underline{w},(a,b))}$

ex

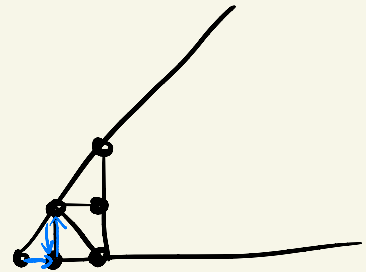
$\underline{w} = |\ |\ |$

plethysm paths for $\underline{w}$ to $(1,0)$

$(\mu_0,\mu_1,\mu_2,\mu_3)$



$((0,0),(1,0),(1,0),(-1,0))$



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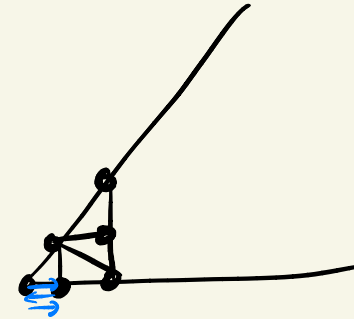

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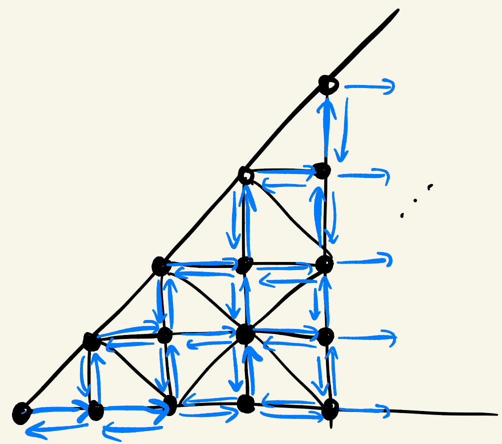

Can we understand morphisms $L(\underline{w}) \longrightarrow L(\underline{u})$?

Can we understand morphisms $L(\underline{\omega}) \longrightarrow L(\underline{u})$?

- $L(\underline{\omega}) \cong \bigoplus L(a,b)^{\#E(\underline{\omega},(a,b))}$ $\qquad L(\underline{u}) \cong \bigoplus L(a,b)^{\#E(\underline{u},(a,b))}$

$$L(\underline{\omega}) \xrightarrow{\text{project}} L(a,b) \xrightarrow{\text{include}} L(\underline{u})$$

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$$\dim \operatorname{Hom}(L(\underline{\omega}), L(\underline{u})) = \sum_{(a,b)} \#E(\underline{\omega},(a,b))\,\#E(\underline{u},(a,b))$$

- Lets understand all projections $L(\underline{\omega}) \longrightarrow L(a,b)$
  and all inclusions $L(a,b) \longrightarrow L(\underline{u})$

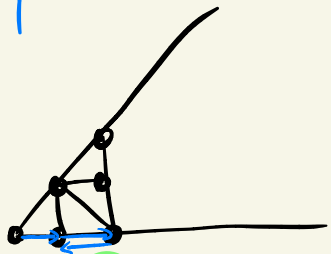
  Then we will put the two together to find all maps $L(\underline{\omega}) \longrightarrow L(\underline{u})$

ex

$\underline{w} = 1\,1\,1$

plethysm paths for $\underline{w}$ to $(1,0)$

$(\mu_0, \mu_1, \mu_2, \mu_3)$



$L(1,0)$

$\uparrow$

$L(2,0) \otimes L(1)$

$\uparrow$

$L(1,0) \otimes L(1) \otimes L(1)$

$\uparrow$

$\mathbb{1} \otimes L(1) \otimes L(1) \otimes L(1) \otimes L(1)$

ex

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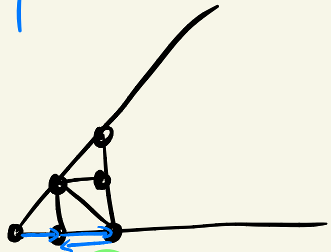
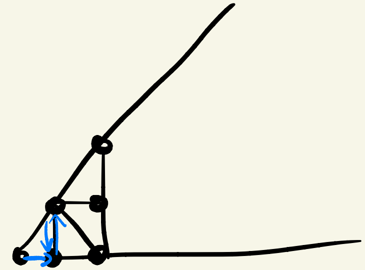


$$\begin{gathered} L(1,0) \\ \uparrow \\ L(2,0) \otimes L(1) \\ \uparrow \\ L(1,0) \otimes L(1) \otimes L(1) \\ \uparrow \\ \mathbb{1} \otimes L(1) \otimes L(1) \otimes L(1) \end{gathered}$$



