Combinatorics, Categorification, and Crystals

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Fall Western Sectional Meeting – UCR

I. The What? Why? When? How? of *Categorification*

II. Focus on combinatorics of symmetric group $\mathcal{S}_n$, crystal graphs, quantum groups $U_q(\mathfrak{g})$

from searching titles . . .

Special Session on Combinatorial Aspects of the Polynomial Ring

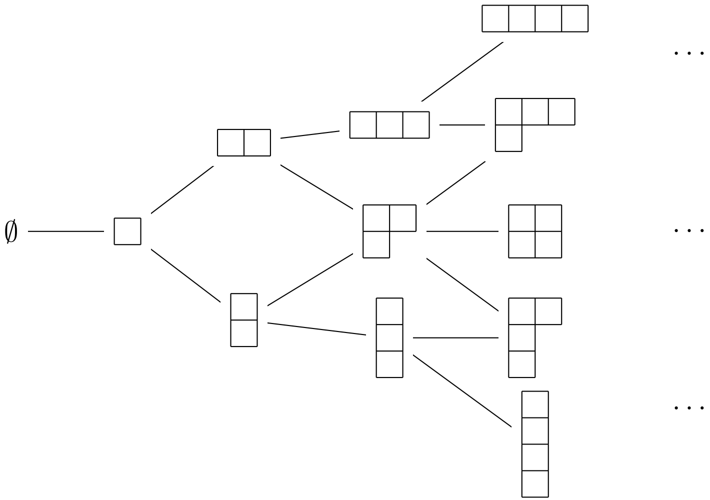
Special Session on Combinatorial Representation Theory

Special Session on Non-Commutative Birational Geometry, Cluster Structures and Canonical Bases

Special Session on Rational Cherednik Algebras and Categorification

and more $\cdots$

Combinatorics



Representation theory of the symmetric group

Branching Rule: Restriction and Induction

The graph encodes the functors Res^n_{n-1} and Ind^n_{n-1} of induction and restriction arising from

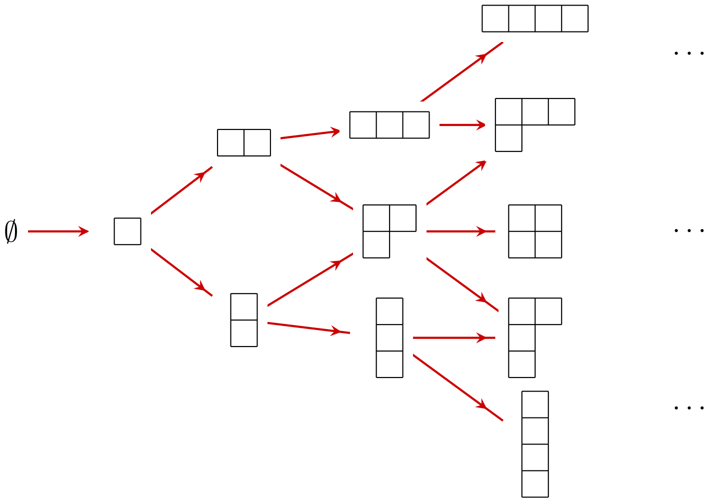
$$\mathcal{S}_{n-1} \subseteq \mathcal{S}_n.$$

Example (multiplicity free)

$$\mathrm{Res}^{\mathcal{S}_4}_{\mathcal{S}_3} S^{(3,1)} = S^{(3)} \oplus S^{(2,1)}$$

or $\dim \mathrm{Hom}(S^{(3)}, \mathrm{Res}^4_3 S^{(3,1)}) = 1, \quad \dim \mathrm{Hom}(S^{(2,1)}, \mathrm{Res}^4_3 S^{(3,1)}) = 1$

