Special parameters for rational Cherednik algebras
(Partly based on joint work with Charles Dunkl and Daniel Juteau)

Stephen Griffeth

Universidad de Talca

January 21, 2021

Outline

1. Cherednik algebras
2. Category $\mathcal{O}_c$
3. Problems

Reflection groups

Let $\mathfrak{h}$ be a finite-dimensional $\mathbb{C}$-vector space and let $W \subseteq \mathrm{GL}(\mathfrak{h})$ be a finite group of linear transformations of $\mathfrak{h}$.

The set of *reflections* in W is

$$R = \{r \in W \mid \mathrm{codim}(\mathrm{fix}(r)) = 1\}.$$

The linear group W is a *reflection group* if it is generated by R. In any case we define

$$\mathscr{A} = \{\mathrm{fix}(r) \mid r \in R\} \quad \text{and} \quad \mathfrak{h}^\circ = \mathfrak{h} \setminus \bigcup_{H \in \mathscr{A}} H.$$

Dunkl operators

For each $r \in R$ we fix a linear form α_r with zero set $\mathrm{fix}(r)$, and a collection $c = (c_r)_{r \in R}$ of complex numbers with the properties

$$c_r = c_{wrw^{-1}} \quad \text{and} \quad c_{r^{-1}} = \overline{c_r} \quad \text{for all } w \in W \text{ and } r \in R.$$

Given a vector $v \in \mathfrak{h}$, the corresponding *Dunkl operator* is defined by

$$D_v = \partial_v - \sum_{r \in R} c_r \frac{\alpha_r(v)}{\alpha_r}(1 - r),$$

which is an element of the algebra $D(\mathfrak{h}^\circ) \rtimes W$ generated by polynomial-coefficient differential operators on $\mathfrak{h}^\circ$ and the group W. Dunkl observed that they commute (for real reflection groups; Dunkl-Opdam later proved it in general).

The rational Cherednik algebra

The *rational Cherednik algebra* is the subalgebra $H_c = H_c(W, \mathfrak{h})$ of $D(\mathfrak{h}^\circ) \rtimes W$ generated by $\mathbb{C}[\mathfrak{h}]$, the group W, and the Dunkl operators D_v for $v \in \mathfrak{h}$.

Since the Dunkl operators commute, they give rise to a map $\mathbb{C}[\mathfrak{h}^*] \to H_c$, and the *PBW theorem* states that multiplication induces an isomorphism

$$\mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*] \cong H_c.$$

First questions

As with any algebra depending on a parameter, one hopes to obtain a description of the parameters c for which the rational Cherednik algebra satisfies (or does not satisfy) various structural properties. The simplest of these is: for which values of c is the ring H_c simple?

Another natural question is the following: we observe that by construction H_c acts on $\mathbb{C}[\mathfrak{h}]$. What is the set of c such that the H_c-module $\mathbb{C}[\mathfrak{h}]$ is simple?

These questions turn out to have nice answers, whose explanation requires a bit more machinery.

The category $\mathscr{O}_c$

The PBW theorem suggests, in analogy with Lie theory, that we should consider the category $\mathscr{O}_c$ consisting of finitely-generated H_c-modules on which each Dunkl operator D_v acts locally nilpotently.

The category $\mathscr{O}_c$ contains, for each $E \in \mathrm{Irr}(\mathbb{C}W)$, the *standard object*

$$\Delta_c(E) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*] \rtimes W}^{H_c}(E) \cong \mathbb{C}[\mathfrak{h}] \otimes E,$$

which comes equipped with a Hermitian *contravariant form* $\langle \cdot, \cdot \rangle_c$ compatible with the H_c-action. The quotient

$$L_c(E) = \Delta_c(E)/\mathrm{rad}(\langle \cdot, \cdot \rangle_c)$$

is the unique irreducible quotient of $\Delta_c(E)$, and these give a complete set of irreducible objects of $\mathscr{O}_c$.

Why category $\mathcal{O}_c$?

The category H_c-mod is very complicated. Certain structural features of H_c depend only on $\mathcal{O}_c$:

(a) Every finite-dimensional H_c-module is in $\mathcal{O}_c$.

(b) If I is a primitive ideal in H_c then $I = \mathrm{Ann}_{H_c}(L)$ for some irreducible object $L = L_c(E)$ of $\mathcal{O}_c$. (Ginzburg's version of Duflo's theorem).

(c) H_c is simple if and only if $\mathcal{O}_c$ is a semi-simple category.

(d) There is an analog of Schur-Weyl duality, known as the *Knizhnik-Zamolodchikov functor*, relating $\mathcal{O}_c$ to the category of modules over the *Hecke algebra* of W.