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$\underline{ex}$

$\underline{w} = 111$

plethysm paths for $\underline{w}$ to $(1,0)$

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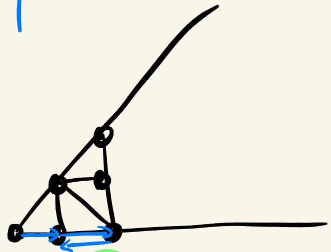

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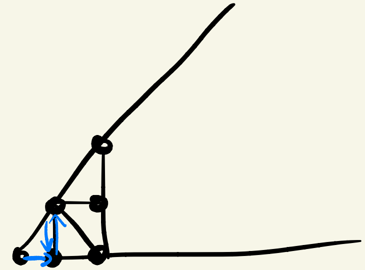
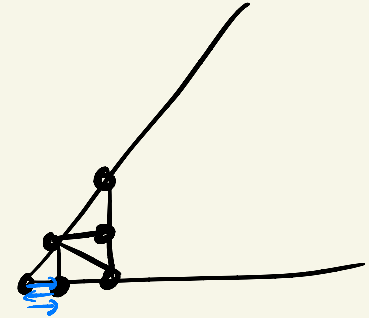


$$\begin{gathered} L(1,0) \\ \uparrow \\ L(0,1) \otimes L(1) \\ \uparrow \\ L(1,0) \otimes L(1) \otimes L(1) \\ \uparrow \\ \mathbb{1} \otimes L(1) \otimes L(1) \otimes L(1) \end{gathered}$$



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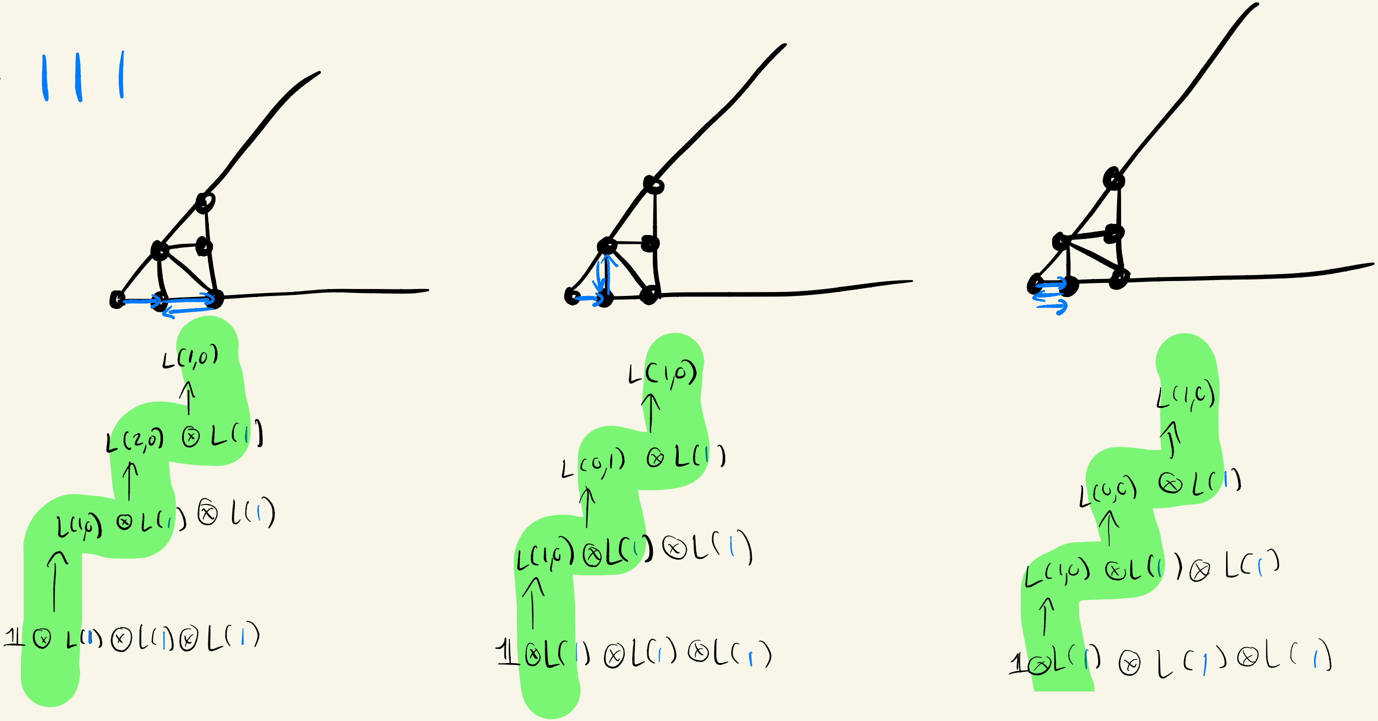
ex

