CONFORMAL GEOMETRY & PHYSICS

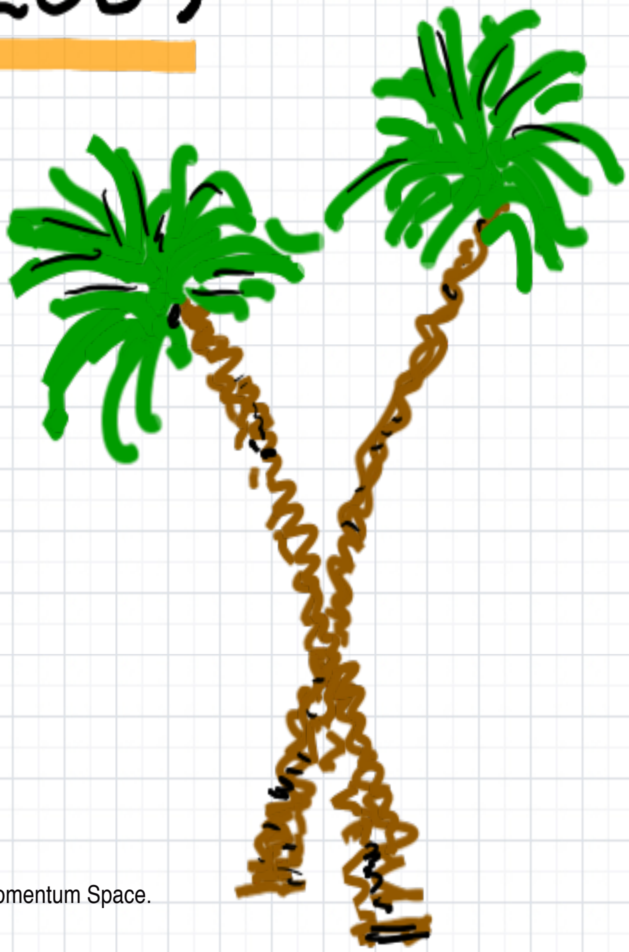
MIAMI 2009

— Andrew Waldron UC Davis

Work with:

— Rod Gover U. Auckland

— Abrar Shaukat UC Davis

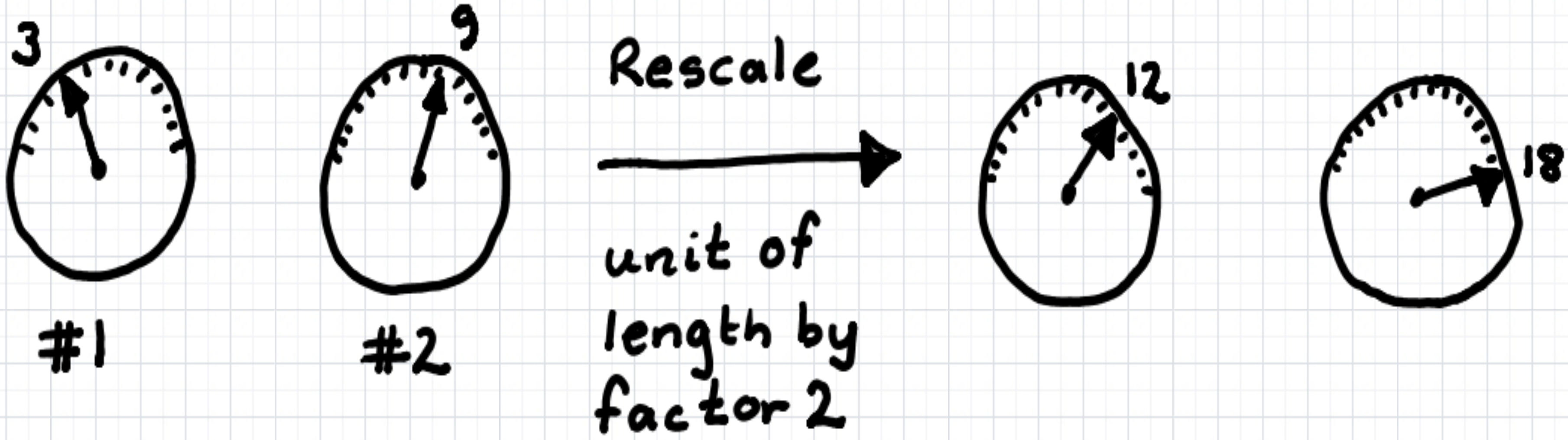


Weyl's Gauge Invariance: Conformal Geometry, Spinors, Supersymmetry, and Interactions.
Nucl. Phys. B, to appear
e-Print: arXiv:0911.2477 [hep-th]

The so(d+2,2) Minimal Representation and Ambient Tractors: the Conformal Geometry of Momentum Space.
e-Print: arXiv:0903.1394 [hep-th]

Weyl Invariance and the Origins of Mass.
Phys. Lett. B
e-Print: arXiv:0812.3364 [hep-th]

Tractors, Mass and Weyl Invariance.
Nucl.Phys.B812:424-455,2009.
e-Print: arXiv:0810.2867 [hep-th]



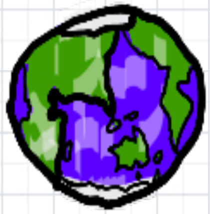
<u>Suspect</u> #1 is an AREA, #2 is a LENGTH

$\Rightarrow$ (#1) + (#2) is nonsense

In physics study actions with a rigid scaling symmetry:

$$S(\Phi_a ; \alpha_i) = S(\Omega^{w_a}\Phi_a ; \Omega^{w_i}\alpha_i)$$

Fields Couplings Weights



EARTH: inches

MOON: meters

WANT A LOCAL RESCALING SYMMETRY

WEYL

EXAMPLE: COORDINATE INVARIANCE

EINSTEIN

- "Gauge field" $g_{\mu\nu}$
- Parallelism ∇_μ
- Tensor calculus – Riemannian Geometry

SCALE

Local unit invariance requires a new, non-dynamical "gauge field" to measure relative rescalings of units

$$\sigma(x) \longmapsto \Omega(x)\,\sigma(x)$$

Local rescalings

In conformal geometry σ is called the SCALE,

In physics it amounts to a WEYL COMPENSATOR or DILATON

WEYL, ZUMINO, DESER

In a canonical choice of scale $\sigma = \text{constant}$

$$\sigma = \kappa^{\frac{2}{d-2}}$$

Newton's constant

This is the only "dimensionful" coupling needed!

GRAVITY

$$S_{EH} = -\frac{1}{2\kappa^2}\int \sqrt{-g}\, R$$

Comes from a locally scale invariant action   DESER, ZUMINO

$$S(g_{\mu\nu}, \sigma) = -\frac{1}{2}\int \frac{\sqrt{-g}}{\sigma^d}\left\{(d-1)(d-2)\left[\nabla_\mu \sigma\right]^2 + R\sigma^2\right\}$$

$$\left.\begin{aligned} g_{\mu\nu} &\longmapsto \Omega^2 g_{\mu\nu} \\ \sigma &\longmapsto \Omega\sigma \end{aligned}\right\} \text{“Weyl invariance”}$$

$$\boxed{S\left(g_{\mu\nu},\, \sigma = \kappa^{\frac{2}{d-2}}\right) = S_{EH}}$$

Invariant because:

- S is action for conformally improved scalar field
- σ is a Weyl compensator $\quad S(g_{\mu\nu},\sigma) = \kappa^2 S_{EH}(\sigma^{-2} g_{\mu\nu})$

or TRACTORS – Einstein Hilbert is square of scale tractor

$$\boxed{\begin{gathered} S(g_{\mu\nu},\sigma) = \frac{d(d-1)}{2}\int \frac{\sqrt{-g}}{\sigma^d}\, I^M \eta_{MN} I^N \\[2ex] I^M = \begin{pmatrix} \sigma \\ \nabla^m \sigma \\ -\frac{1}{d}\left[\Delta + \frac{R}{2(d-1)}\right]\sigma \end{pmatrix} \qquad \eta_{MN} = \begin{pmatrix} & & 1 \\ & \eta_{mn} & \\ 1 & & \end{pmatrix} \end{gathered}}$$

