On the Lattice of Cycles of an Undirected Graph

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ADM Seminar UC Davis
March 4, 2021

Roadmap for today

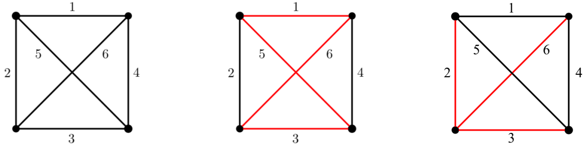
1. Motivation
2. Cycle Spaces of G
3. Determinant and Basis Construction for $\mathrm{Lat}(\mathcal{C})$

Graphs

Definition

- Call the set of vertices and edges, *graph* $G = (V, E)$, an (undirected) connected graph, possibly with loops and multiedges.
- A *cycle of* G is a connected subgraph of G with each vertex having degree two. Denote the set of cycles of G as $\mathcal{C}(G)$.
- A *spanning tree of* G is a acyclic connected subgraph of G containing all of V.

Example



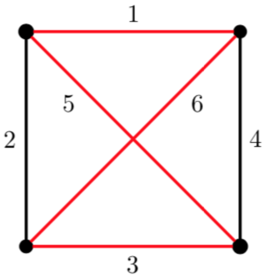
$G = K_4$ cycle s. tree

Graphs

Definition

- For $A \subset E$, let $\chi_A \in \{0,1\}^n$ be the indicator vector of a cycle A of graph G, where $\chi_i = 1$ if $e_i \in E$ and 0 if not.

Example



$\chi_{\{1,3,5,6\}}$ = $(1,0,1,0,1,1)$

An Appetizer: The Cycle Double Cover Conjecture

- A *bridge* is a single edge whose removal disconnects G.
- For any bridgeless graph G, there exists a list of cycles that contains every edge of G exactly twice.

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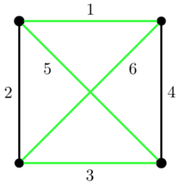

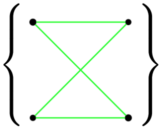

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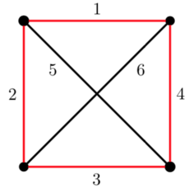

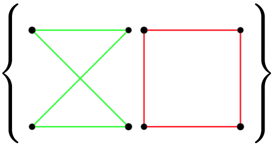

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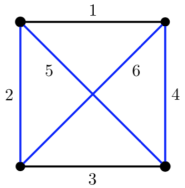

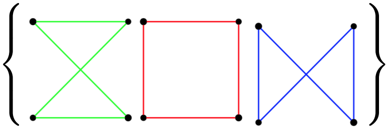

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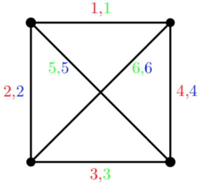

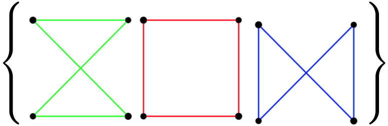

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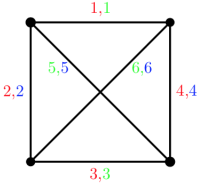

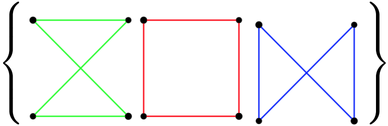

What if we consider positive integer combinations of cycles?

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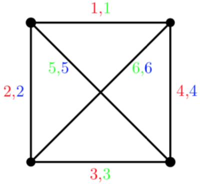

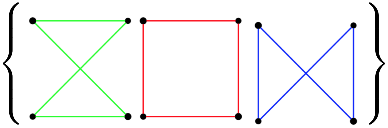

What if we consider positive integer combinations of cycles?

The semi-group generated by the cycles of G

$$SG(\mathcal{C}(G)) = \left\{\sum_{A\in\mathcal{C}} n_A\chi_A \ : \ n_A \in \mathbb{Z}_{\geq 0}\right\}$$

always contains $(2,\ldots,2)$.