$\underline{w} = 111$

plethysm paths for $\underline{w}$ to $(1,0)$

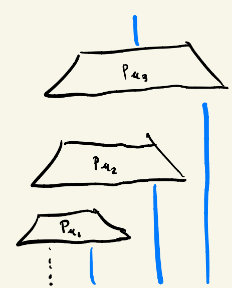
$(\mu_0, \mu_1, \mu_2, \mu_3)$



- schematic using graphical calculus for morphisms

vertical composition is function composition

horizontal composition is ⊗ product



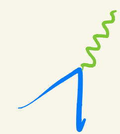
$= p_{\mu_3} \circ (p_{\mu_2} \otimes \mathrm{id}) \circ (p_{\mu_1} \otimes \mathrm{id} \otimes \mathrm{id})$

Kuperberg's Spider

Let $\mathfrak{D}_{C_2}$ be strict, monoidal, pivotal category with

objects words in alphabet $\{1, 2\}$

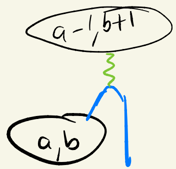
morphisms isotopy classes of planar graphs built

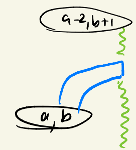
from 

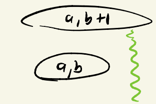
subject to relations: (we ignore this for now)

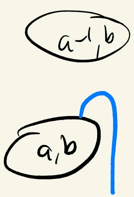
Theorem (Kuperberg '96) $\mathfrak{D}_{C_2} \xrightarrow{\cong} \langle L(1), L(2) \rangle_{\otimes} \overset{\text{full}}{<} \mathrm{Rep}^{\text{f.d.}}(Sp_4)$

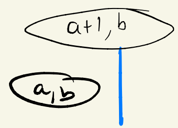
Some candidate P_η's for our plethysm path maps