The braid group of W

The *braid group* of W is the fundamental group

$$B_W = \pi_1(\mathfrak{h}^\circ/W, p) \quad \text{for a fixed base-point } p \in \mathfrak{h}^\circ/W.$$

Given $H \in \mathscr{A}$ a reflecting hyperplane for W, there is a distinguished *generator of monodromy* $T_H \in B_W$. The group algebra of W is the quotient of the group algebra of B_W by

$$\prod_{i=0}^{n_H-1} (T_H - \zeta_H^i) = 0 \quad \text{for } H \in \mathscr{A},$$

where n_H is the order of the point-wise stabilizer W_H of H and $\zeta_H = e^{2\pi i/n_H}$ is a primitive n_Hth root of 1.

The Hecke algebra of W

Let $q = (q_{H,i})_{H \in \mathscr{A}, 0 \leq i \leq n_H - 1}$ be a collection of formal variables with the property

$$q_{H,i} = q_{w(H),i} \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \leq i \leq n_H - 1.$$

The *Hecke algebra* $\mathscr{H}_W$ of W is the quotient of the group algebra of the braid group B_W by the relations

$$\prod_{i=0}^{n_H - 1} (T_H - \zeta_H^i q_{H,i}) = 0 \quad \text{for } H \in \mathscr{A}.$$

Fiber functors

Each object of $\mathscr{O}_c$ is in particular a finitely-generated $\mathbb{C}[\mathfrak{h}]$-module, or in other words a coherent sheaf on $\mathfrak{h}$. Given $p \in \mathfrak{h}$, we therefore obtain a right-exact *fiber functor* to the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional $\mathbb{C}$-vector spaces,

$$F_p : \mathscr{O}_c \to \mathrm{Vect}_{\mathbb{C}}, \quad M \mapsto M(p).$$

When $p \in \mathfrak{h}^\circ$, this functor is actually exact and (by a case-by-case analysis, thanks to work of many mathematicians)

$$\mathrm{End}(F_p) \cong \mathscr{H}_{W,c}$$

where $\mathscr{H}_{W,c}$ is a certain specialization (at "$q = e^{2\pi i c}$") of the finite Hecke algebra $\mathscr{H}_W$ of W, which acts by monodromy on F_p.

The KZ functor

For any choice of $p \in \mathfrak{h}^\circ$ we may and will regard the functor F_p as taking values in $\mathscr{H}_{W,c}$-mod, and we refer to it as the *KZ functor*. The KZ functor:

(a) is represented by a projective object $P_{\mathrm{KZ},c}$ of $\mathscr{O}_c$,

(b) is fully faithful on projectives in $\mathscr{O}_c$ (that is, satisfies the *double centralizer property* familiar from Schur-Weyl duality).

Totally aspherical parameters

A direct description of $P_{\mathrm{KZ},c}$ is possible in certain cases (this is due to Losev): Let $A = \mathbb{C}[\mathfrak{h}^*]/\mathbb{C}[\mathfrak{h}^*]^W_{>0}$ be the co-invariant algebra of W and put

$$M_c = \mathrm{Ind}_{\mathbb{C}[\mathfrak{h}^*]\rtimes W}^{H_c}(A) \cong \mathbb{C}[\mathfrak{h}] \otimes A.$$

Thus by Frobenius reciprocity

$$\mathrm{Hom}(M_c, M) = (eM)^{\mathbb{C}[\mathfrak{h}^*]^W_{>0}}.$$

Losev has observed that this functor is isomorphic to the KZ functor (that is, $M_c \cong P_{\mathrm{KZ},c}$) if and only if the parameter is *totally aspherical*: that is, if and only if each object M of $\mathscr{O}_c$ which is *not* fully supported is killed by $M \mapsto eM$.

Aspherical parameters

A parameter c is *aspherical* if $H_c e H_c \neq H_c$, or in other words if the functor $M \mapsto eM$ from H_c-mod to $eH_c e$-mod is not an equivalence. It turns out to be enough to check this condition on $\mathcal{O}_c$. For the monomial groups $G(\ell, m, n)$, we know the set of aspherical values (Dunkl-G.) is a certain explicit finite union of hyperplanes in the parameter space but for most exceptional groups their calculation remains an important open problem.

The technique that is available for the groups $G(\ell, m, n)$ but not for the exceptional groups has to do with certain representation-valued orthogonal polynomials generalizing Jack polynomials.

A conjecture for totally aspherical parameters

Meanwhile, one might conjecture that c is totally aspherical provided the inequalities

$$-1 \leq c_{H,i} \leq 0 \quad \text{for all } H \in \mathscr{A} \text{ and } 1 \leq i \leq n_H - 1$$

all hold, where

$$c_{H,i} = \frac{1}{n_H} \sum_{r \in W_H} (1 - \det(r)^i) c_r.$$

Since the Hecke parameter in $\mathscr{H}_{W,c}$ is $q_{H,i} = e^{2\pi i c_{H,i}}$, this conjecture would imply the nice algebraic description

$$\mathscr{H}_{W,c} \cong \mathrm{End}_{H_c}(\mathbb{C}[\mathfrak{h}] \otimes A)$$

of the Hecke algebra (for *any* Hecke parameter, we can choose a compatible Cherednik parameter in the range $[-1,0]$).

