

KLR algebras

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The symmetric group

S_m

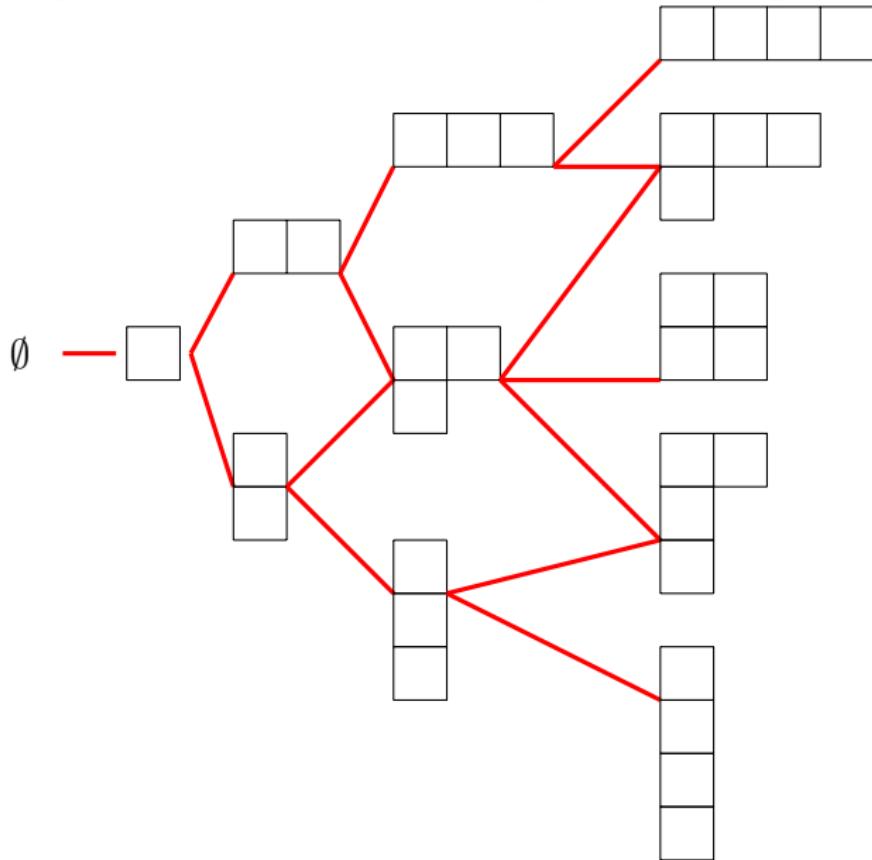
generators $\{s_1, s_2, \dots, s_{m-1}\}$

