Mexico 2013

BGG sequences and Physics.

Rod Gover

Overdetermined linear PDE: [Classical problem $\boxed{\text{on } \mathbb{R}^n}$] — n eq^{n}s / 1 - f^n

Given f^{n}s $\omega_k(x)$, $k=1,\dots,n$ locally: $\exists f$ (potential) which solves $\omega_k = \dfrac{\partial f}{\partial x^k}$? ①

⇝ necessary cond^n: $\dfrac{\partial \omega_k}{\partial x^l} = \dfrac{\partial \omega_l}{\partial x^k}$ ② – also sufficient if $\underbrace{\text{dom. suitable}}_{\text{Poincaré Lemma}}$

Exterior calc: ① : $df = \omega$ ← given 1-form

② $d\omega = 0$ $d^2 = 0$

Similarly locally characterise range d on higher forms ⇝ de Rham cx on a manifold M^n (connected):

$$0 \to \underset{\substack{\uparrow \\ \text{const} \\ \text{f}^n\text{s}}}{\mathbb{R}} \to \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^n \to 0$$

- Poincaré Lemma ⇝ locally exact.
- globally: ⇝ cohomology ⇝ top invts of M

} fund str. on M^n smooth closed.

Q: Geometry $\overset{?}{⇝}$ analogues / refinements / further structure?

Physics: Linear field theories

$$0 \to \text{charges} \to \text{gauges} \xrightarrow{\mathcal{D}_0} \text{potentials} \xrightarrow{\mathcal{D}_0} \text{kinematic fields}$$

& dual

$$0 \leftarrow \text{dual charges} \leftarrow \text{fluxes} \leftarrow \text{sources} \leftarrow \text{dynamic fields}$$

—

E.g. de Rham $n=4$ vs Maxwell

$$\begin{tikzcd} & & & & \Lambda^3 \arrow[d, phantom, "\cong\ *"] & \Lambda^4 \arrow[d, phantom, "\cong\ *"] & & \\ 0 \arrow[r] & \mathbb{R} \arrow[r] & \Lambda^0 \arrow[r] & \Lambda^1 \arrow[r] & \Lambda^2 \arrow[r] & \Lambda_1 \arrow[r] & \Lambda_0 \arrow[r] & \mathbb{R} \arrow[r] & 0 \end{tikzcd}$$

f A F J.

E.M pot. $\parallel$ dA

$A \mapsto A + df$

$\underbrace{*d*}_{\delta}F = J$ $\delta J = 0$

E.g. (M^n, g) ψ-Riemannian. $v \in \mathfrak{X}(M)$ infl conf isometry/symm. if:

$$\mathcal{L}_v g \propto g \iff \underbrace{\mathrm{Tr}^g\text{-free}(\nabla_a v_b + \nabla_b v_a)}_{K_0} = 0$$ overdet if $n \geq 3$

$\boxed{\text{Q: } \exists \text{ ? Poincaré Lemma for } K_0}$? ie, linear diff^l op. W s.t. $\ker W = \text{range } K_0$ – "locally" – integrability cond

Th^m: (Gasqui-Goldschmidt 84) Yes if (M, g) conf flat:

$$\begin{tikzcd} 0 \arrow[r] & \underset{\text{Sol}^n\text{s } K_0}{} \arrow[r] & \underset{\text{gauges}}{TM} \arrow[r, "K_0"] & \underset{\text{pot.}}{S^2_0 T^*M[2]} \arrow[r, "W"] & \underset{\text{Fields}}{\begin{array}{|c|c|}\hline\,&\,\\\hline\,&\,\\\hline\end{array}_{\,0}[2]} \arrow[r, "\text{Bianchi}"] & \begin{array}{|c|c|}\hline\,&\,\\\hline\,&\,\\\hline\,&\,\\\hline\end{array}_{\,0}[2] \arrow[r] & \cdots \arrow[r] & T^*M[-n] \arrow[r] & 0 \end{tikzcd}$$