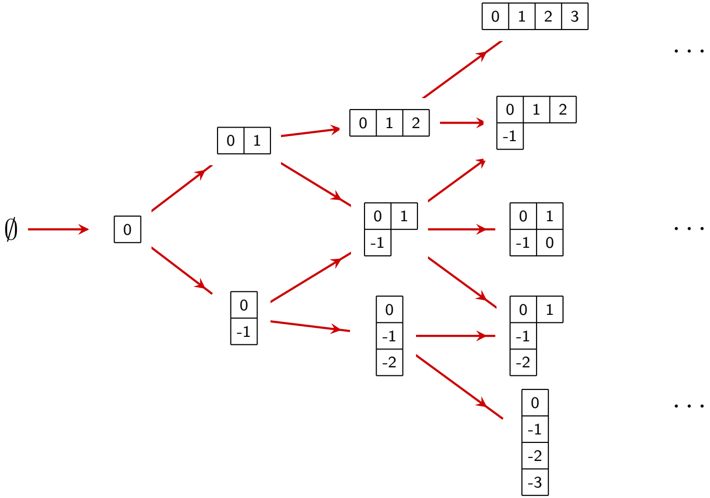
label diagonals



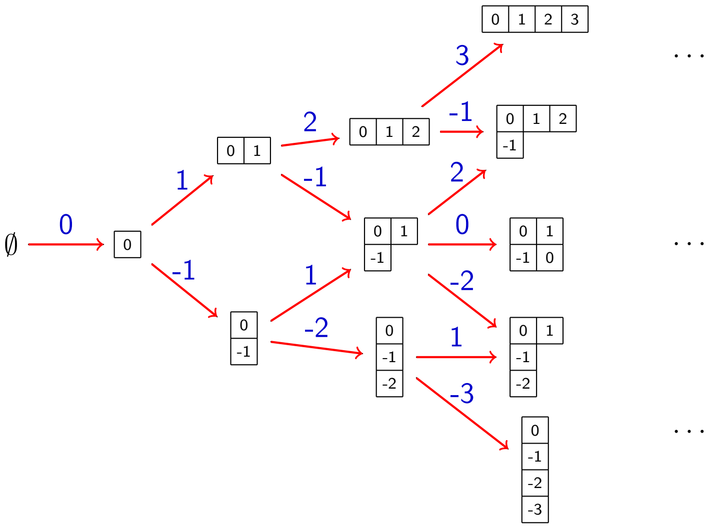
label diagonals

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \\ -1 & 0 & 1 & 2 & \cdots \\ -2 & -1 & 0 & 1 & \cdots \\ & & & 0 & \cdots \\ \vdots & & & & \end{array}$$

label diagonals



label edges via diagonals



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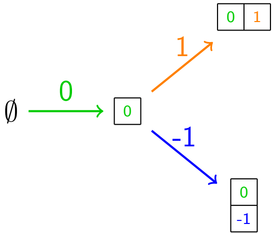
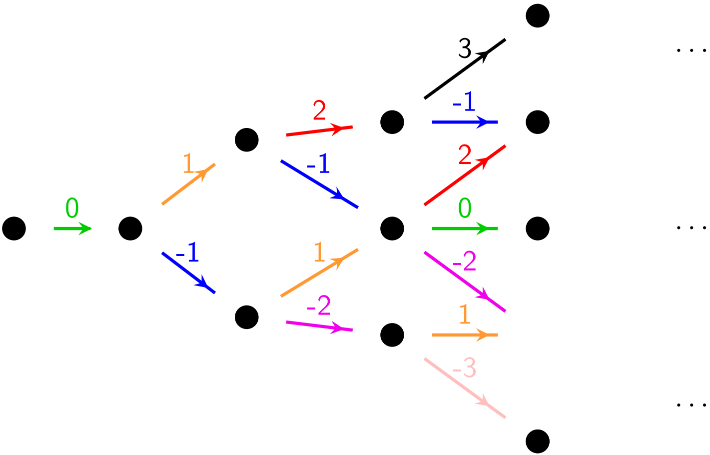
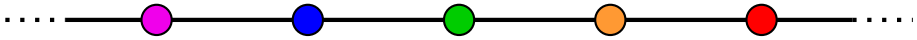


Crystal graph $B(\Lambda_0)$ of $V(\Lambda_0)$ module for $\mathfrak{sl}_\infty$



$\mathbb{I}$-colored graph encoding action of E_i, F_i for $i \in I = \mathbb{Z}$



$\mathfrak{sl}_\infty$ acts on the representation theory of symmetric groups
generators act as induction, restriction

OR start with $V(\Lambda_0)$ and ...

Do we see this anywhere else? Can we repeat/generalize this? Should we?