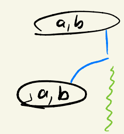


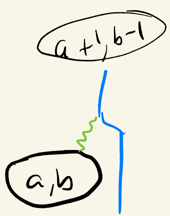


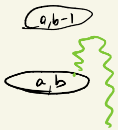


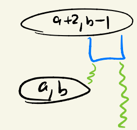








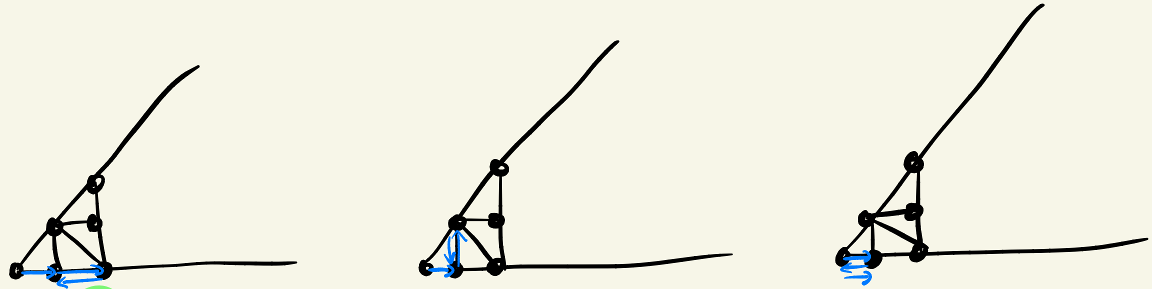




example $\underline{w} = 111$

paths to $(1,0)$

$(\mu_0, \mu_1, \mu_2, \mu_3)$



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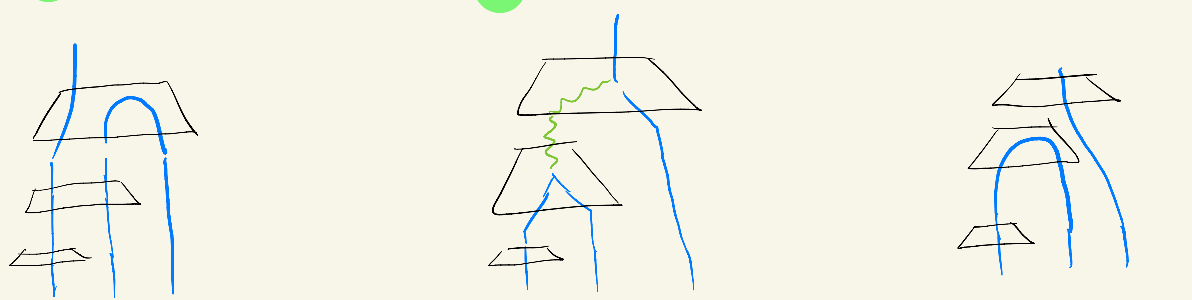
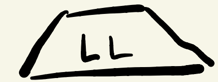
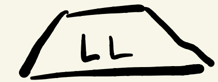
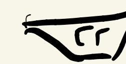
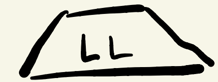
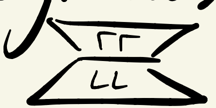


- Thus, we can inductively construct a set of diagrams for $\mathrm{Hom}(\underline{w}, \ )$ (light leaves 

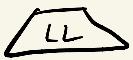
)

- Thus, we can algorithmically construct a set of diagrams for $\mathrm{Hom}(\underline{w}, \ )$ (light leaves )

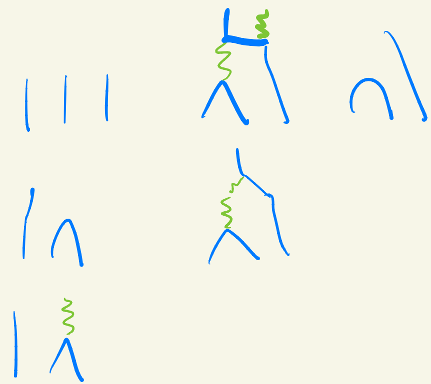
- Carrying out the algorithm for $\underline{u}$, then flipping all those diagrams upside down, we get a set for $\mathrm{Hom}(-, \underline{u})$ (upside down light leaves )

- Thus, we can algorithmically construct a set of diagrams for $\mathrm{Hom}(\underline{w}, \ )$ (*light leaves* )
- Carrying out the algorithm for $\underline{u}$, then flipping all those diagrams upside down, we get a set for $\mathrm{Hom}(-, \underline{u})$ (*upside down light leaves* )
- Putting the two together in all possible ways results in set of diagrams in $\mathrm{Hom}(\underline{w}, \underline{u})$ (*double leaves* $\mathbb{LL} =$ )

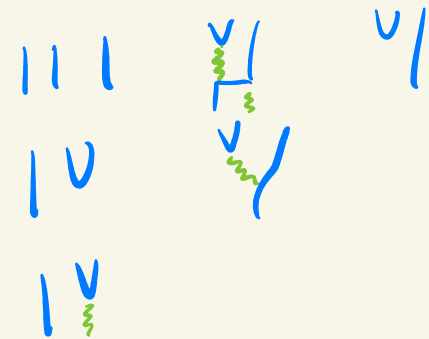
example



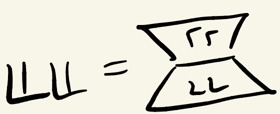
Hom(L(|||), –)

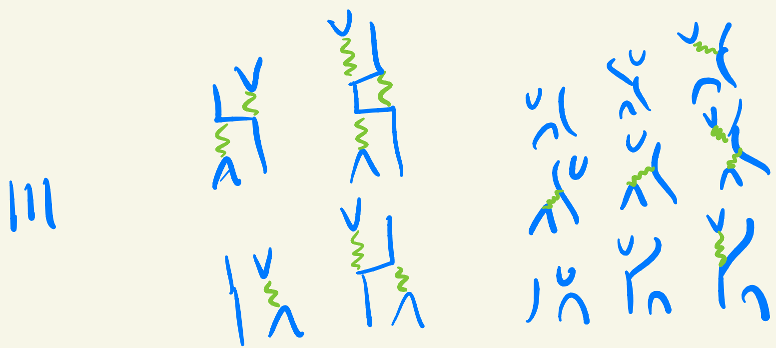


Hom(–, L(|||))



Hom(L(|||), L(|||))





Theorem (B. 2020)

$\sqcup\sqcup$ is a basis for $\mathcal{Q}_{C_2}$

Theorem (B. 2020)

$\mathbb{LL}$ is a basis for $\mathcal{D}_{C_2}$

Proof Make $\mathrm{eval}: \mathcal{D}_{C_2} \longrightarrow \mathrm{Rep}^{f.d.}(sp_4)$ explicit.