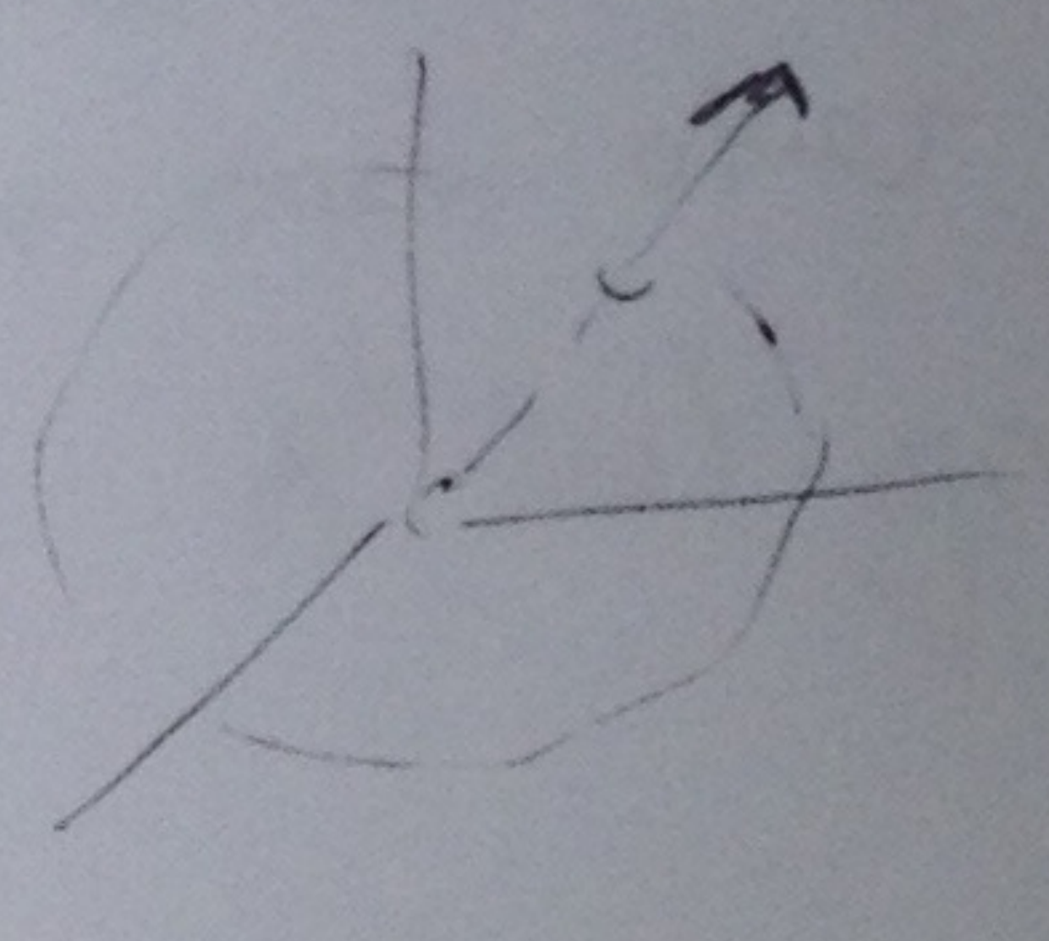
$\nearrow$

- locally exact ✓ • governs deformations of loc. conf flat mflds
- elliptic (symb seq exact) ⇝ cohom. sps finite dim if M cpct..
- H^1 = formal tgt sp moduli sp of def^{n}s of locally conf flat str.
- [cf conformal spin-2 field theory]..
- Proof uses Spencer theory ... but BGG complexes another way:

—

A model – the linear elasticity complex: [cf Eastwood]

On $S^3 = \mathbb{P}_+(\mathbb{R}^4)$
ray projectivisation



Write $\mathcal{E}^i = T\cancel{M}S^3$

Then

$$0 \to \underset{\uparrow \atop f^n\text{s on } S^3}{\mathcal{E}} \xrightarrow{X} \mathcal{E}(\mathbb{R}^4)(1) \longrightarrow \mathcal{E}^i \longrightarrow 0$$

~~$\mathbb{R}$~~ $\{f : S^3 \to \mathbb{R}^4\}$

$f : \mathbb{R}^4 \to \mathbb{R}^4$
$\therefore f(\lambda X) = \lambda f(X)$.

$$\iff \quad 0 \to \underset{\uparrow \atop \text{taut. line bundle}}{\mathcal{E}(-1)} \xrightarrow{X} \mathcal{E}(\mathbb{R}^4) \longrightarrow \mathcal{E}^i(-1) \longrightarrow 0$$

Euler. seq.

<u>Th</u>$\underline{\underline{\text{Th}}}^m$: (BGG, Lepowski, CSS) Let $\mathbb{W}$ irrep of $\overset{G\ \shortparallel}{SO(n+1,1)} \Rightarrow \exists$
a resolution on S^n ← round sphere

$$0 \to \mathbb{W} \to \mathbb{B}^0 \xrightarrow{D_0} \mathbb{B}^1 \xrightarrow{D_1} \cdots \to \mathbb{B}^n \to 0$$

D_i diff^l ops $\mathbb{B}^i$ - tensor/spinor ~~bdle~~ ← bdle.
G-invt - mainly irred.

<u>Proof idea</u>: G acts (transitively) on sphere & $P \to G$ principal bdle (of bdle of frames)

$\therefore S^n = G/P$ ← isotropy subgp - stabilises a pt $\downarrow$ $G/P = S^n$.

① Induced/associated bdle $W := G \times_P \mathbb{W}$ may be canonically trivialised

$$W \xrightarrow[\text{canon.}]{\cong} S^n \times \mathbb{W}.$$

$\therefore \exists$ canonical flat connection ∇ on W [from d on rhs]

$\therefore$ twisted de Rham

$$0 \to \mathbb{W} \to \Lambda^0(W) \xrightarrow{d^\nabla} \Lambda^1(W) \to \cdots$$

is a resol^n of $\mathbb{W}$.

② $\exists$ differential Casimir type operators $\mathcal{C}_i$ which act as $c_i \cdot id$ on $\mathbb{W} \hookrightarrow \Lambda^0(W)$ ← as constant sections

& $\mathcal{C}_i$ commute $\bar{w}$ all d^∇, & each other

$\therefore$ (simult. E. c_i-eigspaces in each $\Lambda^k(W)$ give $\mathbb{B}^k$ $\square$.

$\underline{\underline{\text{So}}}$:· $\exists$ a BGG cx for every irrep $\mathbb{W}$ of $\mathbb{G} = so(n+1,1)$!
· defn cx above from

- But how do we really calculate ?
- Curved case ? - e.g general curved mflds ?
- Other geometries ~

/ The bundles W are called <u>tractor bundles</u> :

Conformal tractors on S^n: [Riem..]

