Cherednik algebras and quasi-invariants
(Following Berest-Chalykh)

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Outline

Reflection groups

Let $\mathfrak{h}$ be a finite-dimensional $\mathbb{C}$-vector space and let $W \subseteq \mathrm{GL}(\mathfrak{h})$ be a finite group of linear transformations of $\mathfrak{h}$.

The set of *reflections* in W is

$$R = \{r \in W \mid \mathrm{codim}(\mathrm{fix}(r)) = 1\}.$$

The linear group W is a *reflection group* if it is generated by R. These are characterized by:

Theorem

A finite group $W \subseteq \mathrm{GL}(\mathfrak{h})$ is a reflection group if and only if the ring $\mathbb{C}[\mathfrak{h}]^W$ of W-invariant polynomial functions on $\mathfrak{h}$ is a polynomial ring (in other words, can be generated by $n = \dim(\mathfrak{h})$ algebraically independent polynomials).

The monomial groups

Here is one large class of examples. Let ℓ and n be positive integers, and let $G(\ell,1,n)$ be the set of n by n matrices with precisely one non-zero entry in each row and each column, such that the non-zero entries are ℓth roots of 1.

Given a positive divisor m of ℓ, we let $G(\ell,m,n)$ be the subgroup of $G(\ell,1,n)$ consisting of those matrices such that the product of their non-zero entries is an (ℓ/m)th root of 1.

These are the *monomial* or sometimes *cyclotomic* or *classical type* reflection groups. The ring of W-invariant polynomials of $W = G(\ell,1,n)$ is

$$\mathbb{C}[x_1,\dots,x_n]^{G(\ell,1,n)} = \mathbb{C}[x_1^\ell,\dots,x_n^\ell]^{S_n}.$$

The reflections in $G(\ell, 1, n)$

The reflections in $G(\ell, 1, n)$ are of two types. We fix ζ, an ℓth root of 1, and for $1 \leq i \leq n$ write ζ_i for the diagonal matrix with ζ in position i and all other diagonal entries equal to 1. We write s_{ij} for the transposition matrix that interchanges the ith and jth basis vectors and fixes all the other basis vectors. Then

$$R = \{\zeta_i^k \mid 1 \leq i \leq n,\ 1 \leq k \leq \ell - 1\} \cup \{\zeta_i^k s_{ij} \zeta_i^{-k} \mid 1 \leq i < j \leq n,\ 0 \leq k \leq \ell - 1\}.$$

Rank one reflection subgroups

Let W be a reflection group, let

$$\mathscr{A} = \{\mathrm{fix}(r) \mid r \in R\}$$

be the set of reflecting hyperplanes for reflections in W, and for $H \in \mathscr{A}$ let

$$W_H = \{w \in W \mid w(p) = p \ \forall \ p \in H\}$$

be the subgroup fixing H point-wise.

The group W_H is cyclic of order $n_H = |W_H|$, and its group of linear characters consists of the powers of the determinant:

$$W_H^\vee = \{\det^k \mid 0 \leq k \leq n_H - 1\} \cong \mathbb{Z}/n_H.$$

Idempotents

For $0 \le i \le n_H - 1$ we write

$$e_{H,i} = \frac{1}{n_H} \sum_{w \in W_H} \det(w)^{-i} w$$

for the idempotent in $\mathbb{C}W_H$ that projects onto the $\det^i$-isotypic component of a representation. Thus a polynomial function $f \in \mathbb{C}[\mathfrak{h}]$ is W-invariant if and only if

$$e_{H,i} \cdot f = 0 \quad \text{for all } H \in \mathscr{A} \text{ and all } 1 \le i \le n_H - 1.$$

We will weaken this condition somewhat to define *quasi-invariant* polynomials.

Quasi-invariants

The definition of quasi-invariants depends on a *multiplicity function* on $\mathscr{A}$, which consists of a collection $m = (m_{H,i})_{H \in \mathscr{A},\ 0 \le i \le n_H - 1}$ of integers $m_{H,i}$ with the property

$$m_{H,i} = m_{w(H),i} \quad \text{for all } w \in W,\ H \in \mathscr{A}, \text{ and } 0 \le i \le n_H - 1.$$

Letting $\mathfrak{h}^\circ$ be the complement to the union of the reflecting hyperplanes for W in $\mathfrak{h}$, a polynomial function $f \in \mathbb{C}[\mathfrak{h}^\circ]$ is an *m-quasi-invariant* provided

$$v_H(e_{H,-i} \cdot f) \ge m_{H,i} \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \le i \le n_H - 1,$$

where $v_H(f)$ is the order of vanishing of f along H.
[**Warning**: this definition is different to and a bit more general than that of Berest-Chalykh; the extra flexibility serves to make the statement of their result more natural and general.]

Representation-valued quasi-invariants

To make the connection with representation theory of Cherednik algebras, we fix a $\mathbb{C}W$-module E, and define an *E-valued quasi-invariant* to be a polynomial function $f \in \mathbb{C}[\mathfrak{h}^\circ] \otimes E$ such that

$$v_H((1 \otimes e_{H,i}) \cdot f) \geq m_{H,i} \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \leq i \leq n_H - 1.$$

We write $Q_m \subseteq \mathbb{C}[\mathfrak{h}^\circ]$ for the space of m-quasi-invariants and $Q_m(E)$ for the space of E-valued m-quasi-invariants, and let $e = \frac{1}{|W|} \sum_{w \in W} w$ be the averaging idempotent for W. The key point relating the definitions is:

Lemma

Taking $E = \mathbb{C}W$ to be the regular representation, we have

$$eQ_m(\mathbb{C}W) = e(Q_m \otimes 1)$$

as subspaces of $\mathbb{C}[\mathfrak{h}^\circ] \otimes \mathbb{C}W$.

