The Minkowski and conformal superspaces

Rita Fioresi, February 14, 2012

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- **Supermodules**: modules over commutative superalgebras.
  $A^{m|n} := A \otimes k^{m|n}$ free A-module of dimension $m|n$.
  $A_0^{m|n} := A_0 \otimes k^m \oplus A_1 \otimes k^n$
  $A_1^{m|n} := A_0 \otimes k^n \oplus A_1 \otimes k^m$

The General Linear Supergroup

The General Linear supergroup $GL(m|n)(A)$: it is the group of invertible parity preserving linear transformations of the module $A^{m|n}$:

$$GL(m|n)(A) := \left\{ \phi : A^{m|n} \longrightarrow A^{m|n},\ \phi \text{ invertible } \right\} = \left\{ \begin{pmatrix} x & \xi \\ \eta & y \end{pmatrix} \right\}$$

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$GL(m|n)$ is a *representable* functor.

$GL(m|n)$: (salg) $\longrightarrow$ (sets), $GL(m|n)(A) = Hom(k[GL(m|n)], A)$.

(salg) = category of commutative superalgebras
(sets) = category of sets

The Berezinian

$$Ber\begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(D)^{-1}det(A - BD^{-1}C)$$
$$= det(A)^{-1}det(D - CA^{-1}B)$$

Ber is multiplicative, i.e. it is a group morphism
$Ber : GL(m|n)(R) \longrightarrow R^*$

$$Ber(XY) = Ber(X)Ber(Y)$$

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Notice the matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible if and only if A and D are invertible, since B and C are nilpotents (nilpotent entries).

The concept of Superspace and Supervariety

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A superspace is a *differentiable supermanifold* if locally isomorphic to the superspace $\mathbb{R}^{p|q}$, its topological space is $\mathbb{R}^p$ and the structural sheaf is

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A superspace is a *algebraic supervariety* or a *superscheme* if its topological space is the topological space of a scheme X_0 and its structural sheaf is a quasicoerent sheaf of $\mathcal{O}_{X_0}$ modules.

Examples: $k^{m|n}$, $GL(m|n) \subset M(m|n)$.

The Funtor of points in supergeometry

For each superspace, supervariety and superscheme we can associate the *functor of points* which characterizes completely the geometric space.

- Differentiable Supermanifolds. Define the funtor of points of M as:

$$\begin{array}{ccll} \text{(smflds)} & \xrightarrow{h_M} & \text{(sets)} & \\ T & \longrightarrow & \mathrm{Hom}_{\text{(smflds)}}(T, M) & = \mathrm{Hom}_{\text{(salg)}}(\mathcal{O}(M), \mathcal{O}(T)) \end{array}$$

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Example revisited: $GL(m|n)$.

- Differentiable Case.

$$GL(m|n)(T) = \mathrm{Hom}_{(\mathrm{salg})}(C^{\infty}(x_{ij}) \otimes \wedge(\xi_{kl})[u^{-1}, v^{-1}], \mathcal{O}(T))$$

$$u = det(x_{ij})_{1 \leq i,j \leq m}, \qquad v = det(x_{ij})_{m+1 \leq i,j \leq n}$$

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Morphisms between supermanifolds: Supersymmetry

Using the functor of points notation we can easily define morphisms:

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$$\begin{array}{cccc} \phi_T : & \mathbb{R}^{1|2}(T) & \longrightarrow & \mathbb{R}^{1|2}(T) \\ & t & \mapsto & t + \theta_1\theta_2 \\ & \theta_1 & \mapsto & \theta_1 \\ & \theta_2 & \mapsto & \theta_2 \end{array}$$

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  Chart's theorem. *There is a bijection between morphisms of superdomains* $\mathbb{R}^{m|n} \longrightarrow \mathbb{R}^{p|q}$ *and the set of* $p+q$*-uples* $(t_1 \dots t_p, \theta_1 \dots \theta_q)$ *of sections in* $\mathcal{O}(\mathbb{R}^{m|n})$.

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- Algebraic case. **Theorem.** *The morphisms between superschemes (supervariety) $X \longrightarrow Y$ correspond bijectively to morphisms of the superalgebras of global sections $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$.*

Supergroups and Quantum Groups

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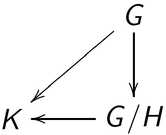
Attention: No functor of points for quantum groups!