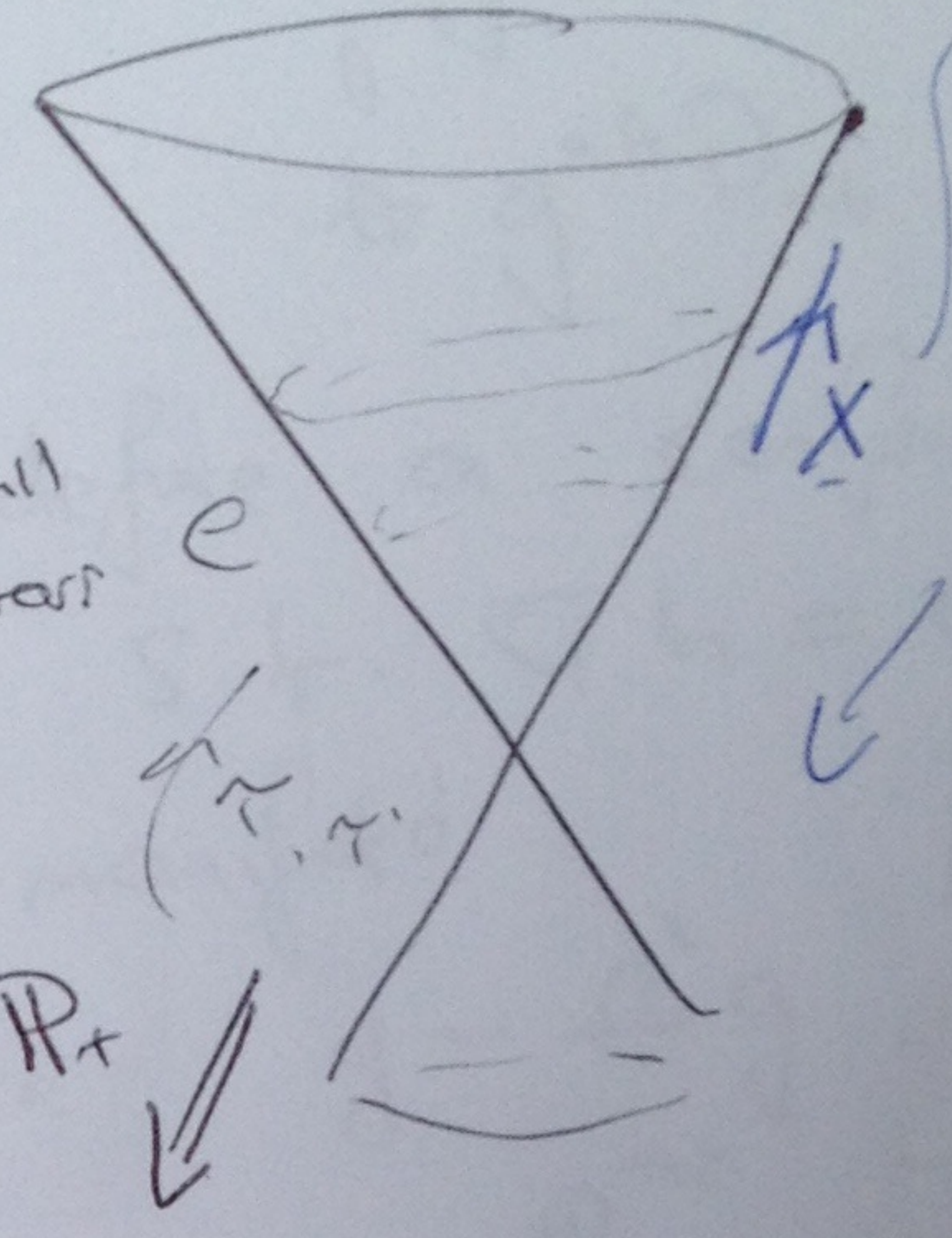
$\mathbb{R}^{n+2}$, X^A

+

$$h = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rightsquigarrow$$

forward Cone of null vectors $\mathcal{C}$

$\underline{X} = X^A \frac{\partial}{\partial X^A}$



Euler v. field $\forall x$

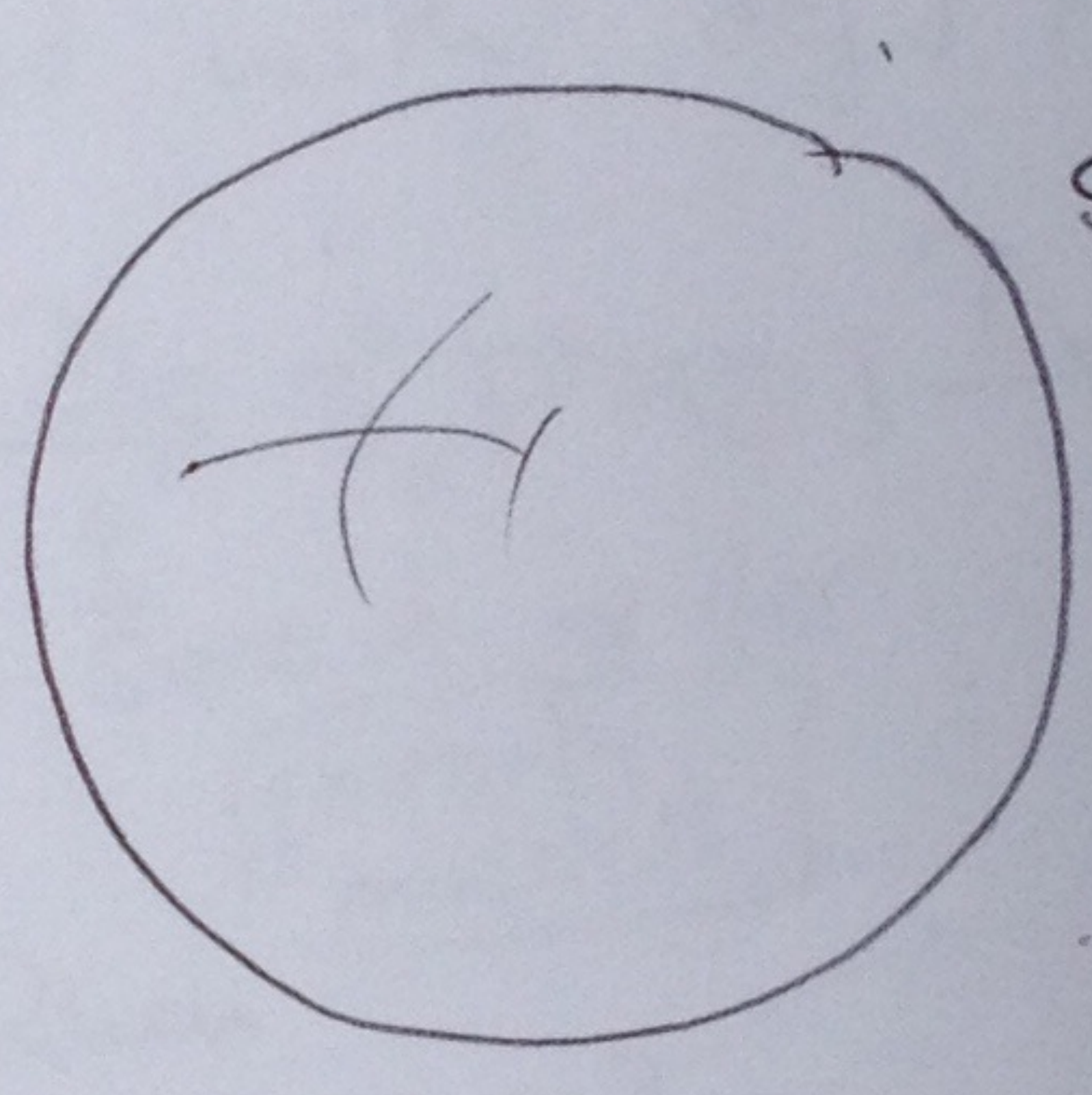
$T_x\mathbb{R}^{n+2} \cong \mathbb{R}^{n+2}$

