R-matrices and Yangians

Algebra and discrete mathematics seminar (UC Davis)

February 25, 2021.

Based on :

[GTL] Meromorphic tensor equivalence for Yangians and quantum loop algebras. (Joint with Valerio Toledano Laredo)
(Publications IHES ; vol. 125 ; 2017)

[GTLW] The meromorphic R-matrix of the Yangian
(joint with Valerio Toledano Laredo & Curtis Wendlandt)
arxiv: 1907.03525
to appear in volume dedicated to N. Reshetikhin's 60th birthday.

$\underline{\text{R-matrix}}$

$$R_{\underline{12}}(u)\, R_{\underline{13}}(u+v)\, R_{\underline{23}}(v) = R_{\underline{23}}(v)\, R_{\underline{13}}(u+v)\, R_{\underline{12}}(u) \qquad \text{(YBE)}$$

$R : \mathbb{C} \to \mathrm{End}(V \otimes V)$ $\quad$ V : f.d. $\mathbb{C}$-v.s.

$N = \dim(V)$

$R_{ij}(u) \circlearrowright \underline{V} \otimes \underline{\underline{V}} \otimes \underline{\underline{V}}$

N^6 cubic eq^{n}s.

N^4 unknowns

e.g. Constant R-matrices $\longleftrightarrow$ quantum groups

Abelian R-matrices $\quad R : \underline{\underline{\mathbb{C}}} \to \underset{\text{comm.}}{\mathcal{A}} \subset \mathrm{End}(V \otimes V)$

$\underline{\underline{\text{Yang}}}$'s R-matrix

$$R(u) = \frac{u \cdot \mathrm{Id} + \hbar \cdot \mathrm{Flip}}{u + \hbar} \circlearrowright V \otimes V$$

$(\hbar \in \mathbb{C}^\times)$

$\boxed{\text{rational R-matrix}}$

$\updownarrow$

Yangian of $\mathfrak{gl}(V)$.

$\boxed{\hbar \in \mathbb{C}^\times}$

Yangian

$\mathfrak{g}$: simple Lie alg / $\mathbb{C}$ $\rightsquigarrow$ $Y_\hbar(\mathfrak{g})$ Yangian of $\mathfrak{g}$

$(\mathfrak{g} = \mathfrak{sl}_2)$

- Unital assoc. alg. / $\mathbb{C}$
  $\Delta : Y \to Y \otimes Y$ alg hom s.t.
  $\Delta \otimes 1 \circ \Delta = 1 \otimes \Delta \circ \Delta$] Hopf alg

- $\{\tau_a \circlearrowright Y\}_{a \in \mathbb{C}}$ 1-p. family of Hopf alg auto's.

$\mathcal{U}(\mathfrak{g}) \subset Y_\hbar(\mathfrak{g})$

$(\hbar = 0)$ $Y_0(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}[z])$

$(\mathrm{Rep}_{fd} Y, \otimes)$

$V \in \mathrm{Rep}_{fd} Y$

$\rightsquigarrow \{V(a)\}_{a \in \mathbb{C}}$, $V(a) = \tau_a^* V$.

Yangian of $\mathfrak{sl}_2$: unital, associative algebra over $\mathbb{C}$.

Generators : $\xi_r \,,\, x_r^{\pm} \qquad r \in \mathbb{Z}_{\geq 0}$.

Relations :

$$[\xi_r, \xi_s] = 0 \qquad \Big(\underbrace{Y^0 = \langle \xi_r : r \geq 0 \rangle \subset Y}_{\text{is a commutative subalg.}}\Big)$$

$$[\xi_0, x_r^{\pm}] = \pm 2 x_r^{\pm} \qquad \Big(\mathcal{U}(\mathfrak{sl}_2) = \langle \underline{\xi_0}, \underline{x_0^{\pm}} \rangle \subset Y$$

$$Y^{\pm} = \langle x_r^{\pm} : r \geq 0 \rangle \subset Y \Big)$$

$$\underline{\text{raising}} \,/\, \underline{\text{lowering operators}}$$

$$\to [\xi_{r+1}, x_s^{\pm}] - [\xi_r, x_{s+1}^{\pm}] = \pm \hbar (\xi_r x_s^{\pm} + x_s^{\pm} \xi_r)$$

$$[x_{r+1}^{\pm}, x_s^{\pm}] - [x_r^{\pm}, x_{s+1}^{\pm}] = \pm \hbar (x_r^{\pm} x_s^{\pm} + x_s^{\pm} x_r^{\pm})$$

$$[x_r^+, x_s^-] = \xi_{r+s}$$

$$\left(\begin{array}{c} \hbar = 0 \rightsquigarrow \mathcal{U}(\mathfrak{sl}_2[z]) \\ \xi_r \rightsquigarrow h \cdot z^r \\ x_r^{\pm} \rightsquigarrow (e/f) \cdot z^r \end{array}\right)$$

$$\boxed{t_1 := \xi_1 - \frac{\hbar}{2} \xi_0^2} \Rightarrow \underline{[t_1, x_r^{\pm}] = \pm 2 x_{r+1}^{\pm}} \qquad \left(\{\xi_0, x_0^{\pm}, \underline{t_1}\} = \text{a set of generators} \right)$$