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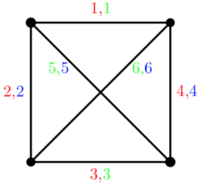

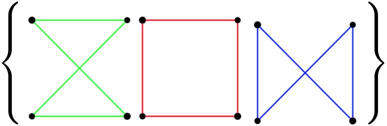

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Moral: Linear Algebra and Graph Theory play well together!

Cycle spaces of a graph

- Let K be a field, usually $\mathbb{Q}$ or $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ for prime p.
- For $A \subseteq E$, let χ_A denote the characteristic vector of A.

Definition

- For a collection $\mathcal{A}$ of subsets of E, define the **linear hull** of $\mathcal{A}$ as

$$\mathrm{Lin.\,Hull}_K(\mathcal{A}) := \left\{ \sum_{A \in \mathcal{A}} n_A \chi_A \; : \; n_A \in K \right\}.$$

For $\mathcal{A} = \mathcal{C}(G)$ and $K = \mathbb{Q}$, this the ***rational cycle space*** of G.

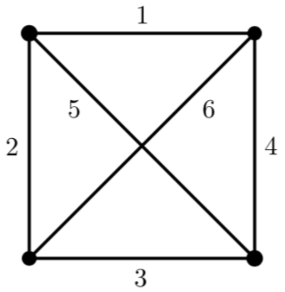
- Considering integer combinations of $\mathcal{A}$, define the **lattice** of $\mathcal{A}$ as

$$\mathrm{Lat}(\mathcal{A}) := \left\{ \sum_{A \in \mathcal{A}} n_A \chi_A \; : \; n_A \in \mathbb{Z} \right\} \subset \mathbb{Z}^E$$

For $\mathcal{A} = \mathcal{C}(G)$ we call this the ***cycle lattice*** of G.

Lattice of undirected K_4

Consider the complete graph K_4 and its set of cycles (as rows of a matrix):



$$\mathcal{C} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

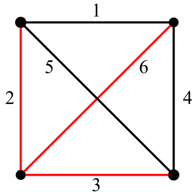
- A basis for the row space of $\mathcal{C}$ forms a basis for $\mathrm{Lat}(\mathcal{C}(K_4))$ whose rational cycle space has dimension 6.
- What about $\mathrm{Lat}(\mathcal{C}(G))$ for a general graph G?
- Is there always a basis of only cycles of G?

Definition

- G is *r-vertex (or r-edge) connected* if the removal of $r-1$ vertices (or edges) does not disconnect it.
- A *fundamental cycle* for a spanning tree T, $\mathrm{ci}(e,T)$, is the unique cycle contained in $T \cup e$ for $e \in E \setminus T$.

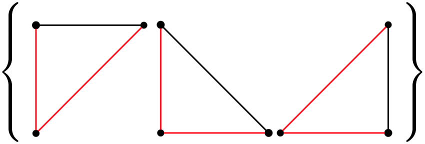
Example (Fundamental Cycles)

Consider the red spanning tree T on K_4 and the remaining edges e_1, e_4, and e_5. Then the corresponding fundamental cycles are:



T

$\Rightarrow$



$\mathrm{ci}(e_1,T)$ $\mathrm{ci}(e_5,T)$ $\mathrm{ci}(e_4,T)$

Prior work

The rational cycle space has been characterized via series classes of E, where $e, f \in E$ are *in series* if they are in the same cycles (an equivalence relation).