trivial ↗ ∴ ∇

π

$G = SO_0(n+1,1)$ acts

$\cancel{\mathcal{C}}\ \cancel{=}\ \mathbb{R}_+$



S^n, $g_\tau \sim g_{\tau'}$

$g_\tau = \Omega^2 g_{\tau'}$ $\Omega > 0$

∴ S^n, c.

Then cf above

$\mathbb{T} = G \times_P \mathbb{R}^{n+2}$

∇ as there . .

$\mathbb{T} := T\mathbb{R}^{n+2}|_{\mathcal{C}} / \sim$, ∇ from on $\mathbb{R}^{n+2}$

h from h.

$\nabla h = 0$

$v_x \sim v'_{x'}$ if (def) parallel and x, x' in same ~~$\pi(v_x) = \pi(v_{x'})$~~ fibre

By construction

tractor bundle, conn, metric $:= (\mathbb{T}, \nabla, h)$

are conf invariant, $\nabla h = 0$

Thm ①:

$$\mathbb{T} = \begin{array}{c} \mathcal{E}[1] \\ \oplus \\ T^*M[+1] \\ \oplus \\ \mathcal{E}[-1] \end{array} \ni V = \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \mapsto \nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + P_{ab}\sigma + g_{ab}\rho \\ \nabla_a \rho - P_{ab}\sigma \end{pmatrix}$$

$\hookrightarrow h(V,V) = 2\sigma\rho + g_\tau(\mu, \mu)$

where

$$\mathrm{Ric}^{g}_{ab} \cancel{\equiv} \doteq (n-2)P_{ab} + J g_{ab}$$

$$J = g^{ab}P_{ab}.$$

$\underline{\mathrm{Th}^m}$ ②: Same formulae define a conformally invariant $(\mathcal{T}, \nabla, h)$ s.t. $\nabla h = 0$ on (M, c) conformal manifold, $c = [g]$ where $g \sim \Omega^2 g$.

— then ∇ not flat.

The BGG associated with def^g rep^n $\mathbb{R}^{n+2}$ of $G = SO_0(n+1,1)$

$$0 \to \mathbb{R}^{n+2} \to \mathcal{E}[1] \xrightarrow{D_0} S^2_0 T^*M[1] \xrightarrow{D_1} \boxed{}_0[1] \to \boxed{}_0 \to \cdots$$

↑ second order..

Claim

$\boxed{\text{partially massless}}$

of nay

Check start:

of twisted de Rham:

tractor con.

$$0 \to \mathbb{R}^{n+2} \to \mathcal{A}^0(\mathcal{T}) \xrightarrow{d^\nabla} \mathcal{A}^1(\mathcal{T}) \xrightarrow{d^\nabla}$$

$$\begin{pmatrix} \sigma \\ \mu_a \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b + P_{ab}\sigma + g_{ab}\rho \\ \nabla_a\rho - P_{ab}\mu^b \end{pmatrix}.$$

<u>kernel</u> ;? : ↑ in kernel if $\mu_a = \nabla_a\sigma$

$$\Rightarrow \boxed{\nabla_a\nabla_b\sigma + P_{ab}\sigma + g_{ab}\rho = 0}$$

$$\& \quad \rho = \frac{-1}{n}(\Delta\sigma + J\sigma)$$

i.e. if ~~∃~~ where

$$D_0\sigma = 0 \quad \left(D_0\sigma = \text{trace-free}(\nabla_a\nabla_b\sigma + P_{ab}\sigma) \right)$$

$\updownarrow$

$\exists\, g \in c$ s.t. Ein. if σ nowhere zero

For next op. : $\Lambda^1(\mathbb{T})$

$$S^2_0T^*[1] \ni h_{ab} \xrightarrow{L} \begin{pmatrix} 0 \\ h_{bc} \\ \delta h \end{pmatrix} \overset{d^\nabla}{\longmapsto} \nabla_{[a} h_{b]c}$$

$= \cancel{h_{ab}}$

h_{bc}

$\uparrow$ conf. splitting operator . . .

The G.G. def^n cx ? – is from rep^n $\mathbb{W} = \Lambda^2\mathbb{R}^{n+2}$, of $\mathfrak{G}$

$\mathfrak{so}(n+1,1) \cong \Lambda^2\mathbb{R}^{n+2}$, $SO(n+1,1) = \mathfrak{G}$

$$0 \to \Lambda^2\mathbb{R}^{n+2} \to TM \xrightarrow[\text{1st order}]{D_0} S^2_0T^*M[2] \xrightarrow[\text{2nd order}]{} \boxed{+}_0[2] \to \cdots$$

bundles are determined by affine action of Weyl gp . . .