relations $s_r^2 = 1$ quadratic

$s_r s_k = s_k s_r$ if $|r - k| > 1$ braid

$s_r s_k s_r = s_k s_r s_k$ if $r = k \pm 1$

Representation theory of the symmetric group



Branching Rule

Refine restriction

$$\mathcal{S}_{m-1} \subseteq \mathcal{S}_m$$

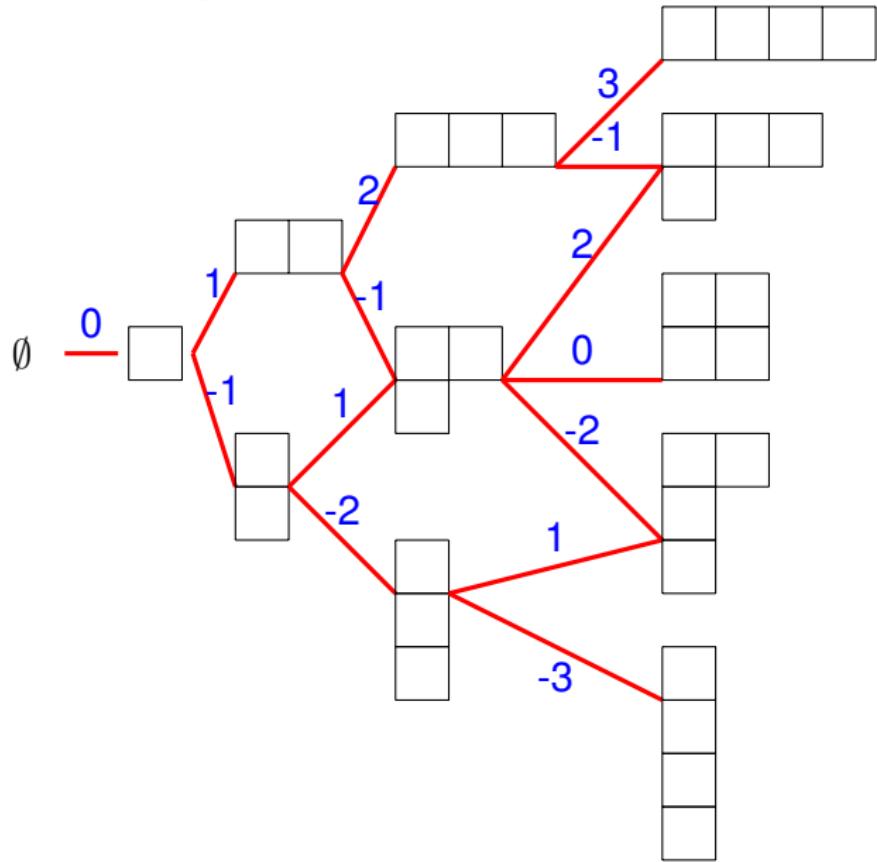
and more generally $\mathcal{S}_m \times \mathcal{S}_n \subseteq \mathcal{S}_{m+n}$.

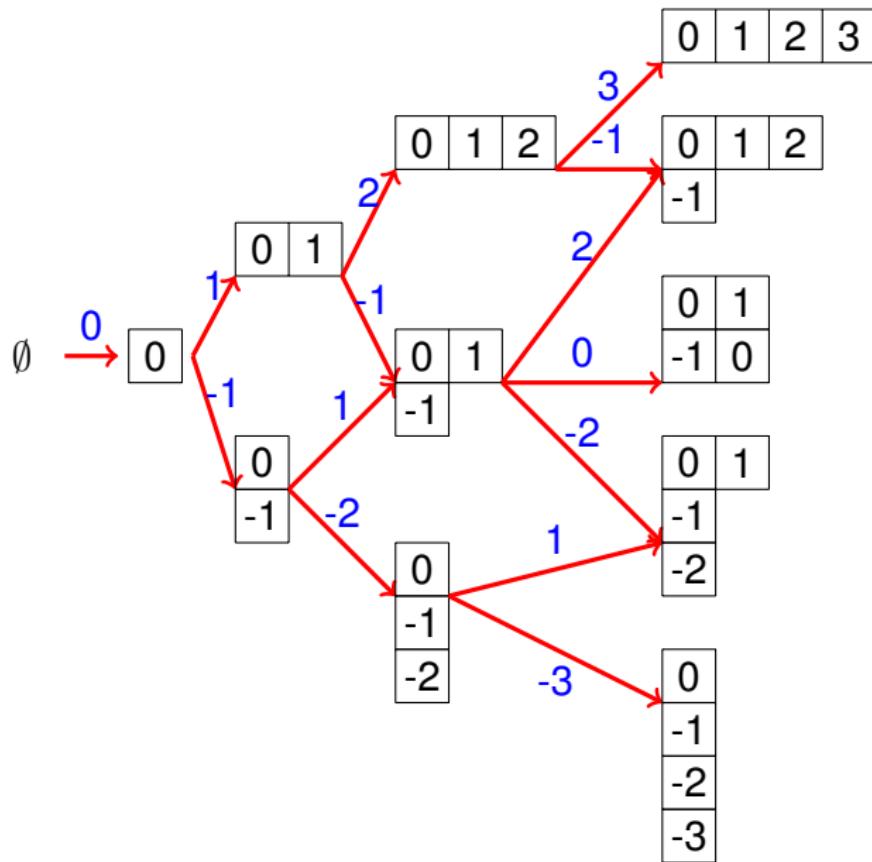
Functors Res_{m-1}^m and Ind_{m-1}^m

Use the center $Z(\mathbb{Q}\mathcal{S}_m)$ to refine restriction/induction by projecting to blocks