Dunkl operators

For each $H \in \mathscr{A}$ we fix a linear form α_H with zero set H, and a collection $c = (c_{H,i})_{H \in \mathscr{A}, 0 \leq i \leq n_H - 1}$ of complex numbers with the property

$$c_{H,i} = c_{w(H),i} \quad \text{for all } w \in W \text{ and } H \in \mathscr{A}.$$

Given a vector $v \in \mathfrak{h}$, the corresponding *Dunkl operator* is defined by

$$D_v = \partial_v - \sum_{H \in \mathscr{A}} \frac{\alpha_H(v)}{\alpha_H} \sum_{i=0}^{n_H - 1} n_H c_{H,i} e_{H,i},$$

which is an element of the algebra $D(\mathfrak{h}^\circ) \rtimes W$ generated by differential operators on $\mathfrak{h}^\circ$ and the group W.
[**Warning**: here again our definition is slightly more general than is standard, since we allow $c_{H,0} \neq 0$. These Dunkl operators commute, but do not preserve the space of polynomial functions in general.]

The rational Cherednik algebra

The *rational Cherednik algebra* is the subalgebra $H_c = H_c(W, \mathfrak{h})$ of $D(\mathfrak{h}^\circ) \rtimes W$ generated by $\mathbb{C}[\mathfrak{h}]$, the group W, and the Dunkl operators D_v for $v \in \mathfrak{h}$. The algebra $D(\mathfrak{h}^\circ) \rtimes W$ acts on $\mathbb{C}[\mathfrak{h}^\circ] \otimes E$ for each representation E of $\mathbb{C}W$, and a calculation shows:

Lemma

If the parameter c and the multiplicity function m satisfy

$$n_H c_{H,i-m_{H,i}} = m_{H,i} \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \le i \le n_H - 1,$$

then $Q_m(E)$ is an H_c-submodule of $\mathbb{C}[\mathfrak{h}^\circ] \otimes E$.

We say that c and m are *compatible* if they satisfy this condition. For a given m there is at most one compatible c, and for a given c there is at most one compatible m.

The spherical subalgebra

The *spherical subalgebra* of the rational Cherednik algebra is the idempotent slice algebra $eH_c e$, where as above e is the averaging idempotent for the group W. On the other hand, we can consider the algebra $D(Q_m)$ of differential operators on $\mathbb{C}[\mathfrak{h}^\circ]$ that preserve Q_m.

Combining the previous lemma with the relationship

$$eQ_m(\mathbb{C}W) = e(Q_m \otimes 1)$$

we obtain

Theorem (Berest-Chalykh)

Suppose that c and m are compatible. As subalgebras of $e(D(\mathfrak{h}^\circ) \rtimes W)e$ we have

$$eH_c e = D(Q_m)^W e.$$

Varying c and m, but not Q_m

The space Q_m is somewhat insensitive to the choice of m. Specifically:

Lemma

Suppose $m = (m_{H,i})_{H \in \mathscr{A},\ 0 \le i \le n_H - 1}$ and $k = (k_{H,i})_{H \in \mathscr{A},\ 0 \le i \le n_H - 1}$ are multiplicity functions such that for all $H \in \mathscr{A}$ and $0 \le i \le n_H - 1$, we have

$$\lceil (m_{H,i} - i)/n_H \rceil = \lceil (k_{H,i} - i)/n_H \rceil .$$

Then $Q_m = Q_k$.

The dot action: equality of spherical algebras

We now define a certain group S_W of permutations of the parameter space as follows: an element of this group is a collection $\phi = (\phi_H)_{H \in \mathscr{A}}$ of permutations ϕ_H, where ϕ_H is a permutation of the set $\{0, 1, 2, \dots, n_H - 1\}$ and $\phi_H = \phi_{w(H)}$ for all $w \in W$ and $H \in \mathscr{A}$. We also define ρ to be the parameter with

$$\rho_{H,i} = i/n_H \quad \text{for all } H \in \mathscr{A} \text{ and } 0 \leq i \leq n_H - 1.$$

The *dot action* of $\phi \in S_W$ on c is given by

$$\phi \cdot c = \phi(c + \rho) - \rho.$$

Theorem

For all parameters c and all $\phi \in S_W$ we have

$$eH_c e = eH_{\phi \cdot c} e.$$

Idea of the proof

The point of the proof is that for a certain set of integral parameters c, there is a multiplicity function m compatibility with c and a multiplicity function k compatible with $\phi \cdot c$ such that

$$Q_m = Q_k \quad \Longrightarrow \quad eH_c e = D(Q_m)^W e = D(Q_k)^W e = eH_{\phi \cdot c} e.$$

This set of parameters is Zariski-dense, and so the preceding equality holds for *all* c.

Consequences for representation theory

The equality $eH_c e = eH_{\phi\cdot c} e$ produces a right-exact functor

$$H_c-\mathrm{mod} \to H_{\phi\cdot c}-\mathrm{mod} \quad \text{given by} \quad M \mapsto H_{\phi\ \cdot c}\, e \otimes_{eH_{\phi\cdot c}e} eM$$

which for abstract reasons is an equivalence if and only if the functor $M \mapsto eM$ is an equivalence (and in any case one might conjecture that it induces a derived equivalence).

A parameter c for which this functor is *not* an equivalence is called *aspherical*. It is an open problem to determine the set of aspherical parameters, but solved (Dunkl-G.) for the case of the groups $G(\ell, m, n)$.