Homogeneous Spaces

Classically if G is a Lie (algebraic) group and H a closed subgroup, there exists a unique manifold (variety) structure on G/H with the universal property:



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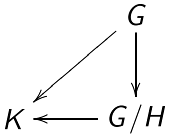

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Theorem(F.-Lledo-Varadarajan, 2007). *Let G is a Lie supergroup and H a closed subgroup. Then there exists a unique supermanifold structure on G/H with the universal property. The functor of points of G/H is:*

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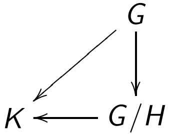

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Algebraic case: Solution proposed by A. Masuoka, A. Zubkov.

The ordinary Minkowski and conformal spaces

Classically the complex Minkowski space M can be realized as big (open) cell inside the grassmannian variety $G(2,4) = SL_4(\mathbb{C})/P_u$ with

$$P_u = \left\{ \begin{pmatrix} L & M \\ 0 & R \end{pmatrix} \in \mathrm{SL}_4(\mathbb{C}) \right\}$$

$$M = \left\{ \begin{pmatrix} 1\!\!1_2 & 0 \\ A & 1\!\!1_2 \end{pmatrix} P_u \right\} \subset SL_4(\mathbb{C})/P_u, \qquad A = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right\}$$

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The group leaving M invariant is:

$$P = \left\{ \begin{pmatrix} L & 0 \\ NL & R \end{pmatrix} \right\} \subset SL_4(\mathbb{C})$$

In this way the actions of the Poincare' P on M and the conformal group $SL_4(\mathbb{C})$ on the grassmannian $G(2,4)$ appear naturally.

$$
\begin{array}{ccc}
P \times M & \longrightarrow & M \\
\begin{pmatrix} L & 0 \\ NL & R \end{pmatrix}, \begin{pmatrix} \mathbb{1}_2 & 0 \\ A & \mathbb{1}_2 \end{pmatrix} P_u & \mapsto & \begin{pmatrix} \mathbb{1}_2 & 0 \\ N + RAL^{-1} & \mathbb{1}_2 \end{pmatrix} P_u
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$P = M_2 \ltimes H$. $M_2 = \{N\}$ acts as translations and

$$H = \left\{ \begin{pmatrix} L & 0 \\ 0 & R \end{pmatrix}, \ \ \det L \cdot \det R = 1 \right\} \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times$$

In the basis of the Pauli matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

an arbitrary matrix A can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = x^0\sigma_0 + x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix},$$

and (x^0, x^1, x^2, x^3) are the ordinary coordinates of Minkowski space.

$$\det A = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

Real Classical Minkowski

There is a natural conjugation θ on the big cell (Minkowski space):

$$\theta : A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto A^\dagger = \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}.$$

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The fixed point set is $M_R^\times = \{A \in M_2(\mathbb{C}) \mid A^\dagger = A\}$, the real Minkowski space. The complex group $\mathrm{SO}(6, \mathbb{C}) \cong P_u$ acting on M corresponds to the real form $\mathrm{SO}(4, 2)$ acting on M_R and its spin group $\mathrm{SU}(2, 2)$. The hermitian form on $\mathbb{C}^4$ left invariant by this real form is

$$(u, v) = u^\dagger F v, \qquad F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad u, v \in \mathbb{C}^4.$$

The subgroup P_R leaving M_R invariant is

$$P_R = \left\{ \begin{pmatrix} L & 0 \\ NL & {L^\dagger}^{-1} \end{pmatrix},\ N \text{ hermitian and } \det(L) \in \mathbb{R} \right\},$$

and its action on M_R is

$$A \mapsto N + {L^\dagger}^{-1} A L^{-1}.$$

Complex Super Flag manifold and Minkowski superspace

$F = F(2|0, 2|1; 4|1) \subset G(2|0; 4|1) \times G_2 = G(2|1; 4|1)$ flag of $2|0$, $2|1$ subspaces in $\mathbb{C}^{4|1}$.

$$F(T) = SL(4|1)(T)/H(T)$$

U_1, U_2 big cells in $G(2|0; 4|1)$ and $G_2 = G(2|1; 4|1)$.