(1) The What?

What is *categorification*?

Replace set-theoretic statements by their category-theoretic analogues. Sets become categories, functions become functors, equations become natural isomorphisms of functors.
The term "categorification" was introduced by L. Crane and I. Frenkel to describe the process of realizing certain algebraic structures as shadows of richer higher ones. Their original motivation was to combinatorially understand geometric constructions of quantum groups in order to produce four-dimensional topological quantum field theories (TQFTs), replacing the algebraic structure of a quantum group with a categorical analog, or "categorified quantum group."

What is *categorification*?

Decategorification

Given $\mathcal{C}$ take its Grothendieck group $G_0(\mathcal{C})$, split Grothendieck group $K_0(\mathcal{C})$, or Trace $\mathrm{Tr}(\mathcal{C})$ aka zeroth Hochschild homology or cocenter

$G_0(\mathcal{C}) =$

$$\{\text{ isom classes of objects } [A]\,\}/\langle\, [B] = [A] + [C] \text{ if } 0 \to A \to B \to C \to 0\,\rangle$$

Often take $G_0(R\text{-fmod})$, $K_0(R\text{-pmod})$

Geometric setting

Given topological space X take homology groups, or equivariant cohomology, or Euler characteristic ...

Decategorification

Example

$$\mathcal{C}\text{ategory} \;= \texttt{Vect}$$

$$G_0(\texttt{Vect}) \simeq \mathbb{Z} \qquad \text{inherits } \bigoplus, \bigotimes$$

Example

$$\mathcal{C}\text{ategory} \;= \texttt{gr}-\texttt{Vect}$$

$$G_0(\texttt{gr}-\texttt{Vect}) \simeq \mathbb{Z}[q, q^{-1}]$$

Decategorification

Example

$$\mathcal{C}\text{ategory} = R\text{−mod}$$

$$G_0(R\text{−mod}) \simeq \mathbb{Z}\text{-span of [simple]s} = \mathbb{Z}\text{-span of characters}$$

Example

$$R = \mathbb{Q}\mathcal{S}_4$$

$$G_0(\mathbb{Q}\mathcal{S}_4\text{−mod}) \simeq \mathbb{Z}[S^\mu \mid \mu \vdash 4] = \mathbb{Z}[\chi^\mu \mid \mu \vdash 4]$$

$$V \simeq \mathbb{C} \otimes_{\mathbb{Z}} G_0(\mathbb{Q}\mathcal{S}_4\text{−mod})$$

comes with an integral form coming from the basis $\{[\mathit{simple}]\}$ indexed by the 5 partitions $\mu \vdash 4$.

What categorifies Symmetric functions?

$\mathcal{S}$ym is a graded algebra with graded dimension
$\mathcal{P}(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots$

$$G_0(?) \simeq {}_{\mathbb{Z}}\,\mathcal{S}\mathrm{ym} = \bigoplus_{n\geq 0} {}_{\mathbb{Z}}\,\mathcal{S}\mathrm{ym}^n$$

The symmetric groups *categorify* the ring of symmetric functions

$$\bigoplus_{n\geq 0} G_0(\mathbb{Q}\mathcal{S}_n\mathrm{-mod}) \simeq \bigoplus_{n\geq 0} {}_{\mathbb{Z}}\,\mathcal{S}\mathrm{ym}^n =: {}_{\mathbb{Z}}\,\mathcal{S}\mathrm{ym}$$

$\chi^\mu \mapsto s_\mu,$ the Schur function

via the Frobenius characteristic map

$$\mathrm{Ind}_{\mathcal{S}_m \times \mathcal{S}_n}^{\mathcal{S}_{m+n}} \chi^\mu \boxtimes \chi^\nu := \chi^\mu \circ \chi^\nu \mapsto s_\mu s_\nu$$

Hopf algebra coproduct $\Delta(\chi^\lambda) = \sum_{m+n=|\lambda|=:\ell} \mathrm{Res}^{\ell}_{m,n}\,\chi^\lambda$

Categorification

Example (Khovanov)

Khovanov homology categorifies the Jones polynomial, a quantum knot invariant.
More specifically it is a graded homology theory giving rise to a collection of graded vector spaces whose graded Euler characteristic is the Jones polynomial.