Check: of twisted de Rham

$$0 \to \underset{\mathfrak{so}(n+1,1)}{\Lambda^2\mathbb{R}^{n+1}} \to \Lambda^0(\Lambda^2\mathbb{T}) \xrightarrow{d^\nabla} \Lambda^1(\Lambda^2\mathbb{T}) \xrightarrow{d^\nabla} \cdots$$

$$\begin{pmatrix} & \Lambda^1[2] & \\ \Lambda^2[2] & & \Lambda^0 \\ & \Lambda^1E & \end{pmatrix} \ni \underbrace{\begin{pmatrix} & \sigma_b & \\ \mu_{bc} & & \kappa \\ & \rho_b & \end{pmatrix}}_{U} \longmapsto \begin{pmatrix} \nabla_a\sigma_b - \mu_{ab} + g_{ab}\kappa & \\ * & * \\ & * \end{pmatrix}$$

∴ $\nabla^+ U = 0 \Rightarrow D_0\sigma = 0$, μ skew, $g \cdot \kappa$ trace.:

t.f $(\nabla_a\sigma_b + \nabla_b\sigma_a) = 0$. conf Killing ✓✓

On S^n ~~this at.~~ $D_0\sigma = 0 \iff \nabla U = 0$

∴ first part of GG BGG ✓✓

A general way to compute: Baston, CSS, Calderbank-Diemer: Mexico.

$\mathrm{Lie}(P) = \mathfrak{p} = \mathfrak{l} + \overset{\cong \mathbb{R}^n \text{ abelian}}{\mathfrak{p}_+}$

$\uparrow \left\{ \left(\begin{array}{c|c|c} 0 & * & * \\ \hline 0 & 0 & * \\ & & * \\ \hline 0 & 0 & 0 \end{array} \right) \right\} < \mathfrak{so}(n+1,1)$

$\mathbb{W}$ rep^n of $G \rightsquigarrow$ rep^n of $\mathfrak{p}_+$ (desibe & sephrizl)

$$\Rightarrow\ 0 \leftarrow \mathbb{W} \xleftarrow{\partial^*} \mathfrak{p}_+ \otimes \mathbb{W} \xleftarrow{\partial^*} \cdots \xleftarrow{\partial^*} \Lambda^n \mathfrak{p}_+ \otimes \mathbb{W} \leftarrow 0$$

↳ Lie alg cohom.

reps of P ✓ & P-equivariant

$\therefore \rightsquigarrow G \times_P$ "

$$\partial^*(Z_0 \wedge \cdots \wedge Z_k \otimes v) = \sum_{i=0}^{k} (-1)^{i+1} Z_0 \wedge \cdots \hat{i} \cdots \wedge Z_k \otimes v)$$

$\Rightarrow$ $W = G \times_P \mathbb{W}$ – tractor bdle

$$\Rightarrow\ 0 \leftarrow \Lambda^0(W) \xleftarrow{\partial^*} \Lambda^1(W) \xleftarrow{\partial^*} \cdots \xleftarrow{\partial^*} \Lambda^n(W) \leftarrow 0.$$

~~cf~~ algebraic $\therefore$ cohom

$\boxed{H^*_k = \ker \partial^* / \operatorname{im} \partial^*}$

↳ bdles on BGG ! ..

cf twisted de Rham

$$0 \longrightarrow \Lambda^1(W) \xrightarrow{d^\nabla} \Lambda^1(W) \xrightarrow{d^\nabla} \cdots \xrightarrow{d^\nabla} \Lambda^n(W) \longrightarrow 0$$

$\partial^* \circ d^\nabla : \Lambda^k(W) \circlearrowleft$, on $\operatorname{im} \partial^* =$

$\subset \Lambda^k(W)$

acts invertible tensor on top part & preserves filtration in there

$\Rightarrow$ $\partial^* \circ d^\nabla$ invertible on $\operatorname{im} \partial^*$ & by diff op [no need ΨDO]

$Q :=$ inverse (and explicit finite formula)

Mexico.

__Define__: $L : \overset{\Gamma(H_k)}{\Gamma(H_k)} \longrightarrow \Lambda^k(W)$ by
$$[\varphi] \mapsto \varphi - Q\partial^* d^\nabla \varphi .$$

well defined ✓ [as $\varphi' \in [\varphi] \Rightarrow \varphi' - \varphi \in \operatorname{im}\partial^*$
& $\therefore Q\partial^* d^\nabla(\varphi' - \varphi) = \varphi' - \varphi$.
$\therefore \varphi' - \varphi$ in kernel of linear map.]

Write ~~$\tilde{\pi}_H$~~ $\boxed{\ker\partial^* \ni \phi \overset{\pi_H}{\longmapsto} [\phi] \in H_k}$.

__Then__:

All works in curved case

__Th__^{m}: $[\varphi] \in H_k$

$\Rightarrow \partial^* L[\varphi] = 0$ ①, $\pi_H L[\phi] = [\phi]$ ② and $\underbrace{\partial^* d^\nabla L[\phi] = 0}_{③}$

[and these properties characterize L.]