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U_1 is inside U_2 iff

$$A = B + \beta\alpha \qquad \text{Twistor relations}$$

(A, α, β) coordinates for the Minkowski superspace.

The subgroup of $\mathrm{SL}(4|1)$ that leaves the big cell invariant is:

$$\begin{pmatrix} L & 0 & 0 \\ NL & R & R\chi \\ d\varphi & 0 & d \end{pmatrix}$$

Its action on the super Minkowski space is:

$$\begin{aligned} A &\longrightarrow R(A+\chi\alpha)L^{-1}+N, \\ \alpha &\longrightarrow d(\alpha+\varphi)L^{-1}, \\ \beta &\longrightarrow d^{-1}R(\beta+\chi) \end{aligned}$$

Real Super Minkowski space

The real form of $SL(4|1)$ is given by the involution:

$$\begin{array}{ccc} SL(4|1)(R) & \xrightarrow{\xi} & \bar{S}L(4|1)(R) \\ g & \longrightarrow & g^{\xi} := L(x^{\theta})^{-1}L \end{array} \quad L = \begin{pmatrix} F & 0 \\ 0 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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The conjugation on the Poincaré supergroup reads:

$$g^{\xi} = \begin{pmatrix} R^{\dagger^{-1}} & 0 & 0 \\ -L^{\dagger^{-1}}M^{\dagger}R^{\dagger^{-1}} - L^{\dagger^{-1}}\varphi^{\dagger}\chi^{\dagger} & L^{\dagger^{-1}} & -jL^{\dagger^{-1}}\varphi^{\dagger} \\ -j\bar{d}^{-1}\chi^{\dagger} & 0 & \bar{d}^{-1} \end{pmatrix}.$$

The fixed points are those that satisfy the conditions:

$$L = R^{\dagger^{-1}}, \qquad \chi = -j\varphi^{\dagger}, \qquad ML^{-1} = -(ML^{-1})^{\dagger} - jL^{\dagger^{-1}}\varphi^{\dagger}\varphi L^{-1}$$

After a translation one gets the more familiar form for the reality conditions:

$$M'L^{-1} \equiv ML^{-1} + \frac{1}{2}jL^{\dagger^{-1}}\varphi^{\dagger}\varphi L^{-1}, \qquad M' = -M'^{\dagger}.$$

On the big cells $U_1 \times U_1$ in F we have:

$$g^{\xi} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ -A^{\dagger} - \alpha^{\dagger}\beta^{\dagger} & \mathbb{1} & -j\alpha^{\dagger} \\ -j\beta^{\dagger} & 0 & 1 \end{pmatrix}.$$

The real points are given by

$$A = -A^{\dagger} - j\alpha^{\dagger}\alpha, \qquad \beta = -j\alpha^{\dagger}.$$

We can make a convenient change of coordinates,

$$A' \equiv A + \frac{1}{2} j\alpha^{\dagger}\alpha,$$

so the reality condition is

$$A' = -A'^{\dagger},$$

So we obtain the real Minkowski space time.

Quantum chiral Super Minkowski space

We need a quantum deformation of the grassmannian and flag supermanifolds.

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We start with the *quantum matrix superalgebra*.

$$M_q(m|n) =_{def} \mathbb{C}_q < a_{ij} > / I_M$$

where the ideal I_M is generated by the Manin relations:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \quad j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \quad i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \quad i < k, j > l \quad or \quad i > k, j < l$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} =$$

$$(-1)^{\pi(a_{ij})\pi(a_{kj})}(q^{-1} - q)a_{kj}a_{il} \quad i < k, j < l$$

$p(i) = 0$ if $1 \leq i \leq m$, $p(i) = 1$ otherwise and $\pi(a_{ij}) = p(i) + p(j)$ denotes the parity of a_{ij}.