Coproduct $\Delta : Y \to Y \otimes Y$: $\Delta(y_0) = y_0 \otimes 1 + 1 \otimes y_0 \quad (y = \xi \text{ or } x^{\pm})$

$$\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1 \underline{- 2\hbar\, x_0^- \otimes x_0^+}.$$

Shift auto. $\tau_a \circlearrowright Y$ $(a \in \mathbb{C})$

$$\tau_a(y_r) = \sum_{k=0}^{r} \binom{r}{k} a^k y_{r-k} \qquad (y = \xi \text{ or } x^{\pm})$$

If $\underline{\xi(u) := 1 + \hbar \sum_{r \geq 0} \xi_r u^{-r-1}}$ then $\tau_a(\xi(u)) = \xi(u-a)$ "shift of argument"

Theorem (Drinfeld 1985) $\exists!\ R(s) = 1\otimes 1 + O(s^{-1}) \in Y\otimes Y[[s^{-1}]]$ s.t.

(i) $\left(\underset{s}{\Delta}(x) = \tau_s\otimes 1(\Delta(x))\ ;\ \underset{s}{\Delta}^{op}(x) = \tau_s\otimes 1(\Delta^{op}(x)) \right)$

$$\underset{s}{\Delta}^{op}(x) = R(s)\,\underset{s}{\Delta}(x)\,R(s)^{-1}. \qquad (\forall\, x\in Y).$$

(ii)
$$\underset{t}{\Delta}\otimes 1(R(s)) = R_{13}(s+t)R_{23}(s)$$
$$1\otimes\Delta(R(s)) = R_{13}(s)R_{12}(s)$$

(iii) $R(s)^{-1} = R_{21}(-s)$ (iv) (YBE).

(v) $\tau_a\otimes\tau_b(R(s)) = R(s+a-b)$

(vi) Let $V,W\in Rep_{fd}(Y)$ be irred. then

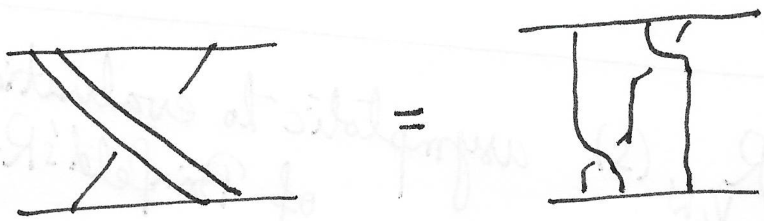
$$R_{V,W}(s) \in End(V\otimes W)[[s^{-1}]]$$
$$\rotatebox{90}{=}\ \underbrace{\boxed{f(s)}}_{\in\,\mathbb{C}[[s^{-1}]]}\ \underbrace{R_{V,W}(s)}_{\text{rat'l in } s.}$$

Moreover it has:
- $f(s)$ is divergent.
- Arbitrary V & W. ($Rep_{fd}Y$ NOT semisimple)
- Construct R?

$(Rep_{fd}Y, \otimes)$ $\quad V\otimes W \not\cong W\otimes V$ $\quad \sigma$: flip

(i) $V(s)\otimes W \xrightarrow[\sigma\circ R_{V,W}(s)]{\sim} W\otimes V(s)$ is Y-linear

"poles in s"

