Gravitational turbulent instability of AdS

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In collaboration with
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Motivation

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- In this work we want to study far from equilibrium dynamics in gravity, and try to understand its field theory interpretation.
  - A poor’s man approach: break down of perturbation theory - onset of interesting dynamics.
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Anti-de Sitter space is a **maximally symmetric** solution to

\[ S = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left[ R + \frac{(d - 1)(d - 2)}{L^2} \right], \]

which in **global coordinates** can be expressed as

\[ ds^2 = \bar{g}_{ab} dx^a dx^b = - \left( \frac{r^2}{L^2} + 1 \right) dt^2 + \frac{dr^2}{r^2 \left( \frac{r^2}{L^2} + 1 \right)} + r^2 d\Omega^2_{d-2}. \]

The **Poincaré coordinates**

\[ ds^2 = R^2 (-d\tau^2 + dx \cdot dx) + \frac{L^2 dR^2}{R^2} \]

**do not cover** the entire spacetime.
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- Poincaré coordinates cover the brown-shaded region.
- The instability described in this talk will occur in global AdS only.
- The dual field theory lives on $\mathbb{R}_t \times S^{d-2}$.
- With energy preserving boundary conditions, waves bounce off infinity and return in finite time.
Minkowski, dS and AdS spacetimes

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Claim:

Generic small (but finite) perturbations of AdS become large and eventually form black holes.

- The energy cascades from low to high frequency modes in a manner reminiscent of the onset of turbulence.
Folklore - 1/2

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The short answer, is NO:

- Positivity energy theorem: if matter satisfies the dominant energy condition, then $E \geq 0$ for all nonsingular, asymptotically AdS initial data, being zero for AdS only. This ensures that AdS cannot decay. It does not ensure that a small amount of energy added to AdS will not generically form a small black hole. That is usually ruled out by arguing that waves disperse. This does not happen in AdS.
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Example (Dafermos):

Consider the following action:

\[ S = \int d^d x \sqrt{-g} \left[ R - (\nabla \phi)^2 \right]. \]

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Consider now the same action, but with the wrong sign for the scalar kinetic term:

\[ S = \int d^d x \sqrt{-g} \left[ R + (\nabla \phi)^2 \right]. \]

There is no positivity energy theorem, but Minkowski space is still nonlinearly stable.
AdS acts like a **confining finite box**. Any generic finite excitation which is added to this box might be expected to **explore all configurations** consistent with the conserved charges of AdS - including small black holes.
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- A perhaps more convincing intuitive picture: **colliding exact plane waves produces singularities** - Penrose - ’71.
Expand the metric as

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At each order in perturbation theory, the Einstein equations yield:

\[ \tilde{\Delta}_L h^{(i)}_{ab} = T^{(i)}_{ab}, \]

where \( T^{(i)} \) depends on \( \{ h^{(j \leq i-1)} \} \) and their derivatives and

\[ 2\tilde{\Delta}_L h^{(i)}_{ab} \equiv -\bar{\nabla}^2 h^{(i)}_{ab} - 2\bar{R}^{c}_{
abla_a b} h^{(i)}_{cd} - \bar{\nabla}_a \bar{\nabla}_b h^{(i)} + 2\bar{\nabla}_{(a} \bar{\nabla}^{c} h^{(i)}_{b)c}. \]
Perturbative construction - 1/3

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- Any smooth symmetric two-tensor can be expressed as a sum of fundamental building blocks, $T^{\ell m}_{ab}$, that have definite transformation properties under the $SO(d-1)$ subgroup of AdS.
For concreteness, set $d = 4$. Perturbations come in three classes:
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- **Vector-type perturbations:** perturbations are constructed from vector harmonics on $S^2$ - these are $\star S^2 \nabla Y_{\ell m}$.
- **Tensor-type perturbations:** only exist in $d \geq 5$. 


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- We go beyond linear order: need **real representation** for $Y_{\ell m}$ - $Y_{\ell m}^c = \cos \phi L^m_\ell (\theta)$ and $Y_{\ell m}^s = \sin \phi L^m_\ell (\theta)$. 

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At each order, we can reduce the metric perturbations to 4 gauge invariant functions satisfying (Kodama and Ishibashi ’03 for $i = 1$):

$$\Box_2 \Phi^{\alpha, (i)}_{\ell m} (t, r) + V^{(i)}_\ell (r) \Phi^{\alpha, (i)}_{\ell m} (t, r) = \tilde{T}^{\alpha, (i)}_{\ell m} (t, r),$$

where $\alpha \in \{c, s\}$ and $\Box_2$ is the w. op. in the $(t, r)$ orbit space.
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where $\alpha \in \{c, s\}$ and $\Box_2$ is the w. op. in the $(t, r)$ orbit space.