SCALE TRACTOR

SO(d,2) INVARIANT METRIC

TRACTOR GAUGE TRANSFORMATIONS

Under changes of scale $g_{\mu\nu} \mapsto \Omega^2 g_{\mu\nu}\,,\ \sigma \mapsto \Omega\sigma$

$$I^M \longmapsto U^M{}_N\, I^N$$

a tractor gauge transformation

$$\boxed{U^M{}_N = \begin{pmatrix} \Omega & 0 & 0 \\ \Upsilon^m & \delta^m_n & 0 \\ -\frac{1}{2}\Omega^{-1}\Upsilon^2 & -\Omega^{-1}\Upsilon^n & \Omega^{-1} \end{pmatrix} \in SO(d,2)}$$

where $\Upsilon_\mu = \Omega^{-1}\partial_\mu\Omega$

$\Rightarrow I^2 = I^M \eta_{MN} I^N$ invariant

EINSTEIN'S EQUATIONS

Introduce a tractor covariant derivative/connexion

$$\boxed{\mathcal{D}_\mu \begin{pmatrix} I^+ \\ I^m \\ I^- \end{pmatrix} \equiv \begin{pmatrix} \partial_\mu I^+ - I_\mu \\ \nabla_\mu I^m + P_\mu{}^m I^+ + e_\mu{}^m I^- \\ \partial_\mu I^- - P_\mu{}^n I_n \end{pmatrix}, \quad \underbrace{\mathcal{D}_\mu I^M}_{\text{Weight zero tractor}} \mapsto U^M{}_N \mathcal{D}_\mu I^N}$$

THOMAS, KAKU, TOWNSEND
VAN NIEUWENHUIZEN

Where the Schouten tensor

$$P_{\mu\nu} \equiv \frac{1}{d-2}\left(R_{\mu\nu} - \frac{1}{2}\frac{1}{d-1} g_{\mu\nu} R\right)$$

THEOREM $(M,[g_{\mu\nu}])$ conformally Einstein

$\Leftrightarrow$ $(M,[g_{\mu\nu}])$ admits a parallel scale tractor

$$\boxed{\mathcal{D}_\mu I^M = 0}$$

$[g_{\mu\nu}] = [\Omega^2 g_{\mu\nu}]$

BAILEY, EASTWOOD, GOVER

TRACTOR CALCULUS

THOMAS, BAILEY, EASTWOOD, GOVER

Conformal tensor calculus:

- Replace tangent bundle TM with weighted tractor bundles $\mathcal{E}[w]$ whose sections ("fields") obey

$$\boxed{V^M = \begin{pmatrix} V^+ \\ V^m \\ V^- \end{pmatrix} \longmapsto \Omega^w U^M{}_N V^N = \Omega^w \begin{pmatrix} \Omega V^+ \\ V^m + \Upsilon^m V^+ \\ \Omega^{-1}\left(V^- - \Upsilon_n V^n + \frac{1}{2}\Upsilon^2 V^+\right) \end{pmatrix}}$$

WEIGHT w TRACTOR VECTORS

- i.e. 4-vectors and $SO(d-1,1)$ Lorentz tensors are replaced by $SO(d+1,1)$ tractor tensors

THOMAS D-OPERATOR

Maps $\mathcal{E}^{M_1 \cdots M_k}[w] \longrightarrow \mathcal{E}^{M_1 \cdots M_{k+1}}[w+1]$

$$\boxed{D^M = \begin{pmatrix} w(d+2w-2) \\ (d+2w-2)\,\mathcal{D}^m \\ -\frac{1}{d}\left(\mathcal{D}^n \mathcal{D}_n + wP\right) \end{pmatrix}}$$