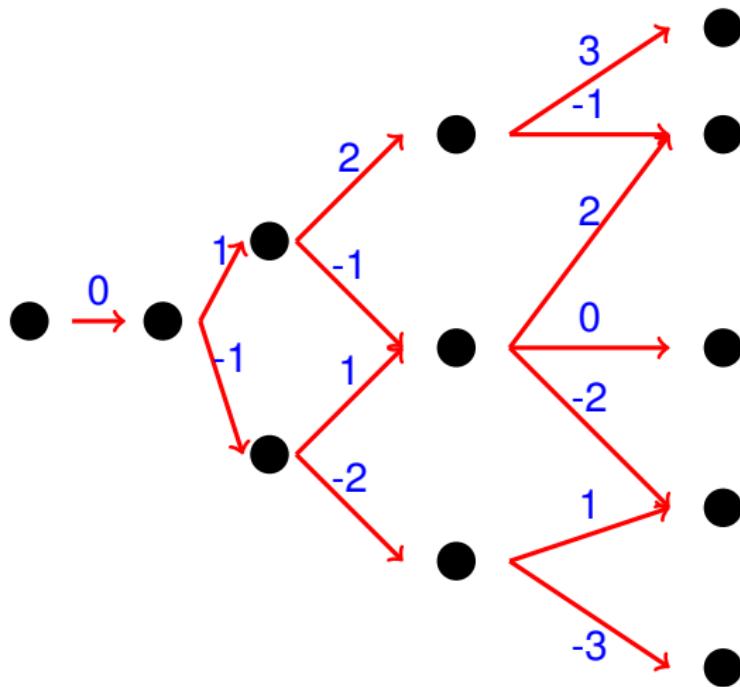
$$\text{Res}_{m-1}^m = \bigoplus_{i \in \mathbb{Z}} E_i \qquad \qquad \text{Ind}_{m-1}^m = \bigoplus_{i \in \mathbb{Z}} F_i^{\Lambda_0}$$

label edges





Crystal graph

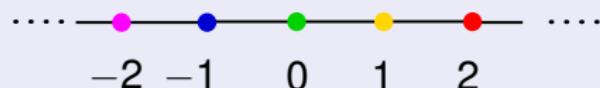


Lie algebra action

Functors E_i and $F_i^{\Lambda_0}$ yield an action of

$$\mathfrak{sl}_\infty \quad \text{on} \quad \bigoplus_{m \geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}\mathcal{S}_m\text{-mod})$$

\mathfrak{sl}_∞ has Dynkin nodes $I = \mathbb{Z}$



yields integral highest weight module

$$V(\Lambda_0) \simeq \bigoplus_{m \geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}\mathcal{S}_m\text{-mod})$$

with crystal graph $B(\Lambda_0)$

Grothendieck ring

$K_0(\text{category}) =$

{ isom classes of objects $[A]$ } / ⟨ $[B] = [A] + [C]$ if
 $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ ⟩

$K_0 \iff$ projectives $G_0 \iff$ simples (finite dimensional)

For $K_0(\mathbb{Q}S_m\text{-mod})$ we can identify $[M]$ with its character χ

Beyond

S_m categorifies the $\mathfrak{g} = \mathfrak{sl}_\infty$ module $V(\Lambda_0)$

What about

- quantum groups $\mathcal{U}_q(\mathfrak{g})$?
- other \mathfrak{g} ?
- other $V(\Lambda)$ for any $\Lambda \in P^+$?

q

Replace $\bigoplus_{m \geq 0} \mathbb{Q}\mathcal{S}_m$ with a *graded* ring R

$$q[M] = [M\{1\}]$$

in the Grothendieck ring, which is now a $\mathbb{Z}[q, q^{-1}]$ -module with basis = isomorphism classes of simples