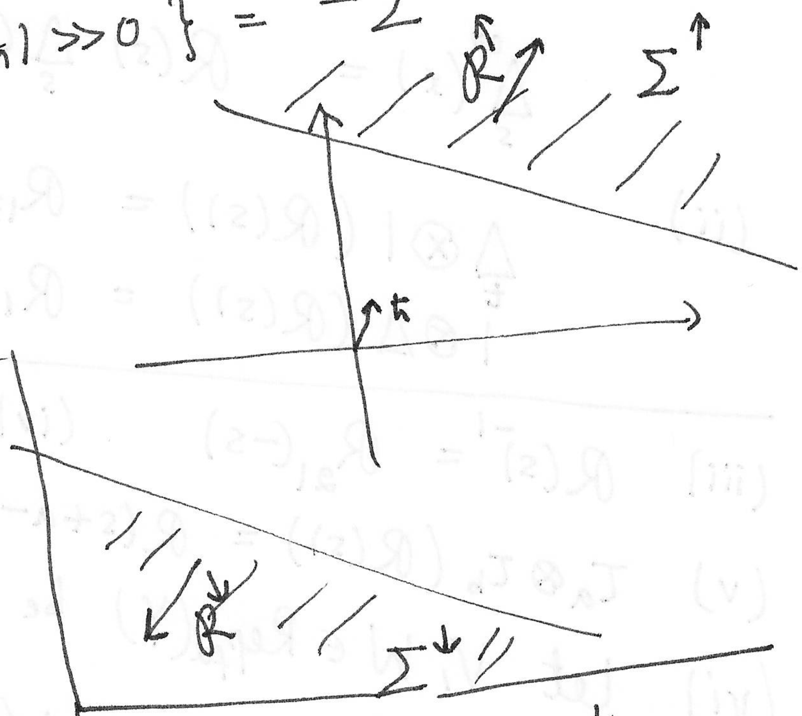


$$\Delta\otimes 1(R) = R_{12}R_{23}$$

Theorem $\forall\ V, W \in \mathrm{Rep}_{fd}(Y)$, there are 2 unique mero. fns. $\mathcal{R}^{0,\uparrow/\downarrow}(s) : \mathbb{C} \longrightarrow \mathrm{End}(V \otimes W)$ s.t (natural in V,W) $\boxed{\text{explicitly}}$

(0) They are holomorphic & invertible in $\eta \in \{\uparrow, \downarrow\}$

$$\Sigma^{\uparrow} = \{\mathrm{Re}(s/\hbar) \gg 0\} = -\Sigma^{\downarrow}$$



$$\lim_{\substack{s \to \infty \\ s \in \Sigma^{\eta}}} \mathcal{R}^{\eta}(s) = \mathrm{Id}_{V \otimes W}.$$

(1) $V(s) \otimes W \xrightarrow{\sigma \circ \mathcal{R}^{0,\eta}(s)} W \otimes V(s)$ is Y-linear

(2) $\mathcal{R}^{0,\eta}_{V_1(s_1) \otimes V_2, V_3}(s_2) = \mathcal{R}^{0,\eta}_{V_1, V_3}(s_1+s_2)\, \mathcal{R}^{0,\eta}_{V_2, V_3}(s_2)$

(3) $\mathcal{R}^{\uparrow}_{V,W}(s)^{-1} = \sigma \circ \mathcal{R}^{\downarrow}_{W,V}(-s) \circ \sigma$

(4) (YBE) ✓

(5) $\mathcal{R}^{\eta}_{V(a), W(b)}(s) = \mathcal{R}^{\eta}_{V,W}(s+a-b)$

(6) $\mathcal{R}^{\eta}_{V,W}(s) = \underset{\text{rat'l}}{R_{V,W}(s)} \cdot \boxed{\mathcal{X}^{\eta}_{V,W}(s)}$ mero. $\in \mathrm{End}_Y(V(s) \otimes W)$

$\underline{\mathcal{R}^{\eta}_{V,W}(s) \sim R_{V,W}(s)}$ asymptotic to evaluation of Drinfeld's R-matrix.

Proof:

$$\underset{(V,W)}{R^{\eta}(s)} = \boxed{\mathcal{R}^{+}(s)}\ \overset{\text{GTL}}{\underbrace{\mathcal{R}^{0,\eta}(s)}}\ \overset{\text{GTLW}}{\boxed{\mathcal{R}^{-}(s)}} \quad \text{Gauss dec}$$

$$\boxed{\mathcal{R}^{-}_{21}(-s)^{-1}}$$

mero. fns.
abelian
R-matrix.

- rat'l in s
- = Id at $s=\infty$
- $\boxed{\text{twist eq}^{n}}$ assoc. of $\underset{D}{\otimes}$.

$$Y \curvearrowright V(s)\otimes W$$
$$\Big\downarrow \mathrm{Ad}(\mathcal{R}^{-}_{V,W}(s))$$
$$V(s)\underset{D}{\otimes} W$$
"Drinfeld coproduct"

$$\begin{tikzcd}
V(s)\otimes W \arrow[r, "\mathcal{R}^{-}_{V,W}(s)"] \arrow[d, dashed, "\sigma\circ R^{\eta}_{V,W}(s)"'] & V(s)\underset{D}{\otimes} W \arrow[d, "\sigma\circ R^{0,\eta}(s)"] \\
W\otimes V(s) = (W(-s)\otimes V)(s) \arrow[r, "\mathcal{R}^{-}_{W,V}(-s)"'] & W\underset{D}{\otimes} V(s) = (W(-s)\underset{D}{\otimes} V)(s)
\end{tikzcd}$$

$$\begin{tikzcd}
V_1(s_1+s_2)\otimes V_2(s_2)\otimes V_3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{twist eq}^{n}"] & V_1(s_1+s_2)\underset{D}{\otimes} V_2(s_2)\otimes V_3 \arrow[d] \\
V_1(s_1+s_2)\otimes V_2(s_2)\underset{D}{\otimes} V_3 \arrow[r] & V_1(s_1+s_2)\underset{D}{\otimes} V_2(s_2)\underset{D}{\otimes} V_3
\end{tikzcd}$$