THOMAS, BAILEY, GOVER, EASTWOOD

- Not Leibnitzian
- Build physics from
  tractor metric η^{MN}
  scale tractor I^M
  Thomas D-operator D^M
- Example $I^M = \frac{1}{d} D^M \sigma$
- Null $D^M D_M = 0$

MASS

Weight w scalar field $\quad \varphi \longmapsto \Omega^{w} \varphi$

Equation of motion

$$\boxed{I_M D^M \varphi = 0}$$

Idea: survey all couplings to scale: "invariant theory"

Conformal scattering
Graham, Zworski, Gover
Čap, Peterson

Explicitly this says

$$\underbrace{\sigma^2 g^{\mu\nu} \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \varphi} + m^2_{grav} \varphi = 0$$

Weyl compensated massless scalar:

$$\widetilde{\nabla}_\mu = \nabla_\mu - w\, b_\mu$$

$$b_\mu = \sigma^{-1} \partial_\mu \sigma$$

Weyl's gauge field

Mass term

$$\boxed{m^2_{grav} = I^M I_M = \frac{-2\sigma^2}{d} w(d+w-1)\left(P + \nabla^\mu b_\mu - \frac{d-2}{2} b^\mu b_\mu\right)}$$

Varies over spacetime

Observe parallel scale tractor $\mathcal{D}_\mu I^M = 0 \Rightarrow$

$$\partial_\mu (I^M I_M) = 0$$

mass constant for conformally Einstein

Canonical scale $\sigma = 1 \Rightarrow$

$$\left\{ \Delta + \frac{2P}{d} w(d+w-1) \right\} \varphi = 0$$

MASS WEYL WEIGHT RELATIONSHIP

$$\boxed{m^2 = -\frac{2P}{d}\left[\left(w+\frac{d-1}{2}\right)^2-\left(\frac{d-1}{2}\right)^2\right]}$$

Masses are Weyl weights

In AdS $P<0 \Rightarrow$

$$\boxed{m^2 \geq \frac{2P}{d}\left(\frac{d-1}{2}\right)^2}$$

Breitenlohner — Freedman bound

(See also Townsend, Mezincescu)

Weyl invariance

At $w = 1 - \frac{d}{2}$

$$D^M = \begin{pmatrix} 0 \\ 0 \\ -\left[\Delta - \frac{d-2}{2}\rho\right] \end{pmatrix}$$

Yamabe operator Δ_Y

Scale tractor I^M decouples

$$D^M \varphi = 0 \iff \Delta_Y \varphi = 0$$

Conformally improved scalar wave equation

Summary

- manifest local unit covariance
- unify massless & massive theories

Scalars:

- masses replaced by weights
- stability bounds for free
- scale tractor decoupling $\longrightarrow$ Weyl invariance

Spins $s \geq \frac{1}{2}$ including spinors

All of the above hold

$$\boxed{m^2 = -\frac{2P}{d}\left[\left(w+\frac{d-1}{2}\right)^2 - s - \left(\frac{d-1}{2}\right)^2\right]}$$

and more...

VECTORS & GRAVITONS

Massive, massless & partially massless

Field content:

$$V^M \quad \text{or} \quad h^{MN}$$

Weight w tractor vector

Weight w tractor metric fluctuations

Gauge principle:

$$\delta V^M = D^M \xi \qquad \delta h^{MN} = D^M \xi^N + D^N \xi^M$$

Tractor Maxwell Tensor

$$\boxed{\mathcal{F}^{MN} = D^M V^N - D^N V^M}$$

Tractor Christoffel Symbols

$$\boxed{2\Gamma^{MNR} = D^M h^{NR} + D^N h^{MR} - D^R h^{MN}}$$

Tractor Maxwell equations

$$\boxed{\mathcal{J}^M = I_N \mathcal{F}^{MN} = 0}$$

yielding:

Generic w

Gauge invariant Stückelberg description of massive Proca

$w = -1$

standard massless Maxwell theory

$\delta A_\mu = \partial_\mu \xi$

$w = 1 - \frac{d}{2}$

$$\Delta A_\mu - \frac{4}{d}\nabla^\nu \nabla_\mu A_\nu + \frac{d-4}{4}\left(2P_{\mu\nu}A^\nu - \frac{d+2}{2}A_\mu\right) = 0$$

WEYL INVARIANT DESER NEPOMECHIE THEORY

Tractor Linear Einstein Equations

$$\boxed{\mathcal{G}^{MN} \equiv I_R \Gamma^{MNR} = 0}$$

Generic w — massive gravitons — helicity $\pm 2, \pm 1, 0$

Stückelberg

$w = -1$ — massless gravitons — helicity ± 2

$$\delta h_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$w = 0$ — partially massless gravitons — helicity $\pm 2, \pm 1$

$$\delta h_{\mu\nu} = (\nabla_\mu \nabla_\nu + \rho g_{\mu\nu})\xi$$

Deser, Nepomechie, Higuchi, Waldron

$w = 1 - \frac{d}{2}$ — new Weyl invariant theory

SUPERSYMMETRY & INTERACTIONS

Can construct

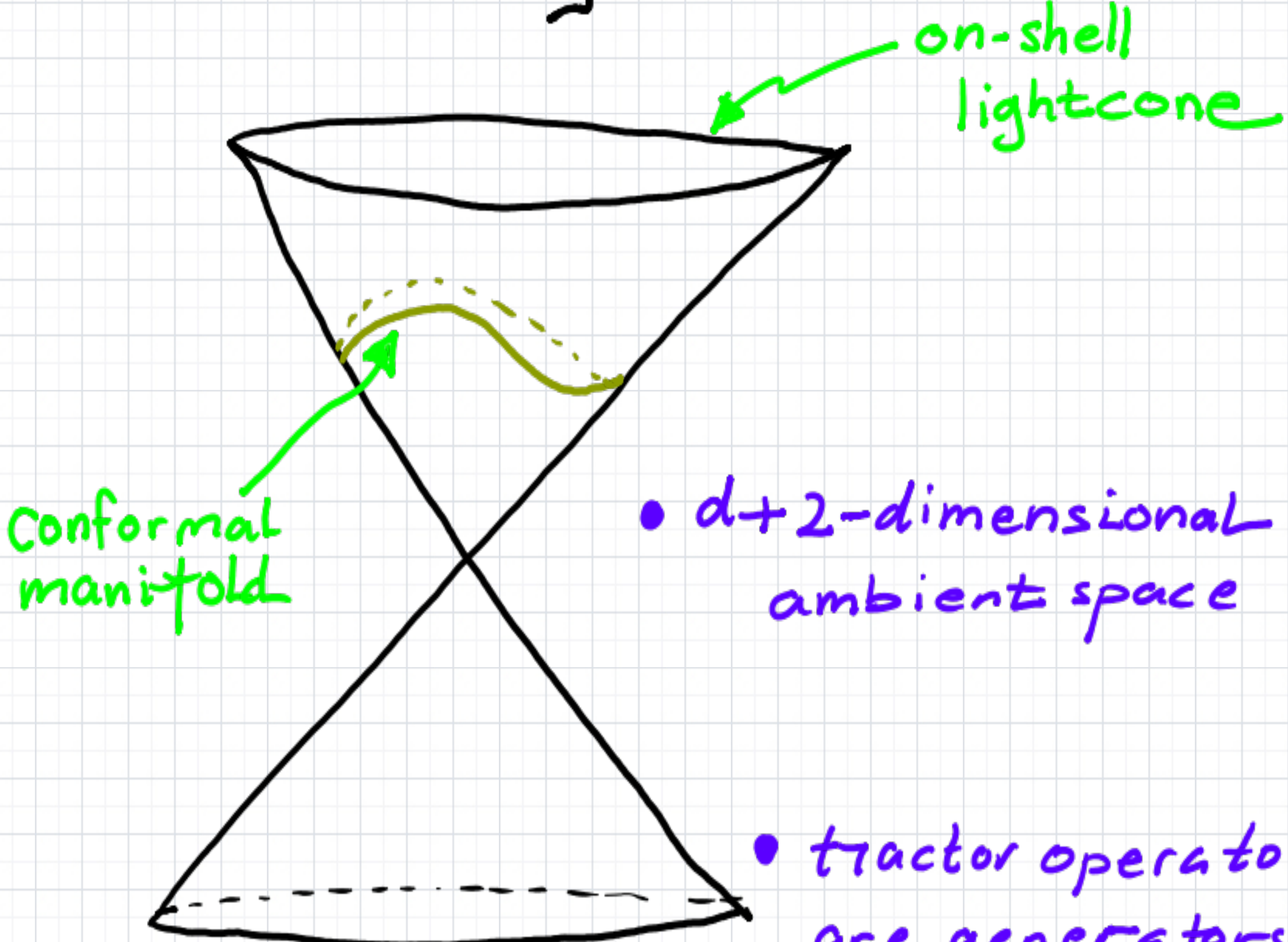
- Dirac equation
- Rarita–Schwinger equation
- Interacting Wess–Zumino model