$\Downarrow$

__Th__^{m}: $\exists$ an invariant (diff^l) operator
$$D_k := \pi_H \circ d^\nabla \circ L : H_k \longrightarrow H_{k+1}$$

$$\begin{tikzcd}
\Lambda^k(W) \arrow[dr, squiggly] & \\
H_k \arrow[u, "L"] \arrow[r, "D_k"] & H_{k+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Lambda^k(W) & \Lambda^{k+1}(W) & \\
\text{①} & \ker\partial^* \arrow[u, phantom, sloped, "\subset"] \arrow[r, "d^\nabla"] & \ker\partial^* \arrow[u, phantom, sloped, "\subset"] \arrow[d, "\pi_H"] & \text{③} \\
 & H_k \arrow[u, "L"] & H_{k+1} &
\end{tikzcd}$$

These give the BGG ops:

$$0 \to W \longrightarrow H_0 \longrightarrow H_1 \longrightarrow \cdots$$

Get $D_{k+1} \circ D_k = 0$ from $d^\nabla \circ d^\nabla = 0$ on __Sphere__.
or loc flat case

In fact __locally exact__ $\therefore$
BGG computes same cohom. as twisted de Rham.

Rod Gover

Detour complexes:

Last time:

M^n smooth: ⇝ de Rham

$$0 \longrightarrow \mathbb{R} \longrightarrow \Lambda^0 \xrightarrow{d} \Lambda^1 \longrightarrow \cdots \longrightarrow \Lambda^n \longrightarrow 0$$

$(M^n, [g])$ conformal: $\exists$ (diff^l ~~ex~~ reso^n on $\boxed{\text{conf flat}}$
sig (p,q)

$$0 \longrightarrow \mathbb{W} \longrightarrow B^0 \xrightarrow{D_0} B^1 \xrightarrow{D_1} \cdots \longrightarrow B^n \longrightarrow 0$$

change here if not conf flat; irred tensor / ~~spinor~~ bdles [except middle forms] even dim.

$\forall\ \mathbb{W}$ an irred rep^n of $SO(p+1, q+1)$.

Problem: On general $(M, [g])$ – i.e. not conf. flat – $\exists$ BGG sequence [D_i still canonical ✓] but they are $\boxed{\text{not complexes. } \ddot{\frown}}$

Idea: Seek weaker integrability cond^{n}s D_i "too tight" etc.

Q: What are all conf invariant (diff^l operators involving the bdles of the BGG cxs?

A: n-odd – just the D_i of BGG nothing new.

But n even:

Th^m: $(S^{n\ \text{even}}, [g_{\text{round}}])$ [Boe-Collingwood, –]

$$\begin{tikzcd}
 & & & & & & & & B^n \arrow[d, phantom, "\cong\ \text{Hodge } *"] \\
B^0 \arrow[r, "D_0"] \arrow[rrrrrrrr, bend right=30, "L_0"'] & B^1 \arrow[r, "D_1"] \arrow[rrrrrr, bend right=30, "L_1"'] & B^2 \arrow[r] \arrow[rrrr, bend right=30, "L_2"'] & \cdots \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend right=40, "(X)"'] & B^{n/2} \arrow[r, shift left] \arrow[r, shift right] & \cdots \arrow[r, "D_2^*"] & B_2 \arrow[r, "D_1^*"] & B_1 \arrow[r, "D_0^*"] & B_0
\end{tikzcd}$$

is a complete list [of conf D.ops involving B^i's

i.e. all compositions vanish, – except (X) is a comp^n.

Coroll: $(M^{n \text{ even}}, [g])$ conf flat. $\exists$ detour complexes:

(*)

$$\begin{tikzcd}
B^0 \arrow[r, "D^0"] & B^1 \arrow[rr, "L_1"] & & B_1 \arrow[r, "D_0^*"] & B_0 & \\
B^0 \arrow[r] & B^1 \arrow[r] & B^2 \arrow[r, "L_2"] & B_2 \arrow[r] & B_1 \arrow[r] & B_0
\end{tikzcd}$$

etc: & L_k weaker inv. cond^n than D_k $\Rightarrow$ **not** locally exact

↑ actually an interesting feature.

Q: $\exists$ curved analogues? A: Yes - interesting, delicate, rich theory emerging.

(Special Case $B^k = \Lambda^k$ - diff^l forms

Th^m: [Branson + G] On any $(M^n, [g])$ $\exists$ complexes (*) for case $B^k = \Lambda^k$ s.t. conf inv., i.e., D_k, L_k invariant (& elliptic) under $g \mapsto \Omega^2 g$ $\Omega > 0$. //

$\sim * d *$ - FA d.

In fact:

$$L_k = \delta \underbrace{((d\delta)^{n/2-k-1} + l.o.t)}_{Q_{k+1}} d$$

Linked to Q-curv.

$$G_k := \delta Q_k$$

Then G_k a gauge-fixing operator for L_k.,

alt. for d

i.e., system & : $\begin{pmatrix} d \\ G_k \end{pmatrix} : \Lambda^k \to \begin{pmatrix} \Lambda^{k+1} \\ \oplus \\ \Lambda_{k-1} \end{pmatrix}$

"conf. inv" &

is, elliptic in Riem signature.

—

$(\delta d \delta + \dots)$

When $n=4$ G_k = Eastwood-Singer gauge operator for Maxwell. : ~~Sd~~ ~~on~~ $n=4$: $\delta d_{\text{Max}} : \Lambda^1 \to \Lambda_1$ conf inv . δA - Coulomb gauge not conf inv G_1 is on ker δd.

Proof (uses in general ~~ambt~~ F. G. ambient metric [for P-E. G- 3. of ([illegible]-[illegible])]
& link to tractors.