Example (Soergel)

The category of Soergel bimodules categorifies the Hecke algebra associated to an arbitrary Coxeter system.
Began with seminal work of Kazhdan and Lusztig (1979)

Example (Khovanov-Lauda, Rouquier)

KLR algebras categorify quantum groups

(1.5) The Who?

I'll miss someone if I try to list them all . . .

Many are here this weekend

See Lauda `arxiv.org/abs/1106.2128` for a nice survey

(2) The Why?

Example (This is explained by the rich structure of morphisms)

Khovanov homology is a strictly stronger knot invariant than the Jones polynomial. It is a functorial knot invariant which means that knot cobordisms induce maps between Khovanov homologies.

Example (surpass geometry)

The Hecke algebra associated to the Weyl group of a reductive algebraic group was geometrically realized by Kazhdan-Lusztig. Soergel introduced an algebraic reinterpretation that allowed for the categorification of the Hecke algebra associated to an arbitrary Coxeter system.
Used by Elias-Williamson to prove Kazhdan-Lusztig conjectures.

The Why?

Why would one want to *categorify*?

Example (surpass geometry)

The categorification of $\mathcal{U}_q^-(\mathfrak{g})$ associated non-symmetric Cartan data gives rise to necessarily positive bases; the canonical bases associated to non-symmetric Cartan data are known to violate positivity, so here, categorification gives rise to new results not accessible from geometry.

Example (applications)

Chuang and Rouquier showed that the higher structure of natural transformations in categorical $U(\mathfrak{sl}_2)$-actions played an important role in the construction of derived equivalences (Rickard); proved Broue's abelian defect conjecture.

Functoriality of Khovanov homology was used by Rassmussen to give a purely combinatorial proof of the Milnor conjecture.

The Why?

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

with $c^\nu_{\lambda,\mu} \in \mathbb{Z}_{\geq 0}$
Easy to see if we know $s_\lambda = [S^\lambda]$.

At its best, categorification in representation theory improves our understanding of the original algebraic structure, explaining positivity and integrality properties, canonical bases, symmetries, and nondegenerate bilinear forms.

The Why?

Why categorify quantum groups?

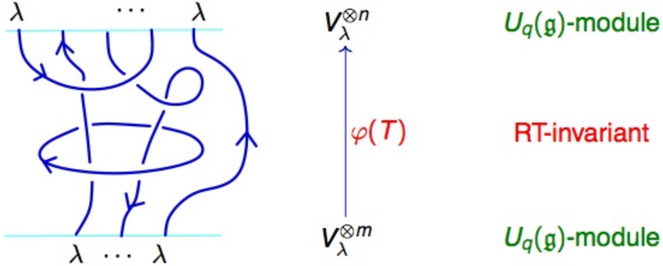
Reshetikhin-Turaev invariants

Quantum knot invariants such as the Jones polynomial, its generalizations to the colored Jones polynomial, and the HOMFLYPT polynomial are special cases of Reshetikhin-Turaev invariants using representation theory of $U_q(\mathfrak{g})$

Reshetikhin-Turaev invariant

For $\mathfrak{g}$ a simple Lie algebra the quantum deformation $U_q(\mathfrak{g})$ of the enveloping algebra of $\mathfrak{g}$ gives link/tangle invariants.

Colour the strands of a tangle by a representation V_λ of $U_q(\mathfrak{g})$

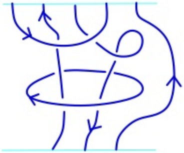


The invariant $\varphi(T)$ is a map of $U_q(\mathfrak{g})$-representations.

Example

- $\mathfrak{g} = \mathfrak{sl}_2$ Jones polynomial, coloured Jones polynomial
- $\mathfrak{g} = \mathfrak{sl}_n$ specializations of the HOMFLYPT polynomial

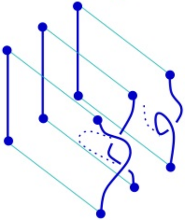
Jones polynomial



Representation theory of quantum $\mathfrak{sl}_2$

Categorification

Categorification



Khovanov homology

Categorified representation theory

(3) The When?

When is it possible to *categorify*?