General programme:

- Compute all possible couplings to scale of existing physical theories
- Survey all possible Weyl invariant scale independent theories
- Interacting massive gravity

QUANTIZATION?

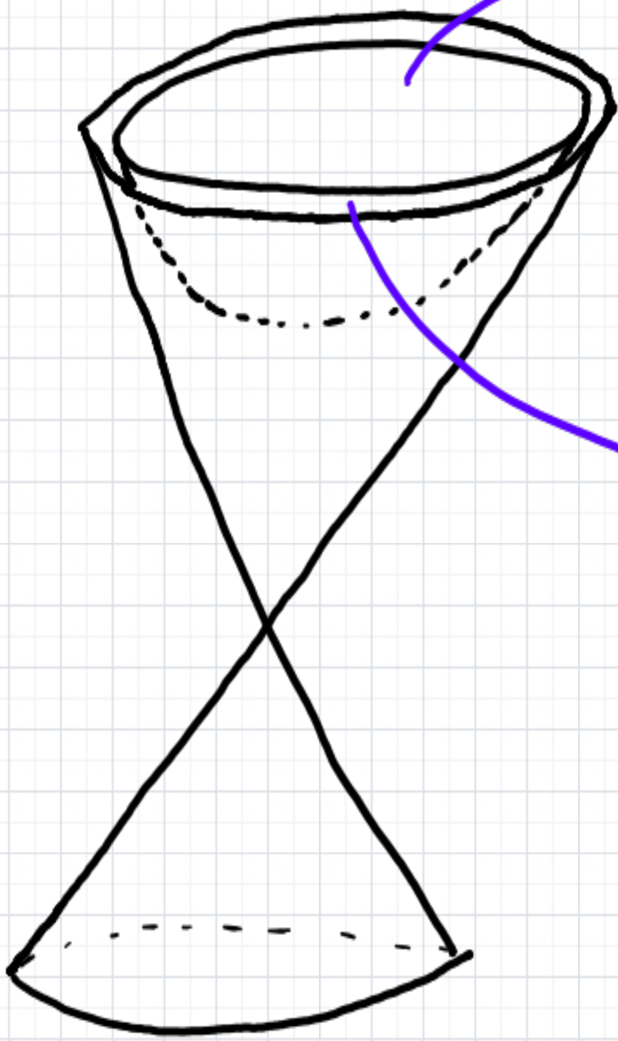
Dual scalar theory



- d+2-dimensional ambient space = momentum space of dual scalar

- tractor operators are generators of ambient conformal group

AdS/CFT



Poincaré disc = AdS

Boundary conformal manifold

Two possible Fefferman–Graham expansions.

(Equivalent at free-level)

Holographic anomalies from asymptotics of I.D

Henningson, Skenderis