Consider $\mathrm{eval}(\mathbb{LL})$

Show independent by unitriangularity argument

This gives $\mathbb{LL}$ is linearly independent.

Spanning of $\sqcup\sqcup$ should be quite hard, but Kuperberg shows relations in $\mathcal{D}_{C_2}$ imply:

- "non-elliptic webs" span
- $\#$ non-elliptic webs $= \dim \mathrm{Hom}_{Sp_4}(-,-)$

$$= \#\sqcup\sqcup \qquad \square$$

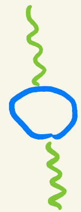
What are the relations?

($[n]_q := q^n - q^{-n} / q - q^{-1}$)

- look at $\dim \mathrm{Hom}_{Sp_4}(\_,\_)$

 $= \dfrac{-[6][2]}{[3]}$

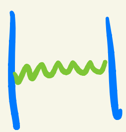
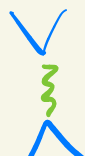
 $= \dfrac{[6][5]}{[3][2]}$

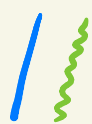
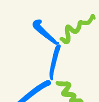
 $= -[2]$ 

 $= 0$

 $= 0$

- need ⊔⊔ spans

 $=$  $+ \dfrac{1}{[2]}$  $- \dfrac{1}{[2]}$ 

exercise: Show  $=$  $+ \dfrac{1}{[2]}$ 

- $\mathcal{D}_{C_2}$ is defined over $\mathbb{Z}[q^{\pm 1}][\frac{1}{[2]_q}]$

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- $\mathcal{D}_{C_2}$ is defined over $\mathbb{Z}[q^{\pm 1}][\frac{1}{[2]_q}]$
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- Lusztig divided powers quantum group $\mathcal{U}_q^{\mathbb{Z}}(\mathfrak{sp}_4)$ defined over $\mathbb{Z}[q^{\pm 1}][\frac{1}{[2]_q}]$
- $\mathcal{U}_q^{\mathbb{Z}}(\mathfrak{sp}_4) \circlearrowright V^{\mathbb{Z}}(a,b)$ "Weyl module"

<u>Note</u>: $L(a,b) = \mathbb{C} \otimes V^{\mathbb{Z}}(a,b)$

- Let $\Bbbk$ be a field and $q \in \Bbbk$ s.t. $q+q^{-1} \neq 0$.

$$\Bbbk \otimes \mathcal{U}_q^{\mathbb{Z}}(\mathfrak{sp}_4) \curvearrowright \Bbbk \otimes V^{\mathbb{Z}}(1) \quad \text{and} \quad \Bbbk \otimes V^{\mathbb{Z}}(2)$$

are irreducible

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are irreducible

- The modules $\Bbbk \otimes V^{\mathbb{Z}}$ will not all be summands of tensors of fundamentals. Instead, get <u>tilting modules</u>:

$$\bullet\ \mathrm{Tilt}\left(\Bbbk \otimes \mathcal{U}_q^{2\ell}(\mathfrak{sp}_4)\right) = \mathrm{Kar}\left(\langle \Bbbk \otimes V^{2\ell}(1),\ \Bbbk \otimes V^{2\ell}(2) \rangle_{\otimes}\right)$$

- $\mathrm{Tilt}(\Bbbk \otimes \mathcal{U}_q^{\mathbb{Z}}(\mathfrak{sp}_4)) = \mathrm{Kar}(\langle \Bbbk \otimes V^{\mathbb{Z}}(1), \Bbbk \otimes V^{\mathbb{Z}}(2) \rangle_{\otimes})$

- $T^{\Bbbk}(a,b) \overset{\oplus}{\subset} \Bbbk \otimes V^{\mathbb{Z}}(\underbrace{11\ldots1}_{a}\underbrace{22\ldots2}_{b})$

indecomposable tilting modules classified by highest weight

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indecomposable tilting modules classified by highest weight

- Lots of general theory about $\mathrm{Tilt}(\Bbbk \otimes \mathcal{U}_q^{\mathbb{Z}})$