Positivity, integrality, diagrammatics, combinatorics, multiplicity one phenomena—a leading term, canonical bases, Kazhdan-Lusztig basis, crystal bases

(4) The How?

It takes some art and insight

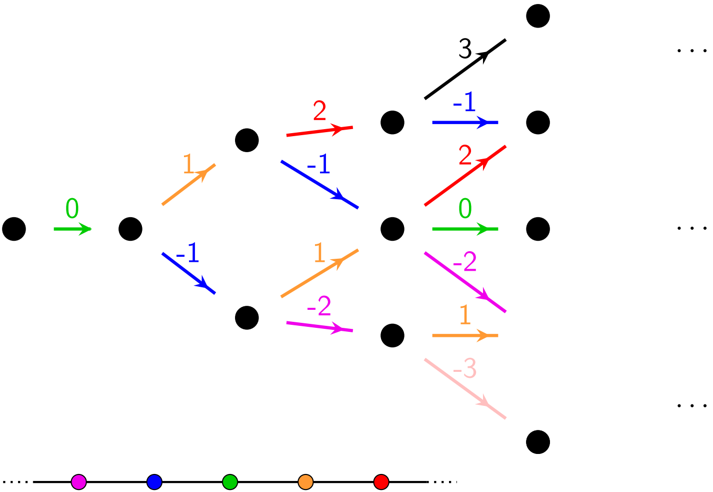
II. Focus on quantum groups $U_q(\mathfrak{g})$, crystal graphs, combinatorics of symmetric group $\mathcal{S}_n$

Start with $U(\mathfrak{sl}_\infty)$ and its basic representation $V(\Lambda_0) \simeq \bigwedge^{\frac{\infty}{2}} V_{\mathbb{Z}} \simeq$ Fock space

All weight spaces have dimension 1.
Draw action of Chevalley generators E_i, F_i on weight basis; coincides with action of Kashiwara crystal operators e_i, f_i

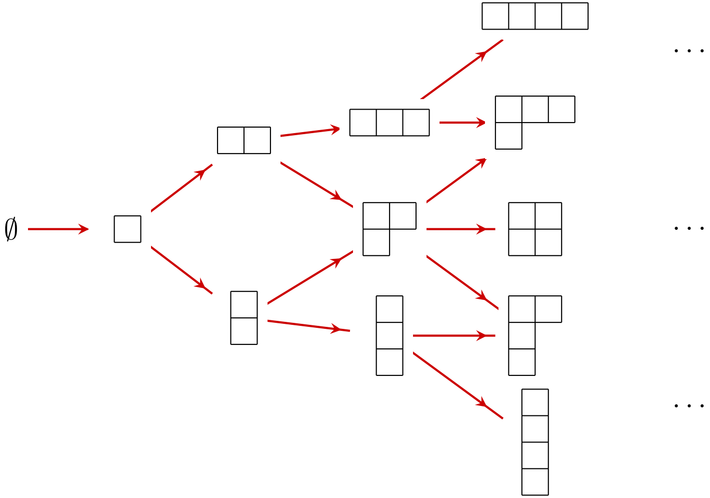
$$a \xrightarrow{i} f_i a \qquad e_i b \xrightarrow{i} b$$

Crystal graph $B(\Lambda_0)$ of $V(\Lambda_0)$



$\mathbb{I}$-colored graph encoding action of E_i, F_i

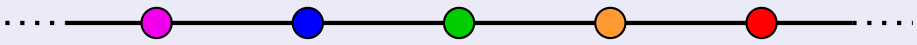
Representation theory of the symmetric group



toward categorifying quantum groups

$$\bigoplus_{n\geq 0} G_0(\mathbb{Q}\mathcal{S}_n\text{−mod}) \simeq V(\Lambda_0) \quad \text{as } \mathcal{U}(\mathfrak{sl}_\infty)\text{-modules}$$

where $\mathfrak{g} = \mathfrak{sl}_\infty$ corresponds to $\mathrm{I} = \mathbb{Z}$



Categorify

$$\bigoplus_{n\geq 0} G_0(\mathbb{Q}\mathcal{S}_n\text{−mod}) \simeq V(\Lambda_0)$$

as $\mathcal{U}(\mathfrak{g})$-modules means the generators E_i, F_i $i \in \mathbb{Z}$ of $\mathfrak{sl}_\infty$ act as decategorified functors [i-Res] and [i-Ind].