Start with

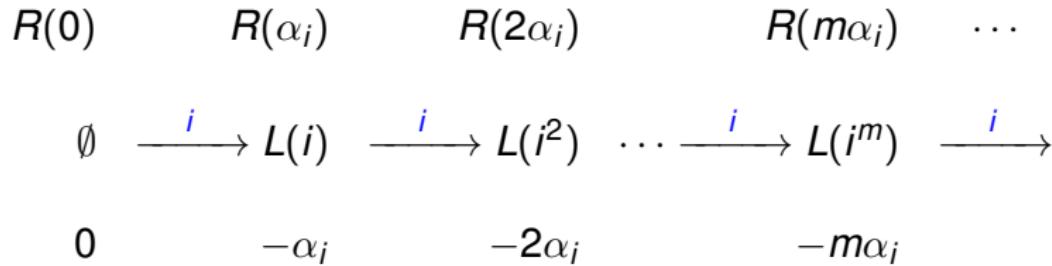
$$\mathfrak{g} = \mathfrak{sl}(2)$$

$$I = \{i\}$$

$$R = \bigoplus_{m \geq 0} R(m\alpha_i)$$

$R(m\alpha_i)$ has one simple (up to overall grading shift) called

$$L(i^m)$$



$$[E_i L(i^m)] = [\text{Res}_{m\alpha_i - \alpha_i}^{m\alpha_i} L(i^m)] = [m]_i [L(i^{m-1})]$$

where $[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}$ and $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$

So $E_i^m [L(i^m)] = [m]_i! \emptyset$ and the divided powers $E_i^{(m)} = \frac{E_i^m}{[m]_i!}$ make sense

- $\bigoplus_{m \geq 0} G_0(R(m\alpha_i)\text{-mod}) \simeq (\mathcal{U}_q^+)^*$ as (\mathcal{U}_q^+) -modules
- $\bigoplus_{m \geq 0} K_0(R(m\alpha_i)\text{-pmod}) \simeq \mathcal{U}_q^-$ as bi-algebras (Khovanov-Lauda)
- ? $\simeq V(\Lambda)$
cyclotomic quotient

affine NilHecke algebra

$$\partial_r^2 = \begin{array}{c} \text{Diagram of two strands crossing twice, labeled } i \text{ at both ends.} \\ \text{A blue diagram showing two strands crossing twice, with a dot at each crossing. The strands are labeled } i \text{ at both ends.} \end{array} = 0$$

$$x_1 \partial_1 - \partial_1 x_2 = \begin{array}{c} \text{Diagram of two strands crossing once, with a dot at the top-left crossing, labeled } i \text{ at both ends.} \\ \text{A blue diagram showing two strands crossing once, with a dot at the top-left crossing. The strands are labeled } i \text{ at both ends.} \end{array} - \begin{array}{c} \text{Diagram of two strands crossing once, with a dot at the bottom-right crossing, labeled } i \text{ at both ends.} \\ \text{A blue diagram showing two strands crossing once, with a dot at the bottom-right crossing. The strands are labeled } i \text{ at both ends.} \end{array} = \begin{array}{c} \text{Two vertical strands labeled } i \text{ at both ends.} \\ \text{Two vertical blue lines labeled } i \text{ at both ends.} \end{array}$$

$$\partial_r \partial_{r+1} \partial_r - \partial_{r+1} \partial_r \partial_{r+1} = \begin{array}{c} \text{Diagram of three strands crossing twice, with dots at both crossings, labeled } i \text{ at both ends.} \\ \text{A blue diagram showing three strands crossing twice, with dots at both crossings. The strands are labeled } i \text{ at both ends.} \end{array} - \begin{array}{c} \text{Diagram of three strands crossing twice, with dots at both crossings, labeled } i \text{ at both ends.} \\ \text{A blue diagram showing three strands crossing twice, with dots at both crossings. The strands are labeled } i \text{ at both ends.} \end{array} = 0$$

affine NilHecke algebra

$$\deg \left(\begin{array}{c} | \\ \bullet \end{array} \right) = (\alpha_i, \alpha_i) = 2$$

$$\deg \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = -(\alpha_i, \alpha_i) = -2$$

$R(m\alpha_i)$ is the affine NilHecke algebra (aka NH_m)
 ∂_r acts on polynomials f via

$$\partial_r(f) = \frac{f - s_r(f)}{x_r - x_{r+1}}$$

$R(m\alpha_i) \cong \text{NH}_m$ with identity the idempotent $1_{i^m} = \begin{array}{c|c|c|c} & & \cdots & \\ i & i & \cdots & i \end{array}$, where
 $i^m := i \dots i$.