Quantum General Linear Supergroup

$$\mathrm{GL}_q(m|n) =_{def} M_q(m|n)\langle {D_1}^{-1}, {D_2}^{-1}\rangle$$

where ${D_1}^{-1}$, ${D_2}^{-1}$ are even indeterminates such that:

$$D_1 D_1^{-1} = 1 = {D_1}^{-1} D_1, \qquad D_2 {D_2}^{-1} = 1 = {D_2}^{-1} D_2$$

and

$$D_1 =_{def} \sum_{\sigma \in S_m} (-q)^{-l(\sigma)} a_{1\sigma(1)} \dots a_{m\sigma(m)}$$

$$D_2 =_{def} \sum_{\sigma \in S_n} (-q)^{l(\sigma)} a_{m+1,m+\sigma(1)} \dots a_{m+n,m+\sigma(n)}$$

are the quantum determinants of the diagonal blocks.
$M_q(m|n)$, $\mathrm{GL}_q(m|n)$ are bialgebras with the usual comultiplication and counit:

$$\Delta(a_{ij}) = \sum a_{ik} \otimes a_{kj}, \qquad \epsilon(a_{ij}) = \delta_{ij}.$$

$\mathrm{GL}_q(m|n)$ is also an Hopf superalgebra.

The Supergrassmannian

We define *quantum super Grassmannian* of $2|0$ planes in $4|1$ dimensional superspace as the non commutative superalgebra Gr_q generated by the following quantum super minors in $\mathrm{GL}_q(4|1)$:

$$D_{ij} = a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1}, \qquad 1 \le i < j \le 4, \qquad D_{55} = a_{51}a_{52}$$

$$D_{i5} = a_{i1}a_{52} - q^{-1}a_{i2}a_{51}, \qquad 1 \le i \le 4.$$

The quantum minors are all of the invariants under the natural $GL_2(2)$ coaction.

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We define *quantum super Grassmannian* of $2|0$ planes in $4|1$ dimensional superspace as the non commutative superalgebra Gr_q generated by the following quantum super minors in $\mathrm{GL}_q(4|1)$:

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Theorem: Gr_q is a quantum homogeneous superspace for the quantum supergroup $\mathrm{GL}_q(4|1)$, i. e., we have a coaction given via the restriction of the comultiplication of $\mathrm{GL}_q(4|1)$:

$$\Delta|_{Gr_q} : Gr_q \longrightarrow \mathrm{GL}_q(4|1) \otimes Gr_q.$$

Quantum Super Minkowski

The quantum super Minkowski M_q is realized as the big cell inside the quantum super grassmannian: we invert the quantum minor D_{12}.

M_q is defined as the subring of Gr_q generated by:

$$t = \begin{pmatrix} -D_{23}D_{12}^{-1} & D_{13}D_{12}^{-1} \\ -D_{24}D_{12}^{-1} & D_{14}D_{12}^{-1} \end{pmatrix} \qquad \tau = (-D_{25}D_{12}^{-1}, D_{15}D_{12}^{-1})$$

M_q admits the following presentation:

$$t_{i1}t_{i2} = q\, t_{i2}t_{i1}, \qquad t_{3j}t_{4j} = q^{-1}\, t_{4j}t_{3j}, \qquad 1 \le j \le 2, \quad 3 \le i \le 4$$

$$t_{31}t_{42} = t_{42}t_{31}, \qquad t_{32}t_{41} = t_{41}t_{32} + (q^{-1} - q)t_{42}t_{31},$$

$$\tau_{51}\tau_{52} = -q^{-1}\tau_{52}\tau_{51}, \qquad t_{ij}\tau_{5j} = q^{-1}\tau_{5j}t_{ij}, \qquad 1 \le j \le 2$$

$$t_{i1}\tau_{52} = \tau_{52}t_{i1}, \qquad t_{i2}\tau_{51} = \tau_{51}t_{i2} + (q^{-1} - q)t_{i1}\tau_{52}.$$

Theorem There is a natural coaction of the quantum super Poincare' group P_q on the quantum super Minkowski space:

$$\tilde{\Delta}|_{M_q} : M_q \quad \longrightarrow \quad P_q \otimes M_q$$

where

$$P_q = \mathrm{GL}_q(4|1))/I_q$$

where I_q is the (two-sided) ideal in $\mathrm{GL}_q(4|1)$ generated by

$$g_{1j}, g_{2j}, \quad \text{for} \quad j = 3, 4 \quad \text{and} \quad \gamma_{15}, \gamma_{25}$$