Proposition (Goddyn et al., 1999)

For any graph $G = (V, E)$ with cycles $\mathcal{C}$,

$$\begin{aligned}\mathrm{Lin.\,Hull}_{\mathbb{Q}}(\mathcal{C}) =& \{p \in \mathbb{Q}^E : p(e) = 0 \text{ for any bridge } e, \\ & \text{and } p(f) = p(g) \text{ for } f \text{ and } g \text{ in series}\}.\end{aligned}$$

Remark: This implies $\mathrm{Lin.\,Hull}_{\mathbb{Q}}(\mathcal{C})$ and $\mathrm{Lat}(\mathcal{C})$ are full dimensional when G is 3-edge-connected (equivalently, bridgeless and has only trivial series classes).

What about an explicit basis?

Prior work

Computing dimension and specific bases for directed graphs and undirected graphs over $\mathbb{F}_2$ is well-understood:

Proposition (Can be found in Diestel, Section 1.9)

Consider the connected graph G with cycles $\mathcal{C}$ over $K = \mathbb{Z}/2\mathbb{Z}$. Then

- *for T a spanning tree of G, the collection of fundamental cycles of T with respect to $E \backslash T$ forms a basis of $\mathrm{Lin.\,Hull}_K(\mathcal{C})$.*
- $\dim(\mathrm{Lin.\,Hull}_K(\mathcal{C})) = |E| - |V| + 1$.

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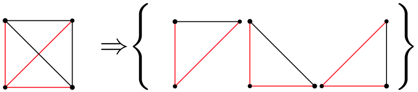
- $\dim(\mathrm{Lin.\,Hull}_K(\mathcal{C})) = |E| - |V| + 1$.

Example (Recall: dim$(\mathrm{Lin.\,Hull}_{\mathbb{Q}}(\mathcal{C}(K_4))) = 6$ for undirected K_4)

The dimension of $\mathrm{Lin.\,Hull}_{\mathbb{Z}/2\mathbb{Z}}(\mathcal{C}(K_4)) = 6 - 4 + 1 = 3$, corresponding with the fundamental cycle basis below:



Undirected graphs and non-binary fields

- For the rational cycle space and cycle lattice over undirected cycles and for non-binary fields, our goals are:

Goal 1: Determine the
  - ⋆ determinant of the cycle lattice, and
  - ⋆ dimension of the rational cycle space.

Goal 2: Describe the associated bases using only cycles.

Goal 3: Determine the complexity of finding bases.

Determinant of $\mathrm{Lat}(\mathcal{C})$

Proposition (Averkov, C., De Loera, Gillespie - 2020)

If $G = (V, E)$ *is a* 3*-edge-connected graph with cycles* $\mathcal{C}$*, then* $\det(\mathrm{Lat}(\mathcal{C})) = 2^{|V|-1}$*.*

Determinant of $\mathrm{Lat}(\mathcal{C})$

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If $G = (V, E)$ is a 3-edge-connected graph with cycles $\mathcal{C}$, then $\det(\mathrm{Lat}(\mathcal{C})) = 2^{|V|-1}$.

Example

Considering the matrix of cycles of K_4 that span $\mathrm{Lat}(\mathcal{C})$,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

we have $\det(A) = 8 = 2^{4-1}$.

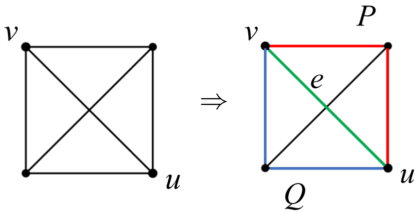
Characterizing edge (vertex) connectivity on a graph

Theorem (Menger's theorem)

Let G be an undirected graph and $u, v \subset V$. Then the minimum number of edges (vertices) to disconnect u and v is equal to the maximum number of edge-disjoint (vertex-disjoint) paths connecting u and v.

Example (Cycle Double Cover Conj. for cycle lattice!)