simples

Can realize $L(i^m)$ as $\mathbb{F}[x_1, \dots, x_m]/\Lambda_m^+$ where Λ_m^+ is the ideal generated by symmetric polynomials with 0 constant term. (geometry \rightarrow grading)

Can realize $L(i^m)$ as $\text{Ind}_{(1,1,\dots,1)}^m L(i) \boxtimes L(i) \boxtimes \dots \boxtimes L(i)\{k\}$ (glue)

$$x_r^m L(i^m) = 0 \quad x_r^{m-1} L(i^m) \neq 0$$

cyclotomic quotient

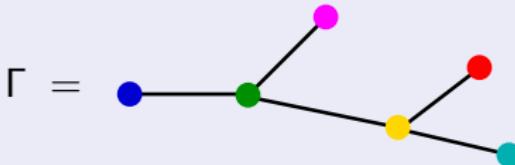
$$x_r^m L(i^m) = 0 \quad x_r^{m-1} L(i^m) \neq 0$$

$R^\lambda(m\alpha_i) = R^n(m\alpha_i) = R(m\alpha_i)/\langle x_1^n \rangle$ is finite dimensional and will collapse if $m > n$

$$\begin{array}{ccccccc} R^\lambda(0) & R^\lambda(\alpha_i) & R^\lambda(2\alpha_i) & \cdots & R^\lambda(n\alpha_i) \\ \emptyset & \xrightarrow{i} L(i) & \xrightarrow{i} L(i^2) & \xrightarrow{i} \cdots & \xrightarrow{i} L(i^n) \\ \lambda & \lambda - \alpha_i & \lambda - 2\alpha_i & & \lambda - n\alpha_i \\ n & n - 2 & n - 4 & & -n \end{array}$$

\mathbf{U}_q^+ for any Γ

Let Γ be an unoriented graph with set of vertices I .



\mathbf{U}_q^+ is a $\mathbb{Q}(q)$ -algebra; its integral form ${}_{\mathcal{A}}\mathbf{U}_q^+$ is the $\mathbb{Z}[q, q^{-1}]$ -algebra with:

- generators: $E_i^{(n)} \quad i \in I$
- quantum Serre relations: $\sum_0^{a+1} (-1)^n E_i^{(a+1-n)} E_j E_i^{(n)} = 0 \quad \text{if}$



$$\text{where } a = a_{ij} = -\langle h_i, \alpha_j \rangle = -2 \frac{\langle i, j \rangle}{\langle i, i \rangle}$$

\mathbf{U}_q^+ is $Q^+ = \mathbb{N}[I]$ graded with $\deg(E_i) = \alpha_i$.

Definition of the algebra $R(\nu)$

Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ .

Let $\nu = \sum_{i \in I} \nu_i \cdot \alpha_i$, for $\nu_i = 0, 1, 2, \dots$

ν keeps track of how many strands of each color occur in a diagram



Form an abelian group by taking \mathbb{Z} -linear (or \mathbb{k} -linear) combinations of diagrams:

$$5 \begin{array}{c} \text{Diagram: two strands (green and pink) crossing twice, with a dot on the pink strand.} \\ \text{Diagram: two strands (green and pink) crossing twice, with a dot on the pink strand.} \end{array} - 2 \begin{array}{c} \text{Diagram: three strands (blue, green, pink) crossing in a complex way, with dots on all strands.} \end{array} - 17 \begin{array}{c} \text{Diagram: two strands (pink and green) forming a tight helix, with a dot on the pink strand.} \\ \text{Diagram: a single vertical blue line with a dot at the top.} \end{array}$$

Multiplication is given by stacking diagrams on top of each other when the colors match:

$$\begin{array}{ccc} \text{Diagram 1} & * & \text{Diagram 2} \\ \text{Diagram 1} & * & \text{Diagram 2} \end{array} = \begin{array}{c} \text{Stacked Diagram} \end{array}$$

The first diagram shows two strands (one pink, one green) crossing over each other. The second diagram shows two strands (one blue, one green) crossing over each other. The result is a stacked diagram where the strands from both diagrams cross over each other in a combined manner.