Finite-dimensional representations

One of the most natural questions one might ask is: what are the finite-dimensional irreducible H_c-modules? It amounts to the same thing to ask: for which pairs E and c is $L_c(E)$ finite-dimensional?

For the group $G(\ell, m, n)$ and a fixed parameter c, one can compute the set of E for which $L_c(E)$ is finite-dimensional using crystal operators on level ℓ Fock space (Shan-Vasserot, previously conjectured by Etingof).

It seems to be a difficult problem in general to fix E first, and then find the set of c for which $L_c(E)$ is finite dimensional. But for $E = \mathrm{triv}$ (and for certain other cases for the monomial groups, due to Gerber-Norton) we have an answer.

Varagnolo-Vasserot and Etingof's work

For W a Weyl group and c_r constant, Varagnolo-Vasserot proved that $L_c(\mathrm{triv})$ is finite-dimensional if and only if c is a positive rational number whose denominator is an *elliptic number* for W (technique: realize the Cherednik algebra as a convolution algebra coming from an affine Steingberg variety).

Etingof settled the case in which W is a real reflection group, and the parameter is no longer assumed constant. He used a formula relating the contravariant form on $L_c(\mathrm{triv})$ to the *Macdonald-Mehta integral* that arising in statistical mechanics. Etingof's work gives Varagnolo-Vasserot's as a corollary (though this is not trivial).

Poincaré polynomials

The *Poincaré polynomial* $P_W(q)$ of W is the graded dimension of the coinvariant algebra of W. If W is a Coxeter group, this is also equal to the *length generating function* for W, and it factors as

$$P_W(q) = \prod_{i=1}^{n} [d_i]_q \quad \text{where } d_1, \ldots, d_n \text{ are the } \textit{degrees} \text{ of } W$$

and $[d_i]_q = 1 + q + q^2 + \cdots + q^{d_i - 1}$ is the q-analog of d_i. In the Coxeter case with two conjugacy classes of reflections, there is a two-variable version of the length-generating function P_W in which one distinguishes between conjugacy classes of simple reflections.

The principal Schur elements

For a complex reflection group, there is another q-analog of the order of W which turns out to be important for us. Namely, assuming the existence of a *symmetrizing trace* on the Hecke algebra with certain favorable properties (as conjectured by Broué), one may associate a *Schur element* to each irreducible representation of W, which is a polynomial in the Hecke parameters. The Schur element S_W of the trivial representation is then the desired q-analog of $|W|$, and is equal to P_W if W is a Coxeter group.

When is $L_c(\mathrm{triv})$ finite dimensional?

With Daniel Juteau we prove:

Theorem (G. - Juteau)

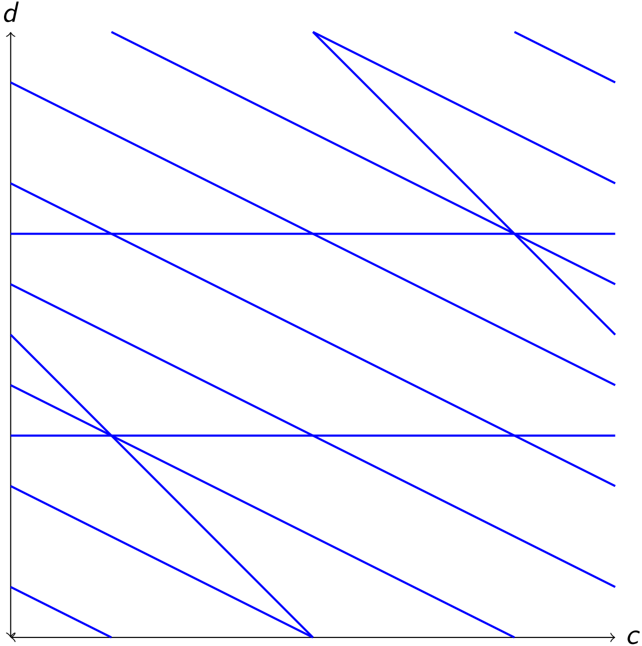
The representation $L_c(\mathrm{triv})$ is finite dimensional if and only if for each maximal parabolic subgroup $W' < W$, the parameter c lies on a positive hyperplane H such that $(S_W/S_{W'})(H) = 0$.

The method obtains this theorem as a corollary of a more general theorem which is independent of the symmetrizing trace conjecture and relates the support of a module in category $\mathcal{O}_c$ to certain multi-variable analogs of Bessel functions, the *Dunkl exponential functions*. Etingof's result is a consequence.

It would be very interesting to carry out a similar program for other representations E (but this will require several new ideas).

The group G_{13}

As an example, we might consider the two-dimensional irreducible reflection group labeled G_{13} in the Shephard-Todd notation. It has two conjugacy classes of reflections, all of order 2, and we write c and d for the corresponding Cherednik parameters. On the next page we draw the set of points $0 \leq c, d \leq 1$ for which $L_c(\mathrm{triv})$ is finite-dimensional in this case.



d
c