For u and v of K_4, by Menger's theorem, we can use these paths to show that $2\chi_e \in \mathrm{Lat}(\mathcal{C})$.



$$\Rightarrow 2\chi_e = \chi_{P\cup e} + \chi_{Q\cup e} - \chi_{P\cup Q}$$

Proof that $\det(\mathrm{Lat}(\mathcal{C})) = 2^{|V|-1}$

- We can express $\mathcal{L} = \det(\mathrm{Lat}(\mathcal{C}))$ in terms of group indices, $\left[\mathbb{Z}^{|E|} : \mathcal{L}\right]$.
- This implies

$$\frac{2^{|E|}}{\mathcal{L}} = \left[\mathcal{L} : 2\mathbb{Z}^{|E|}\right] = 2^{\dim_{\mathbb{Z}/2\mathbb{Z}}\left(\mathcal{L}/2\mathbb{Z}^{|E|}\right)}.$$

- The first equality is by comparison of determinants. The second follows from interpreting $\mathcal{L}/2\mathbb{Z}^E$ as a vector space of $\mathbb{Z}^E/2\mathbb{Z}^E$.
- The space of cycles over $\mathbb{Z}/2\mathbb{Z}$ has dimension $|E| - |V| + 1$ (Diestel).
- Thus, we can compute $\mathcal{L}$ directly:

$$\frac{2^{|E|}}{\mathcal{L}} = 2^{|E|-|V|+1} \Rightarrow \mathcal{L} = 2^{|E|-(|E|-|V|+1)} = 2^{|V|-1}.$$

A basis for $\mathrm{Lat}(\mathcal{C})$

Theorem (Averkov, C., De Loera, Gillespie - 2020)

If $G = (V, E)$ is a 3-edge-connected graph with circuits $\mathcal{C}$, and if T is a spanning tree of G, let

$$\mathcal{C}_T := \left\{\chi_{\mathrm{ci}(e,T)} \,:\, e \in E \setminus T\right\},$$

and let

$$X_T := \{2\chi_t \,:\, t \in T\},$$

where $\mathrm{ci}(e, T)$ denotes the unique cycle contained in $e \cup T \subset E$. Then the collection $\mathcal{C}_T \cup X_T$ is a basis for the cycle lattice of G.

One drawback: these are not all cycles! . . . but we can handle that.

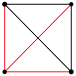
A basis of cycles for $\mathrm{Lat}(\mathcal{C})$

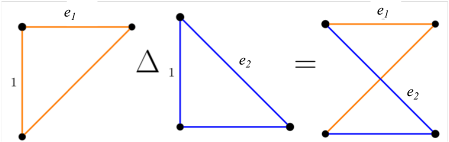
Some notation first:

Definition

A *semi-fundamental cycle* of G is the symmetric difference of two fundamental cycles, denoted $\mathrm{ci}(e_1, e_2, T)$, of a given spanning tree T and edges $e_1, e_2 \in E \backslash T$.

Example

For K_4 with spanning tree, , the symmetric difference of fundamental cycles whose intersection is edge 1 is:



A basis of cycles for $\mathrm{Lat}(\mathcal{C})$

Theorem (Averkov, C., De Loera, Gillespie - 2020)

If G is a connected graph, then a cycle lattice basis of G exists. If T is a spanning tree of G, then the basis may be chosen to be semi-fundamental with respect to T.

A basis of cycles for $\mathrm{Lat}(\mathcal{C})$

Theorem (Averkov, C., De Loera, Gillespie - 2020)

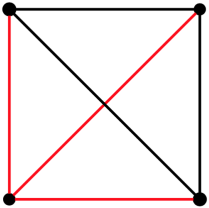
If G is a connected graph, then a cycle lattice basis of G exists. If T is a spanning tree of G, then the basis may be chosen to be semi-fundamental with respect to T.

