KHOVANOV HOMOLOGY
sl(n) HOMOLOGIES
and
RIBBON CONCORDANCE

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joint work with

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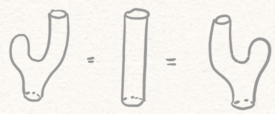
I. LINK HOMOLOGY

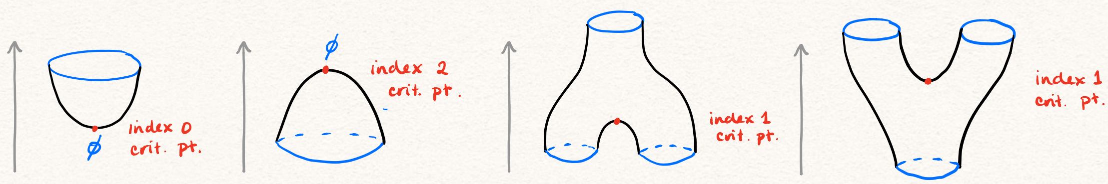
$\mathbb{Z} \hookrightarrow \mathbb{Z}[x]/x^2$

Each rank 2 Frobenius system $\mathcal{F} = (R, A, \varepsilon, \Delta)$ defines a TQFT:

topological quantum field theory



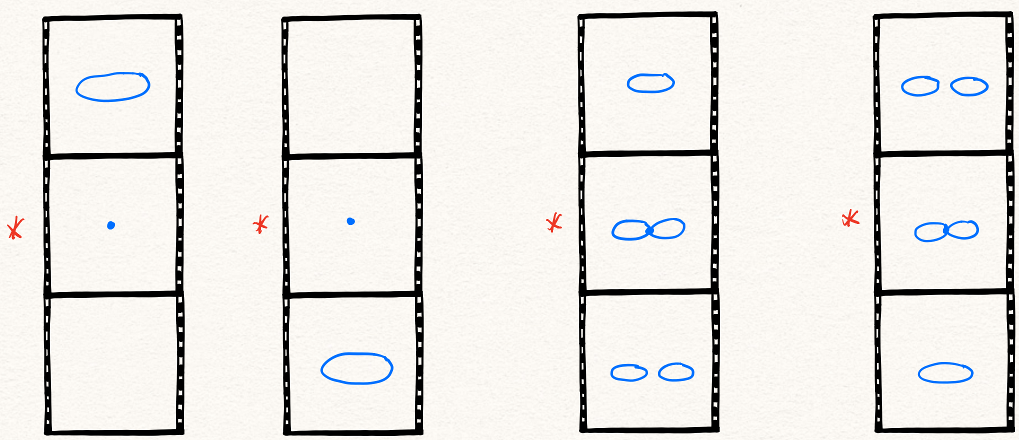
$\iota : R \hookrightarrow A$ $\qquad$ $\varepsilon : A \longrightarrow R$ $\qquad$ $m : A \otimes A \longrightarrow A$ $\qquad$ $\Delta : A \longrightarrow A \otimes A$



"saddles"

As a movie:

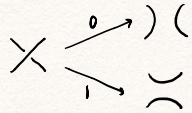
"birth of a circle" $\qquad$ "death of a circle" $\qquad$ "merging of two circles" $\qquad$ "splitting of a circle"

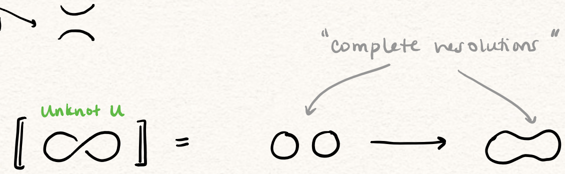


LINK HOMOLOGIES

$$(\text{Links}_{\text{in } S^3}, \text{Cobordisms}_{\text{in } S^3 \times I}) \longrightarrow \mathbb{Z}\text{-mod}$$

① **Khovanov Homology**: $R = \mathbb{Z}$, $A = \mathbb{Z}[X]/(X^2)$ [Khovanov '00]





$$A \otimes A \xrightarrow{m} A$$

$1 \otimes 1 \longmapsto 1$

$X \otimes 1$, $1 \otimes X \longmapsto X$

$X \otimes X$

$Kh(U)$

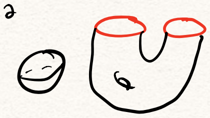
② **Universal Khovanov Homology** [Khovanov '04]

$$R = \mathbb{Z}[h,t] \qquad A = \mathbb{Z}[h,t,X]/\underbrace{(X^2 - hX - t)}_{\text{"potential"}}$$

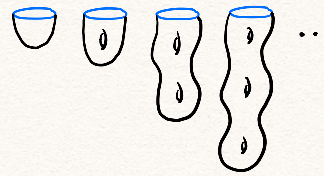
Can define and compute these topologically via the TQFT: [Bar-Natan '05]