Ex. $Y = Y_\hbar(\mathfrak{sl}_2) \circlearrowright \mathbb{C}^2$ $\quad x_0^- \left(\begin{array}{l} |1\rangle \\ |2\rangle \end{array} \right) x_0^+$ $\quad \begin{array}{l} +1 \\ -1 \end{array}$

ξ_0

$$\boxed{t_1 = -\frac{\hbar}{2} \cdot \mathrm{Id}} \rightsquigarrow \xi_r = 0 = x_r^\pm \quad (r \geq 1)$$

$$\boxed{[t_1 \otimes 1 + 1 \otimes t_1 + s \cdot \xi_0 \otimes 1, \mathcal{R}^-(s)] = -2\hbar \mathcal{R}^-(s) x_0^- \otimes x_0^+}$$

$$\mathcal{R}^-(s) = \underline{1} \otimes \underline{1} + \sum_{n \geq 1} \underset{2n}{\downarrow} \underset{2n}{\uparrow} \quad \in Y^- \otimes Y^+[[s^{-1}]].$$

$\mathcal{R}^-(s) \circlearrowright \mathbb{C}^2 \otimes \mathbb{C}^2$

$|11\rangle \longmapsto |11\rangle$

$|12\rangle \quad |21\rangle \longmapsto$ $\begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$

$|22\rangle \longmapsto |22\rangle$

$$\left[s \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \right] = -2\hbar \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$-2s\gamma = -2\hbar \Rightarrow \boxed{\gamma = \hbar/s} \checkmark$$

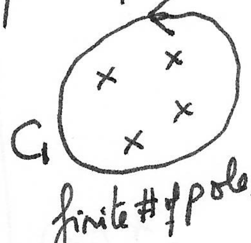
$$R^0(s+2\hbar) = \boxed{A(s)}\, R^0(s)$$

$$F(s+1) = \underbrace{R^0(s)_a F(s)}_{R^0_{trig}(e^{2\pi i s})}$$

$$\boxed{\xi(u) = \frac{u+\hbar}{u} \text{ on } |1\rangle}$$

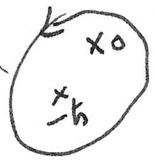
$$A(s) \to \exp\left(-\oint_{C_1} \underline{\underline{\xi(u)^{-1}\xi'(u)}} \otimes \boxed{\log \xi(u+\underline{\underline{s}}+\hbar)}\, du\right)$$

hol. outside of C_1



$A(s)$.

$$|11\rangle \quad \exp\left(-\int \left(\frac{1}{u+\hbar} - \frac{1}{u}\right) \log\left(\frac{u+s+2\hbar}{u+s+\hbar}\right)\right)$$



$$\frac{s+2\hbar}{s+\hbar}\left(\frac{s+\hbar}{s}\right)^{-1} = \frac{s(s+2\hbar)}{(s+\hbar)^2}$$

$$R^0(s)|11\rangle = \frac{\Gamma_{2\hbar}(s)\,\Gamma_{2\hbar}(s+2\hbar)}{\Gamma_{2\hbar}(s+\hbar)^2}$$

$$\Gamma_{2\hbar}(z) = \Gamma\left(\frac{z}{2\hbar}\right)$$

$$f(z+2\hbar) = \frac{z}{2\hbar} f(z)$$

$\uparrow \quad \Gamma\left(\frac{z}{2\hbar}\right)$

$\dfrac{1}{\Gamma\left(1-\frac{z}{2\hbar}\right)} \quad \downarrow$

$$\mathcal{R}(s) = \mathcal{R}^+(s)\,\boxed{\mathcal{R}^0(s)}\,\underline{\mathcal{R}^-(s)}$$

$$\mathcal{R}(\zeta) = \mathcal{R}^+(\zeta)\,\boxed{\mathcal{R}^0(\zeta)}\,\mathcal{R}^-(\zeta)$$

$$\zeta = e^{2\pi i s} \qquad \lim_{N\to\infty} \prod_{k=-N}^{N} \mathcal{R}^0(s+k)$$

$$\mathcal{R}^0(\zeta) = \boxed{\cdots\ \mathcal{R}^0(s-1)}\,\mathcal{R}^0(s)\,\boxed{\mathcal{R}^0(s+1)\ \cdots}$$

$(\mathcal{U}_q(\mathcal{L}g)$