The second diagram shows two strands (one pink, one green) crossing over each other. The second diagram shows two strands (one blue, one pink) crossing over each other. The result is 0.

Definition

Given $\nu \in Q^+$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν , modulo planar braid-like isotopies and the following local relations:

$R(\nu)$ local relations

$$\deg \left(\begin{array}{c} | \\ \bullet \end{array} \right) = (\alpha_i, \alpha_i)$$

$$\deg \left(\begin{array}{cc} & \\ \diagup & \diagdown \\ i & j \end{array} \right) = -(\alpha_i, \alpha_j)$$

$$\begin{array}{ccc} \text{Diagram: } & = & \text{Diagram: } \\ \text{green arc from } i \text{ to } j & & \text{green arc from } i \text{ to } j \text{ with dot at } i \end{array}$$

$$\begin{array}{ccc} \text{Diagram: } & = & \text{Diagram: } \\ \text{green dot at } i \text{ and blue dot at } j & & \text{blue dot at } i \text{ and green dot at } j \end{array} \quad \text{for } i \neq j$$

$$\begin{array}{ccc} \text{Diagram: } & = & \left\{ \begin{array}{ll} 0 & \text{if } i = j \\ a_{ij} \begin{array}{c} | \\ \bullet \end{array} + \begin{array}{c} | \\ \bullet \end{array} a_{ji} & \text{if } (\alpha_i, \alpha_j) = 0 \\ & & \text{if } (\alpha_i, \alpha_j) \neq 0 \end{array} \right. \\ \text{green arc from } i \text{ to } j & & \text{green arc from } i \text{ to } j \\ & & \text{green arc from } j \text{ to } i \end{array}$$

$$\begin{array}{c} \text{Diagram 1: } \\ \text{Two strands } i \text{ and } j \text{ cross twice.} \\ \text{Left diagram: } i \text{ (blue) over } j \text{ (green), } i \text{ (blue) under } j \text{ (green).} \\ \text{Right diagram: } i \text{ (blue) under } j \text{ (green), } i \text{ (blue) over } j \text{ (green).} \end{array}
 - \quad = \quad \sum_{d+b=a_{ij}-1} d \cdot \begin{array}{c} \text{dot} \\ | \\ i \end{array} \quad \begin{array}{c} \text{dot} \\ | \\ j \end{array} \quad b \cdot \begin{array}{c} \text{dot} \\ | \\ i \end{array} \quad \begin{array}{c} \text{dot} \\ | \\ j \end{array} \quad \text{if } \begin{array}{c} i \\ \text{---} \\ j \end{array}$$

$$\begin{array}{c} \text{Diagram 2: } \\ \text{Three strands } i, j, k \text{ cross three times.} \\ \text{Left diagram: } i \text{ (blue) over } j \text{ (green) over } k \text{ (pink).} \\ \text{Right diagram: } i \text{ (blue) under } j \text{ (green) under } k \text{ (pink).} \end{array}
 = \quad \text{otherwise,}$$

some of i, j, k may be equal

Cyclotomic quotients

For a given dominant integral weight $\lambda = \sum_{i \in I} \lambda_i \cdot \Lambda_i$ define the cyclotomic quotient $R^\lambda(\nu)$ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero \longrightarrow $\lambda_{i_1} \bullet | | | \dots | = 0$



This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_m^\lambda := H_m / \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Cyclotomic quotients

The category of finitely-generated graded modules over the ring

$$R^\lambda = \bigoplus_{\nu \in Q^+} R^\lambda(\nu)$$

categorifies the integrable version of the representation V_λ of $\mathcal{U}_q(\mathfrak{g})$ of highest weight λ .

Theorem

- ① (Webster, Kang-Kashiwara) As $\mathcal{U}_q(\mathfrak{g})$ -modules $V(\lambda) \cong K_0(R^\lambda)$.
- ② (Lauda-Vazirani) The simple R^λ -modules carry the structure of the corresponding crystal graph $B(\lambda)$.