Instead of chain complex:

$$\text{Homology} = \frac{\ker \partial}{\text{im } \partial}$$

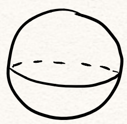
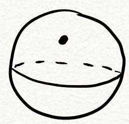
Use a bilinear form: 

eg. resolution =

surfaces = 

$$\text{Homology} = \frac{\left\{ \begin{array}{c} \text{all resolutions and all the} \\ \text{surfaces they bound} \\ \text{(with dots to represent ``x'')} \end{array} \right\}}{\text{a bilinear form}}$$

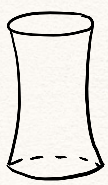
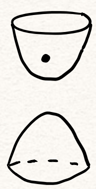
describe as relations, ie evaluation of closed surfaces

 = 0  = 1

 = 0.

(Sphere relation S)

↳ evaluates

 =  + 

(Neck-cutting relation NC)

↳ reduces genus

$$\mathbb{Z}[x]/(x^2) = \mathbb{Z}\langle 1, x \rangle$$

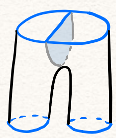
What about higher rank Frob. system?

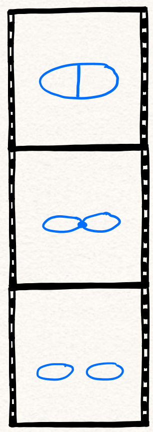
③ $sl(3)$ homology:

eg. $A = \mathbb{Z}[X]/(X^3)$ topologically: need to use webs and foams:

"potential"

[Khovanov '04]





④ <u>Remark</u>. $sl(n)$ homologies categorify the $sl(n)$ link polynomials.

[Khovanov] $sl(2)$ homology (aka Khovanov Homology)

[Khovanov-Rozansky] $sl(n)$ homology (matrix factorizations)

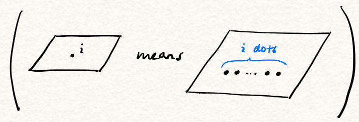
Functoriality for these theories: see work of

Jacobsson,
Bar-Natan,
Caprau,
Mackaay-Vaz,
Clark,
Mackaay-Stošić-Vaz
...

In fact, there is a topological description (webs + foams + dots) for:

any potential {
- universal $sl(2)$ [Bar-Natan]
- universal $sl(3)$ [MacKaay - Vaz]
- $sl(n)$ homology for $\mathbb{Z}[X]/(X^n)$ [MacKaay - Stošić - Vaz]

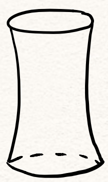
There are many more relations, (eg. more dots = more ways to decorate; we have singular trivalent edges now) but we still have the sphere and neck-cutting relations:



S_n

$= \begin{cases} 1 & i = n-1 \\ 0 & \text{otherwise} \end{cases}$

NC_n



$= \sum_{i=0}^{n-1}$

All we need is the above relations + functoriality to show that these theories give an obstruction to a ribbon concordance from one link to another.

II. KNOTS IN S^3, SURFACES IN B^4

3D: $K \subset S^3$ $\qquad$ $(i_K : S^1 \hookrightarrow S^3)$ $\qquad$ (many uses)

4D: $(F, \partial F = K) \subset (B^4, \partial B^4 = S^3)$

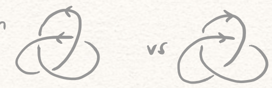
(!!!)

We can make the set of knots into a group if we mod out by concordance:

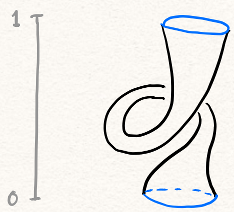
<u>defn.</u> A knot K_0 is <u>concordant</u> to a knot K_1 if there exists an annulus $A \approx S^1 \times I \hookrightarrow S^3 \times I$ such that

$$\partial A = K_0 \cup -K_1$$

$-$ ↖ opp. orientation



<u>notice</u> This is an equivalence relation. $(S^3 \times I \longrightarrow S^3 \times (-I) \approx S^3 \times I)$



$\rightsquigarrow$ knot concordance group $\mathcal{C}$

eg. If K is concordant to the unknot, then K is <u>slice</u>, ie bounds a disk in B^4
(very significant property in applications to 3-, 4- manifolds)
(Then $[K]$ is the identity in $\mathcal{C}$.)

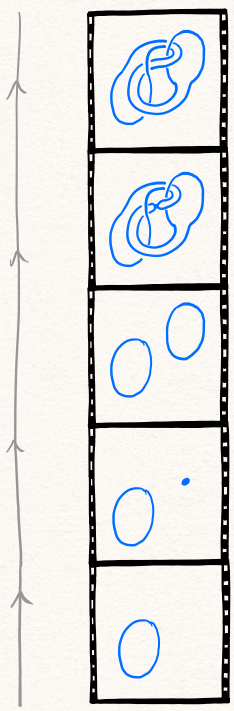
Q. Within each equivalence class, can we partially order the knots?