E.g. Eastwood-Singer case.

$$\begin{tikzcd}
\mathcal{A}^0(\mathcal{T})[-1] \arrow[rr, "\square = \overbrace{\Delta + \text{lots}}^{\text{conf Laplacian coupled tractor}}"] & & \mathcal{A}_1(\mathcal{T})[-3] \\
\cup & \begin{pmatrix} 0 \\ A \\ -\frac{1}{2}\delta A \end{pmatrix} \arrow[r, mapsto, "\square\begin{pmatrix} 0 \\ A \\ -\frac{1}{2}\delta A \end{pmatrix}"'] & \begin{pmatrix} 0 \\ \delta dA \\ G\star A \end{pmatrix} \in \begin{pmatrix} \Lambda_1 \\ \oplus \\ \Lambda_0 \end{pmatrix} \\
\text{subbundle of } \mathcal{T}[-1] \arrow[r, squiggly] & \text{“}\ker \delta^*\text{”} & \\
& L \arrow[u] \quad ? & \\
\Lambda^1 \arrow[rr, bend right, "\mathcal{A}^1"'] & \ni A \arrow[uuu] &
\end{tikzcd}$$

Many applications to constr of global invts ..

Other curved detours? / General theory?

Simplest de Rham detour: Maxwell case~~[illegible]~~:

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\delta d} \Lambda_1 \xrightarrow{\delta} \Lambda_0 \quad (*).$$

- $\delta dA = 0$ Maxwell eq^ns on potential A.

  Gauge transf^n A ~~*~~ $\mapsto A + df$

- conf. invariant $\boxed{\text{if } n = 4}$, [metric not involved really!]

Q: Can we twist/couple to get other cxs?

of constr of BGGs.

Let D. be any ~~other~~ connection. Coupling

$d \mapsto d^D$. fails, if D curved

4
M-exico $\overline{\text{II}}$

But: Suppose $\begin{matrix} V \\ \downarrow \\ M \end{matrix}$ v. bdle with

a connection. D. ($\rightsquigarrow$ D on V^* etc) + curvature

F..

Write: $F\bullet : \Lambda^1(V) \overset{= \Lambda^1 \otimes V}{} \longrightarrow \Lambda_1(V)$ for curv. action

$\overset{\Psi}{\varphi_a} \longmapsto F\bullet\varphi := \underbrace{F_a{}^b}_{} \varphi_b$.

$\uparrow$ V ~~ea~~ indices suppressed

not include End(V) - curvature action on φ.

$\uparrow$ wh depends on nature of V

Define: $M^D : \Lambda^1(V) \longrightarrow \Lambda_1(V)$

by $\varphi \longmapsto \delta^D d^D \varphi - F^D \bullet \varphi$.

Then (exercise): $M^D d^D \psi = \varepsilon(\delta^D F^D)\psi$ $\quad \psi \in \Lambda^0(V)$.

alg. action

$(\Rightarrow)$ & $\delta^D M^D \varphi = \iota(\delta^D F^D)\cdot\varphi \quad \varphi \in \Lambda^1(V)$.

$\uparrow$ contracts 1-form index

$\therefore$

Thm: (G-Sam.-Sau)

$$\Lambda^0(V) \xrightarrow{d^D} \Lambda^1(V) \xrightarrow{M^D} \Lambda_1(V) \xrightarrow{\delta^D} \Lambda_0(V)$$

- is a complex (nec. elliptic in Riem sig case) iff $\delta^D F = 0$ - ie, D is Yang-Mills (source free)
- If D unitary $\Rightarrow$ seq. FSA
- If $n=4$ conf. inv.

NB: Many examples.

- E.g. ∇ = Levi-Civita of g is Harmonic i.e., $\nabla^a R_{ab}{}^c{}_d = 0$ [e.g. Ein.]
- If D YM $\Rightarrow$ D on $\otimes$ powers V YM, etc.

What's going on?

[Two special cases suggest part of the story]

① Dim 4: oriented. Hodge $*: \Lambda^2 \circlearrowleft$ & this bundle map conf inv. $*$ real symm. on Λ^2

Riem case – other sig sim use cx

$*^2 = 1$, $\Rightarrow \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ e'spc ± 1 wrt $*$ [cf Mon talk]

$\Rightarrow F = F_+ + F_-$

Say F anti-self-dual (ASD) if $F_+ = 0$ i.e. $*F = -F$.

Then well-known that

$$\begin{tikzcd} & & \Lambda^2_+(V) \arrow[d, phantom, "\overset{?}{\|}"] & & \\ \Lambda^0(V) \arrow[r, "d^D"] & \Lambda^1(V) \arrow[ur, "d^D_+"] & \Lambda^2_+(V) \arrow[r, "\delta^D_+"'] & \Lambda^*_1(\Lambda) \arrow[r] & \Lambda_0(V) \end{tikzcd}$$

$\leftarrow$ elliptic cx

Donaldson theory/gauge th

Adjoint

$\therefore$ get

$$\begin{tikzcd} \Lambda^0(V) \arrow[r] & \Lambda^1(V) \arrow[r, "\delta^D_+ d^D_+", "\overset{\|\,(Ex)}{M_D}"'] & \Lambda_1(V) \arrow[r] & \Lambda_0(V) \end{tikzcd}$$

as special case. as ASD $\overset{\text{Bianchi}}{\Rightarrow}$ YM, SD $\Rightarrow$ YM

[But **NB**: Previous thm did not need ASD or SD! – just YM]

A variational construction ! [of class of cases]
(source free)

– YM eq^n's are Euler-Lagrange eq^ns of action. $S = \int_M |F|^2$ ← [illegible] F ≈ F.

– now look at variation of YM eq^ns – so 2^nd variation of S. $D \mapsto \delta^D F$ · [illegible]

– i.e., family $D^t v := Dv + A^t v$ $A^0 = 0$, $A^t \in \Lambda^1(\operatorname{End} V)$.

$$\overset{\text{Ex.}}{\Longrightarrow} \boxed{\frac{d}{dt}\, \delta^{D} F^t \Big|_{t=0} = M^D \dot{A}} \quad (*)$$

∴ **Lemma**: D YM ⟹ inf^l deformation $\dot{D} = \dot{A}$ is thru' YM connections iff $M^D \dot{A} = 0$.