Most important for us :

$$\dim \mathrm{Hom}_{\Bbbk \otimes \mathcal{U}_q^{\mathbb{Z}}}(\Bbbk \otimes V^{\mathbb{Z}}(\underline{w}), \Bbbk \otimes V^{\mathbb{Z}}(\underline{u})) = \dim \mathrm{Hom}_{Sp_4(\mathbb{C})}(L(\underline{w}), L(\underline{u}))$$

__Theorem__ (B 2020)

$$\Bbbk \otimes \mathrm{eval} : \mathrm{Kar}(\Bbbk \otimes \mathcal{D}^{\mathbb{Z}}_{c_2}) \xrightarrow{\cong} \mathrm{Tilt}(\Bbbk \otimes \mathcal{U}_q^{\mathbb{Z}}(\mathfrak{sp}_4))$$

Theorem (B. 2020)

$$\mathbb{k}\otimes \mathrm{eval} : \mathrm{Kar}(\mathbb{k}\otimes \mathcal{D}^{\mathbb{Z}}_{C_2}) \xrightarrow{\cong} \mathrm{Tilt}(\mathbb{k}\otimes \mathcal{U}^{\mathbb{Z}}_q(\mathfrak{sp}_4))$$

Proof Non elliptic webs span over $\mathbb{Z}[q^{\pm 1}][\frac{1}{[2]}]$.

Need to show $\mathrm{eval}(\mathbb{LL})$ independent over $\mathbb{k}$

Unitriangularity reduces this to analysis of "diagonal" of matrix which we compute explicitly.

$\square$

- Our result for q a root of unity and $\mathbb{C} = \mathbb{k}$ says $\mathfrak{D}_{C_2}$ does compute RT 3 manifold invariant.

- There is more to say about relation b/w

  $\mathbb{k} \otimes \mathfrak{D}_{C_2}$ and Tilt

  but no time.

- Our result for q a root of unity and $\mathbb{C} = \mathbb{k}$ says $\mathcal{D}_{C_2}$ does compute RT 3 manifold invariant.

- There is more to say about relation b/w $\mathbb{k} \otimes \mathcal{D}_{C_2}$ and Tilt but no time.

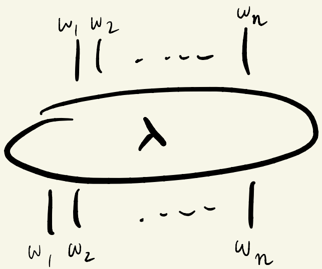
- $\mathbb{LL}$ basis new even over $\mathbb{C}(q)$.

- We use $\mathbb{LL}$ to derive formulas for clasps (alias: Jones Wenzl projector) in Sp_4 Webs.
  - a complete answer to one question Dongseok Kim grappled with in his thesis (at Davis!)

Clasps

$\Bbbk = \mathbb{C}(q)$ $\lambda \in X_+$

$\underline{w} = w_1 \dots w_n$, $w_i \in \{1, 2\}$ & $\sum w_i = \lambda$



$\longleftarrow$ linear combination of LL's projecting to $L(\lambda) \overset{\oplus}{\subseteq} L(\underline{w})$

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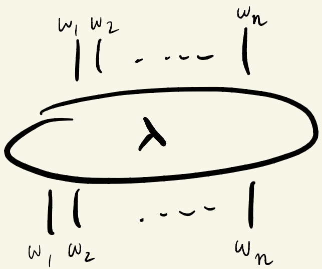
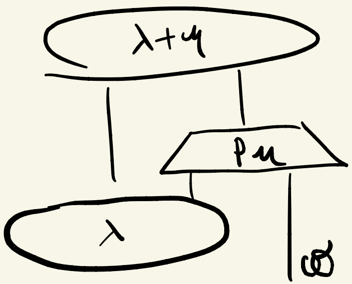


$\leftarrow$ linear combination of LL's projecting to $L(\lambda) \subseteq L(\underline{w})$

$\varpi \in \{1, 2\}$



is basis for $\mathrm{Hom}_{Sp_4}\big(L(\lambda)\otimes L(\varpi),\ L(\lambda+\mu)\big)$