Conjecture [Gordon '81] Ribbon concordance gives a partial ordering.

defn. A ribbon concordance from K_0 to K_1 is a concordance with only index 0, 1 critical points, ie. its movie has only births and saddles (no deaths)

(concordance: no genus either!)

The movie should look like



join to make Stevedore knot (6_1)

meet

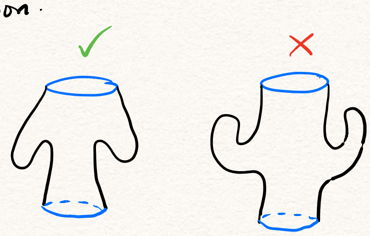
$$\chi(F) = \#\text{ births} - \#\text{saddles} = 0.$$

$\Rightarrow$ only $\underbrace{\text{births}}_{\text{create circles}}$ and $\underbrace{\text{merges}}_{\text{bands connect circles}}$

birth of a friend (very happy movie)

lonely circle (unknot)

Cartoon:



Conjecture [Gordon '81] Ribbon concordance gives a partial ordering,

Q. If there are ribbon concs. $K_0 \longrightarrow K_1$ and $K_1 \longrightarrow K_0$, then does $K_0 = K_1$?

as (isotopy classes of) knots in S^3

"Classical" results (80s)

[Gordon] If F is a ribbon conc from $K_0 \to K_1$, then

- $\pi_1(S^3 \setminus K_0) \hookrightarrow \pi_1(B^4 \setminus F)$
- $\pi_1(B^4 \setminus F) \twoheadrightarrow \pi_1(S^3 \setminus K_1)$

February 2019

[Zemke] If $K_0 \xrightarrow{\text{ribbon}} K_1$, then $\widehat{HFK}(K_0) \hookrightarrow \widehat{HFK}(K_1)$ as a direct summand.

many other papers followed; notably, for today:

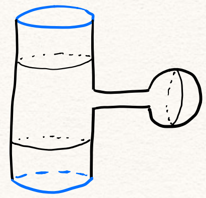
- [Levine-Zemke] Kh
- [Sarkar] a measure of "ribbon distance" using Kh
- [Kang] TQFTS that satisfy a neck-passing relation (very general!)
- [CGLLSZ] $sl(n)$ web/foam homologies

They all relied on Zemke's critical topological lemma:

Lemma [Zemke]
Let C be a ribbon concordance $L_0 \to L_1$ (happy movie)
Let $\bar{C}$ be the upside down concordance $L_1 \to L_0$. (very sad movie)

Then $\bar{C} \circ C : L_0 \to L_0$ is isotopic to the surface (concordance) $F: L_0 \to L_0$ obtained by

- starting with the identity cobordism $L_0 \times I$
- "tubing on" some unknotted, unlinked spheres.



III. RESULT

thm. [CGLLSZ, Kang; cf. Levine-Zemke]

Let $\mathcal{H}$ denote one of the following homology-type invariants:

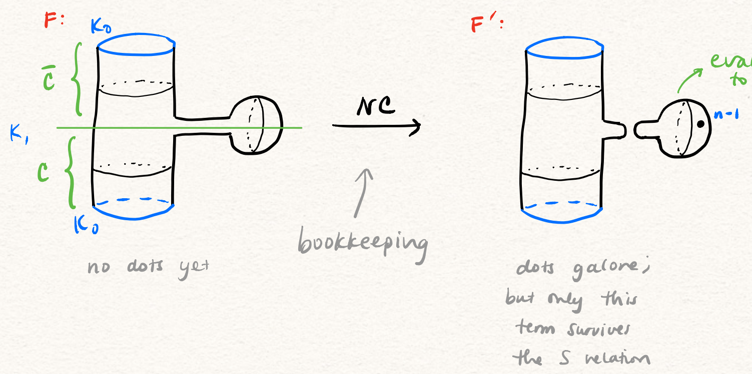
- universal $sl(2)$
- universal $sl(3)$
- web/foam $sl(n)$

Let C be a ribbon concordance $C: K_0 \longrightarrow K_1$.
Then, for any bigrading $(gr_{homological}, gr_{quantum}) = (i,j)$,
$\mathcal{H}^{i,j}(K_0)$ embeds in $\mathcal{H}^{i,j}(K_1)$ as a direct summand.

eg. $U \longrightarrow 6_1$

In practice, these would be used as obstructions to ribbon concordance.

Proof Sketch



$\Rightarrow \mathcal{H}(L_0 \times I) = \mathcal{H}(F')$

$\Rightarrow \mathcal{H}(C): \mathcal{H}(K_0) \longrightarrow \mathcal{H}(K_1)$ is injective, with left-inverse given by $\mathcal{H}(\bar{C})$. $\square$