– **Next**, gauge transf^ns: $u \in \Gamma(\operatorname{Aut}(V))$ acts on D by

$D \mapsto D_a + (u^{-1} D_a u)$, now family of such u_s^s, $s \in \mathbb{R}$, $u_0^0 = id$

⇝ $\dot{A} = d^D \dot{u}$ if $\dot{A}$ from gauge. ⑪ – into [illegible]. &

⇝ $\boxed{\delta^D F \rightarrow u_s^{-1}(\delta^D F) u_s}$! – gauge covariant ②

YM current

Compare $d/ds|_{s=0}$ this with (*) in special case

$\dot{A} = d^D \dot{u}$ $\dot{u} = \frac{d}{ds} u\Big|_{s=0}$

$$\Longrightarrow \boxed{M^D d^D \dot{u} = \mathcal{E}(\delta^D F) \cdot \dot{u}}$$ obtain again in this special case of End V.

∴ directly

Th^m: $0 \to \Lambda^0(\operatorname{End}(V)) \xrightarrow{d^D} \Lambda^1(\operatorname{End}(V)) \xrightarrow{M^D} \Lambda_1(\operatorname{End}(V)) \to \Lambda_0(\operatorname{End}(V)) \to 0$

a cx iff D YM, & F.S.A since this a second variation.

- H^1 = Formal tgt sp to moduli sp of YM deformations

<u>Translation</u> (of BGG constr^n).

<u>Prop^n</u>: Suppose • D YM & preserves a metric on V, and
• $\exists$ $\mathcal{B}^0, \mathcal{B}^1$ & L_0, L_1 diff^l ops s.t

$$\begin{tikzcd}
\Lambda^0(V) \arrow[r, "d^D"] & \Lambda^1(V) \arrow[r, "M^D"] & \Lambda^1(V) \arrow[r] \arrow[d, "L_1^*"] & \Lambda_0(V) \arrow[d, "L_0^*"] \\
\mathcal{B}^0 \arrow[u, "L_0"] \arrow[r, "\mathcal{D}"] & \mathcal{B}^1 \arrow[u, "L_1"'] \arrow[r, dashed, "L_1^* M^D L_1"] & \mathcal{B}_1 \arrow[r, "\mathcal{D}^*"] & \mathcal{B}_0
\end{tikzcd}$$

sq commutes. Then last sq. commutes and

$$\mathcal{B}^0 \longrightarrow \mathcal{B}^1 \longrightarrow \mathcal{B}_1 \longrightarrow \mathcal{B}_0$$

a (FSA) <u>complex</u>.

<u>Example</u>: $V = \mathcal{T} = \begin{matrix} \Lambda^0[1] \\ \oplus \\ \Lambda^1[1] \\ \oplus \\ \Lambda^0[-1] \end{matrix}$, $\overset{\text{tractor connection}}{D_a} \begin{pmatrix} \sigma \\ \mu \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab}\rho + P_{ab}\sigma \\ \nabla_a \rho - P_{ab}\mu^b \end{pmatrix}$

$\exists$ L_0, L_1 s.t 1st sq. commutes, (& $\exists$ metric h on $\mathcal{T}$ 〃) with

$$\mathcal{D}\sigma = t.f.(\nabla_a\nabla_b + P_{ab})\sigma.$$

$$\mathcal{D}\sigma = 0 \iff \sigma^{-2}g \text{ Ein.} \quad \text{if } \sigma \text{ nowhere zero}$$

<u><u>Th^m</u></u>: On (M^4, g) the seq.

$$\Lambda^0[1] \xrightarrow{\mathcal{D}} S^2_0\Lambda^1[1] \xrightarrow{M^B} S^2_0\Lambda_1[-1] \xrightarrow{\mathcal{D}^*} \Lambda^0[-5]$$

is conf inv^t, FSA and.

$$M^B \circ \mathcal{D}\,\sigma = \underbrace{(\text{Bach})}_{} \cdot \sigma \quad \text{conf inv}^t \text{ Bach tensor}$$

– so is a cx iff Bach = 0 !

<u><u>and</u></u> Coroll. g Ein $\Rightarrow$ Bach $= 0$ [known before ☺)

Another ex. :

Th^m : (M^4, g).

$\subseteq \Lambda^1 \otimes \mathbb{S}$ s.t. $\gamma^a \phi_a = 0$ ← third order

$$\mathbb{S} \xrightarrow{\underline{I}} Tw \xrightarrow{N} Tw^* \xrightarrow{\underline{I}^*} \mathbb{S}^*$$

$\mathbb{S}$ ← basic spinor $\otimes \mathcal{E}[1/2]$

$\underline{I}s = \mathrm{Proj}_{Tw} (\nabla_a s)$.

$\mathbb{S}^* = \mathbb{S} \otimes \mathcal{E}[-n] \otimes [1/2$

And. $N \circ \underline{I} = B_{ab} \gamma^b$

Bach ; γ matrices ~ Clifford action

Th^m : [Br + G.] related const^n. $(M^{n\text{-even}}, g)$: Tof G.G

There is a detour (conf invt) ~~cx~~ seq . for def^n cx ,

$$\Lambda^1[2] \xrightarrow{K_0} S^2_0 \Lambda^1[2] \xrightarrow{\mathbb{B}} S^2_0 \Lambda^1[2-n] \xrightarrow{K_0^*} \Lambda^1[-n]$$

with. $\mathbb{B} \circ K_0 (\cdot) = \mathcal{L}_{(\cdot)}$ (Feff. Graham obst^n tensor).

$v \in TM$

generalise Bach to higher dim

So this a cx iff $\boxed{g \text{ FG obst}^n\text{-flat}}$

seq. governs deformations of FG flat mflds.

Also ∃ non-conformal detours related to def^{n}s of Ein mflds - ~~[illegible]~~ due to Calabi & using Spencer theory. ← so (partially massless) spin-2