Clasps

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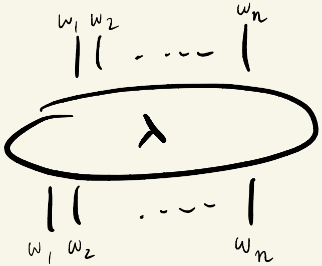
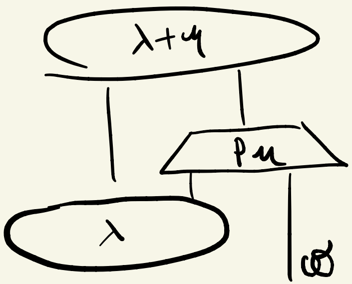
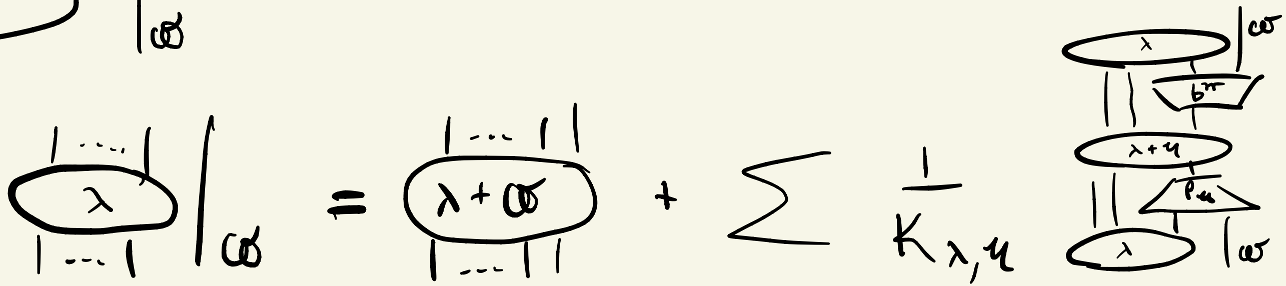


← linear combination of $\mathbb{LL}$'s projecting to $L(\lambda) \overset{\oplus}{\otimes} L(\underline{w})$

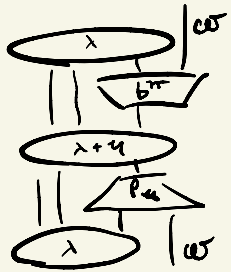
$\varpi \in \{1, 2\}$



is basis for $\mathrm{Hom}_{Sp_4}(L(\lambda) \otimes L(\varpi), L(\lambda+\mu))$

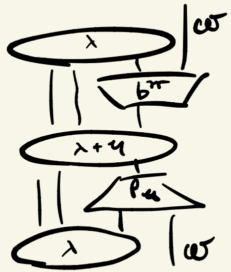
$\Longrightarrow$ 

$$L(\lambda) \otimes L(\varpi) \cong L(\lambda+\varpi) \oplus \bigoplus_{\mu \in wt(L(\varpi))} L(\lambda+\mu)$$

$\frac{1}{K_{\lambda,\mu}}$ 

want this map to be an idempotent

$$\left(X = X_1 \oplus X_2 \iff \mathrm{id}_X = e_1 + e_2 \right)$$

$\frac{1}{K_{\lambda,\mu}}$ 

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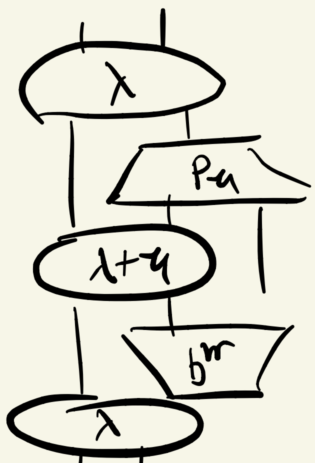
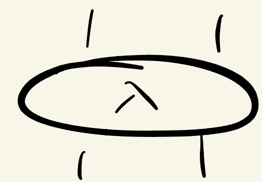
$(X = X_1 \oplus X_2 \iff id_X = e_1 + e_2)$

easy to check that this follows from

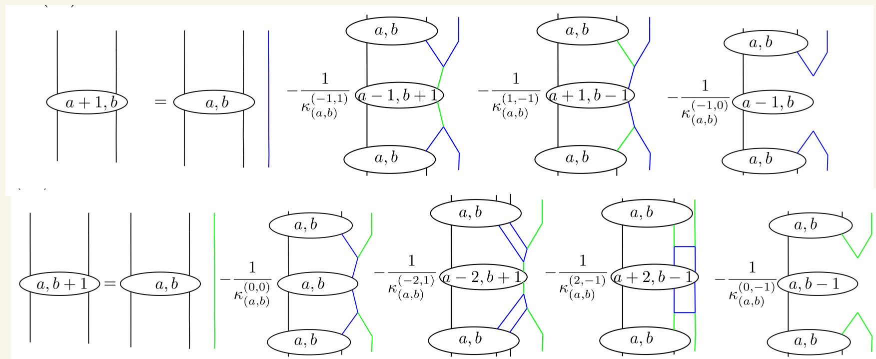


$= K_{\lambda,\mu}$ 

Theorem (B. 2021) Let d_μ = minimal length element of Weyl group s.t. $d_\mu(\mu) \in X_+$