i-Res refines restriction by projecting to a block or by using a Jucys-Murphy operator

Even when weight spaces have high multiplicity, i.e. $\mathbb{F}_p\mathcal{S}_n-\mathrm{mod}$, $\dim \mathrm{Hom}(D^\lambda, \mathrm{Res}^n_{n-1} D^\mu) \leq 1$

Crystal operators $e_i = [\mathrm{soc}\ i\text{-Res}]$, $f_i = [\mathrm{cosoc}\ i\text{-Ind}]$ for $\mathfrak{g} = \widehat{\mathfrak{sl}}_p$

Can we generalize this?

- other $V(\Lambda)$ for any $\Lambda \in P^+$?
- other $\mathfrak{g}$?
- quantum groups $\mathcal{U}_q(\mathfrak{g})$?

Categorify quantum groups

Theorem (Khovanov-Lauda, Rouquier)

$$\mathcal{U}_q^-(\mathfrak{g}) \simeq K_0(R\mathrm{-mod})$$

as bi-algebras for their constructed R (KLR algebra). They categorified half the quantum group.

Theorem (Lauda-V)

$$[M] \xrightarrow{\ i\ } [N] \text{ if } \dim \mathrm{Hom}(M, \mathrm{Res}_{\nu}^{\nu+\alpha_i} N) = 1$$

- The simple R^{Λ}-modules carry the structure of the corresponding crystal graph $B(\Lambda)$ where R^{Λ} is a cyclotomic quotient, $\Lambda \in P^+$
- The simple R-modules carry the structure of the crystal graph $B(\infty)$ corresponding to $\mathcal{U}_q^-$.

KLR algebras categorify quantum groups

Theorem (Webster, Kang-Kashiwara)

$$V(\Lambda) \simeq K_0(R^\Lambda - \mathrm{mod})$$

as $\mathcal{U}_q(\mathfrak{g})$-modules

KLR algebras categorify quantum groups

q

Replace $\bigoplus_{m\geq 0} \mathbb{Q}\mathcal{S}_m$ with a *graded* ring $R = \bigoplus_{\nu \in Q^+} R_\nu$

$$q[M] = [M\{1\}]$$

in the Grothendieck ring, which is now a $\mathbb{Z}[q, q^{-1}]$-module with basis = isomorphism classes of simples

KLR algebras categorify quantum groups

Features:

- $\mathfrak{g}$ is of arbitrary symmetrizable type
- R is a family of algebras, for which $\mathrm{Res}^{m+n}_{m,n}$, $\mathrm{Ind}^{m+n}_{m,n}$ make sense (or more finely $\mathrm{Ind}^{\mu+\nu}_{\mu,\nu}$ and $\mathrm{Res}^{\mu+\nu}_{\mu,\nu}$, $\mu, \nu \in Q^+$)
- R is *graded*

Multiplicity one phenomena

$e_i = [\mathrm{soc}\ \ \mathrm{Res}^{\mu+\alpha_i}_{\mu,\alpha_i}], \quad f_i = [\mathrm{cosoc}\ \ \mathrm{Ind}^{\mu+\alpha_i}_{\mu,\alpha_i}]$

Kleshchev's description

For $\mathfrak{g}$ of type E_8, R is type A (like the symmetric group) but in "characteristic E_8"
For $\mathfrak{g} = \widehat{\mathfrak{sl}}_p$, R is like the symmetric group but in characteristic p
leading terms of local (rank 2) relations are braid relations
leading terms of rank 1 relations are affine nil-Hecke

The When?

Combinatorialists have been categorifying $\mathbb{Z}_{\geq 0}$ in $\texttt{Sets}$ for decades.
See a positive structure constant like $c^{\nu}_{\lambda,\mu}$ —What does it count?
See positive coefficients of a polynomial or power series

$$\mathcal{P}(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \cdots ,$$

What do they count?

Positivity, integrality, diagrammatics, combinatorics, multiplicity one phenomena—a leading term or with morphisms, canonical bases, Kazhdan-Lusztig basis, crystal bases

When there is good combinatorics (crystals) look to categorify

How?

Be brilliant