We prove this by giving an algorithm to construct a basis consisting of cycles of G for a given spanning tree T:

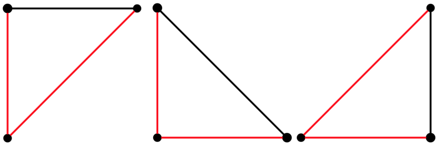
- S1: Use $\mathrm{ci}(e_1, T)$ and $\mathrm{ci}(e_2, T)$ to form a semi-fundamental cycle.
- S2: Use same $\mathrm{ci}(e_i, T)$ to indicate a unique edge t of G.
- S3: Contract G by t and repeat process on G/t.
- End: This inductively produces cycles, that along with the fundamental cycles of T, form a basis.

Generating a Cycle Lattice Basis for K_4

Recall K_4:

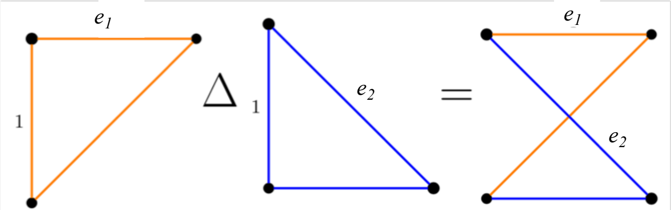


and the fundamental cycles we computed before:



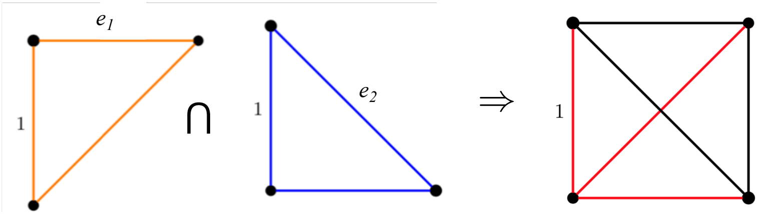
Generating a Cycle Lattice Basis for K_4

Step 1: Use the symmetric difference of $\mathrm{ci}(e_1, T)$ and $\mathrm{ci}(e_2, T)$ to uniquely find a semi-fundamental cycle:



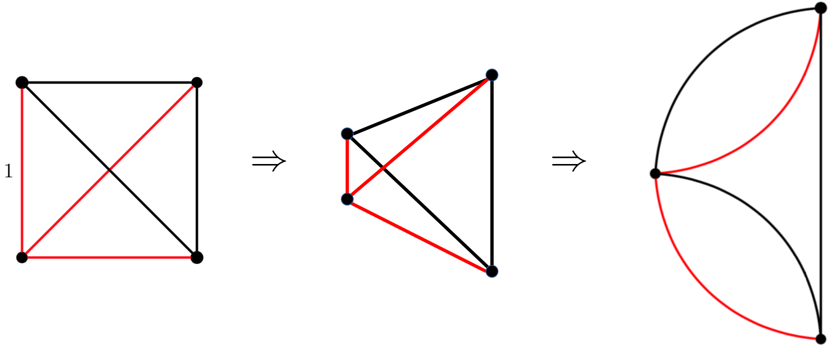
Generating a Cycle Lattice Basis for K_4

Step 2: Identify the unique edge that the fundamental cycles share. This is the edge labeled 1:



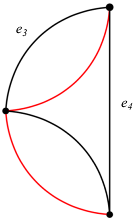
Generating a Cycle Lattice Basis for K_4

Step 3: Contract K_4 by 1 and repeat process on $K_4/1$. Contraction means identify the vertices adjacent to the edge 1:

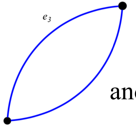


Generating a Cycle Lattice Basis for K_4

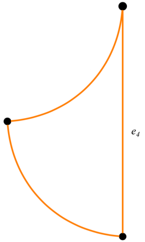
Step 1: We now identify two fundamental cycles on $K_4/1$:



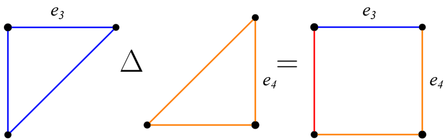
produces



and

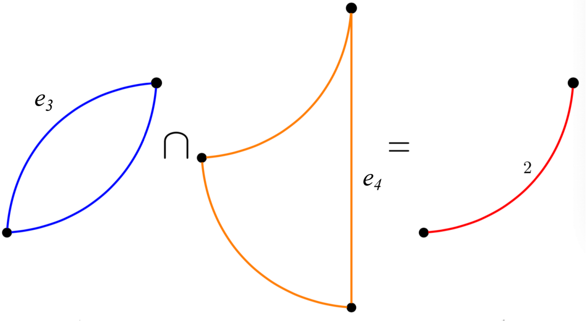


which identifies the symmetric difference of fundamental cycles:



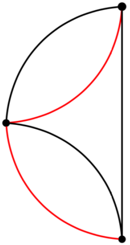
Generating a Cycle Lattice Basis for K_4

Step 2: Identify the unique edge that the fundamental cycles share. This is the edge labeled 2:

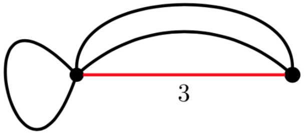


Generating a Cycle Lattice Basis for K_4

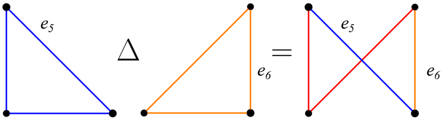
Step 3: Contract $K_4/1$ by 2 and repeat process on $K_4/1/2$:



contracts to

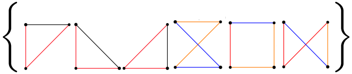


Final Steps: This identifies two final fundamental cycles and their symmetric difference gives us the last basis:

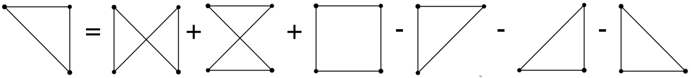


Example of Cycle Lattice Basis for K_4

The basis constructed contains the following fundamental and semi-fundamental cycles:

$\mathcal{B} =$ 

Check: the missing cycle is a linear combination of basis elements,



Basis size and computational complexity

Corollary (Averkov, C., De Loera, Gillespie - 2020)

Let $G = (V, E)$ is a connected graph with m edges and $n \geq 2$ vertices, and let T be a spanning tree of G. Then a lattice cycle basis of G may be constructed in time $O(mn)$ such that each cycle has length at most $2\,\mathrm{diam}(T)$.

Proof.

Any semi-fundamental cycle lattice basis with respect to T has the desired property. In particular, note that a fundamental cycle of T has length at most $\mathrm{diam}(T) + 1$, and thus a semi-fundamental cycle, as the symmetric difference of two intersecting fundamental cycles, has length at most $2\,\mathrm{diam}(T)$. □

Notes on field K of characteristic p

Theorem (Averkov, C., De Loera, Gillespie - 2020)

Let $G = (V, E)$ be a 3-edge-connected graph with m edges and n vertices, and let K be a field of characteristic p.

- *Then $\mathrm{Lin.\,Hull}_K(\mathcal{C}(G))$ is a K-vector space of dimension*

$$\dim_K(\mathrm{Lin.\,Hull}_K(\mathcal{C}(G))) = \begin{cases} m, & \text{if } p \neq 2, \\ m - n + 1, & \text{if } p = 2. \end{cases}$$

- *If $p \neq 2$, then any lattice basis of $\mathrm{Lat}(\mathcal{C}(G))$ reduces modulo p to a linear basis of $\mathrm{Lin.\,Hull}_K(\mathcal{C}(G))$.*
- *If $p = 2$, then any basis of the classical binary cycle space maps to a linear basis of $\mathrm{Lin.\,Hull}_K(\mathcal{C}(G))$ under the natural inclusion map.*

Future Work

- Generalize cycle-basis approach to matroids.
- Explore the semi-group generated by cycles of graphs ... seems difficult!
- Can refined structural results be useful for addressing the Cycle Double Cover Conjecture???

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