and $\Phi_+(\mu) = \{\alpha \in \Phi_+ \mid d_\mu(\alpha) \in \Phi_-\}$.

Then



where
$$K_{\lambda,\mu} = \prod_{\alpha \in \Phi_+(\mu)} \frac{[(\alpha^\vee, \lambda+\rho)]_{q_\alpha}}{[(\alpha^\vee, \lambda+\mu+\rho)]_{q_\alpha}}$$

$*$ $q_\alpha = q^{l(\alpha)}$ $\qquad l(\alpha) = \begin{cases} 2 & \alpha \text{ long} \\ 1 & \alpha \text{ short} \end{cases}$

Proof Solve the recursive formulas:

$$\kappa_{(a,b),(1,0)} = 1 \tag{2.6}$$

$$\kappa_{(a,b),(0,1)} = 1 \tag{2.7}$$

$$\kappa_{(a,b),(-1,1)} = -[2] - \kappa^{-1}_{(a-1,b),(-1,1)} \tag{2.8}$$

$$\kappa_{(a,b),(2,-1)} = -\frac{[4]}{[2]} - \kappa^{-1}_{(a,b-1),(2,-1)} \tag{2.9}$$

$$\kappa_{(a,b),(0,0)} = \frac{[5]}{[2]} - \kappa^{-1}_{(a-1,b),(-1,1)} \cdot \kappa_{(a-2,b+1),(2,-1)} - \kappa^{-1}_{(a-1,b),(1,-1)} \tag{2.10}$$

$$\kappa_{(a,b),(1,-1)} = \frac{[5]}{[2]} - \kappa^{-1}_{(a,b-1),(2,-1)} \cdot \kappa_{(a+2,b-2),(-1,1)} - \frac{1}{[2]^2}\kappa^{-1}_{(a,b-1),(0,0)} \tag{2.11}$$

(2.12)

$$\kappa_{(a,b),(-1,0)} = -\frac{[6][2]}{[3]} - \kappa^{-1}_{(a-1,b),(-1,0)} - \kappa^{-1}_{(a-1,b),(-1,1)} \cdot \kappa_{(a-2,b+1),(1,-1)} - \kappa^{-1}_{(a-1,b),(1,-1)} \cdot \kappa_{(a,b-1),(-1,1)}$$

$$\kappa_{(a,b),(0,-1)} = \frac{[6][5]}{[3][2]} - \frac{1}{\kappa_{(a,b-1),(0,-1)}} - \frac{\kappa_{(a+2,b-2),(-2,1)}}{\kappa_{(a,b-1),(2,-1)}} - \frac{\kappa_{(a,b-1),(0,0)}}{\kappa_{(a,b-1),(0,0)}} - \frac{\kappa_{(a-2,b),(2,-1)}}{\kappa_{(a,b-1),(-2,-1)}} \tag{2.13}$$

(2.14)

$$\kappa_{(a,b),(-2,1)} = \frac{[5]}{[2]} \cdot \kappa_{(a-1,b),(-1,1)} - (-[2] - \kappa^{-1}_{(a-2,b),(-1,1)}) \cdot \frac{\kappa_{(a-1,b),(-1,1)}}{\kappa_{(a-1,b),(-1,0)}} - \frac{\kappa_{(a-2,b+1),(0,0)}}{\kappa^{2}_{(a-2,b),(-1,1)} \cdot \kappa_{(a-1,b),(-1,1)}}$$

- Evidence for a conjecture of Elias in type A.
- G_2 clasp formulas work in progress with Haihan Wu

- Finally, an advertisement for work w/ Elias/Rose/Tatham

We find generators and relations for Sp_{2n} webs and can show $Sp_{2n}\text{-webs} \xrightarrow{\cong} \mathrm{Fund}(Sp_{2n})$

- proof is inspired by Kuperberg's rank 2 paper!

Thank You!