Choice of initial data relates $\Phi_{\ell m}^{c,(i)}$ and $\Phi_{\ell m}^{s,(i)}$: 2 PDEs to solve.
Boundary conditions:

- Regularity at the origin \((r = 0)\) requires

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  \[ \Phi_{\ell m}^{\alpha, (i)}(t, r) \sim R_{\ell m}(t) + \frac{S_{\ell m}(t)}{r} + \ldots. \]

Surprisingly, if we want to keep the boundary metric fix, we need to choose
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- This is also the choice that **conforms with finite energy** for the standard definition of “gravitational energy”.
At the linear level \((i = 1)\) we can further decompose our perturbations as

\[
\Phi^{\alpha,(i)}_{\ell m}(t, r) = \Phi^{\alpha,(i),c}_{\ell m}(r) \cos(\omega \ell t) + \Phi^{\alpha,(i),s}_{\ell m}(r) \sin(\omega \ell t).
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Linear Perturbations

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- Because AdS acts like a confining box, **only certain frequencies are allowed to propagate**

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  \omega_{\ell}^2 L^2 = (1 + \ell + 2p)^2,
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  where \(p\) is the radial overtone. These are the so-called **normal modes of AdS**. The fact that \(\omega^2 L^2 > 0\) means that AdS is **linearly stable**.
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- For simplicity, we will take \(p = 0\), in which case one finds

\[
\Phi^{\alpha,(1),\kappa}(r) = A^{\alpha,(1),\kappa} \frac{r^{\ell+1}}{(r^2 + L^2)^{\frac{\ell+1}{2}}},
\]

where \(A^{\alpha,(1),\kappa}\) is a normalization constant.
General Structure

1. Start with a given perturbation $\Phi^{\alpha,(i),\kappa}_{\ell m}(r)$, and determine the corresponding $h^{(i)}_{\ell m}(t, r, \theta, \phi)$ through a linear differential map.
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4. If $\tilde{T}^{\alpha,(i+1)}_{\ell m}(t, r)$ has an harmonic time dependence $\cos(\omega t)$, then $\Phi^{\alpha,(i+1)}_{\ell m}(t, r)$ will exhibit the same dependence, **EXCEPT** when $\omega$ agrees with one of the normal frequencies of AdS:

$$\Phi^{\alpha,(i+1)}_{\ell m}(t, r) = \Phi^{\alpha,(i+1),c}_{\ell m}(r) \cos(\omega t) + \Phi^{\alpha,(i+1),s}_{\ell m}(r) t \sin(\omega t).$$

This mode is said to be **resonant**.
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   This mode is said to be resonant.

5. If for a given perturbation one can construct $\Phi_{\ell m}^{\alpha,(i)}$ to any order, without ever introducing a term growing linearly in time, the solution is said to be stable and is unstable otherwise.
Example I - 1/2

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- The structure of the equations indicate that there is only one resonant term at each odd order, and that the amplitude of the growing mode can be set to zero by correcting the frequency.
- One can compute the asymptotic charges to fourth order, and they readily obey to the first order of thermodynamics:

$$E_g = \frac{3J_g}{2L} \left(1 - \frac{4901 J_g}{7560\pi L^2}\right), \quad \omega_2 = \frac{3}{L} \left(1 - \frac{4901 J_g}{3780\pi L^2}\right),$$

where we defined $\epsilon$ by $J_g = \frac{27}{128} \pi L^2 \epsilon^2$. 
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Resonances occur because normal modes of AdS take integer values:
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- Resonances occur because normal modes of AdS take integer values:
  - **Geons** are likely to be “more” stable than AdS because the normal modes of the Geons correspond to continuous deformations of the AdS normal modes.
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- If $E < E_s$, the cascades stops at frequencies $\omega = E$, and one gets a gas of particles in AdS.
Field theory implications - 1/2:

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In 2+1 dimensions, classical turbulence has an inverse energy cascade due to an extra conserved quantity - the enstrophy. Our results indicate that in a strongly coupled quantum theory, there is a standard energy cascade.

Caveat: This intuition comes from solving the Navier Stokes equations in 2+1 dimensions. Because our regime is non-hydro, we don’t know how to define enstrophy.
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  \[ K = \frac{\partial}{\partial t} + \frac{\omega}{m} \frac{\partial}{\partial \phi}, \]

  which is timelike near the poles but spacelike near the equator.
Gravitational hairy black holes with a single $U(1) - 1/2$.

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- The Kerr-AdS is **not the unique** stationary black hole in AdS.
These black holes can be seen as metastable configurations in a time evolution towards the endpoint of superradiance.
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**Superradiance:**

- If a wave $e^{-i\omega t} + m\phi$ scatters off a rotating black hole with $\omega < \Omega_H$, it can return with a larger amplitude - superradiance.

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**What is the endpoint?**

- Simple systems with a single unstable mode: the final state will be the **rotating black hole with a single $U(1)$** or oscillations thereof.
- Superposition of modes: **superradiance cause them to grow**; **turbulent instability will cause higher frequency modes** to be created.
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1. If the black hole absorbs the higher frequency modes faster than they can be created, **might stabilize with gravitational waves sloshing around outside the black hole**.
2. The black hole exterior might continue to evolve **toward higher and higher frequency - black moon?**
Conclusions & Open questions

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- Anti-de Sitter spacetime is nonlinearly unstable: generic small perturbations become large and (probably) form black holes.
- For each linearized gravity mode, there is an exact, nonsingular geon.
- Dual field theory shows generic turbulent cascade to maximum entropy state but there are special states (geons) that do not thermalize.

Open questions:

- Understand why the energy cascade in 2+1 quantum theory is different from the classical theory.
- Prove a singularity theorem for anti-de Sitter.
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