Homework 10

Q1 §11.2

q2 $e^{-x} = 1 - x + \frac{x^2}{2!} \cdots + \frac{(-x)^N}{N!}$

$$+ \frac{(-x)^{N+1}}{N+1!} e^{-x_1}$$

where $0 \leq x_1 \leq x$.

But $\lim_{N\to\infty} \frac{(-x_1)^{N+1}}{(N+1)!} = 0$ ✓

q4 $\frac{1}{1+x} = 1 - x + x^2 \cdots + (-x)^N$

$$+ (-)^{N+1} x^{N+1} \left(\frac{1}{1+x_1}\right)^{N+2}$$

Here we want $-\frac{1}{2} < x < 1$

$\Rightarrow \lim_{N\to\infty} \frac{x^{N+1}}{(1+x_1)^{N+2}} = \frac{1}{1+x_1} \lim_{N\to\infty} \left(\frac{x}{1+x_1}\right)^{N+1}$

but $\left|\frac{x}{1+x_1}\right| < 1$ check this for $-\frac{1}{2} < x \leq 0$ and $0 < x < 1$ separately

hence remainder tends to zero as $N \to \infty$.

Q6 Similar to exponential case.

Note that $|f^{(N+1)}(x_1)| \leq 1$

because derivatives of cosine is either cosine or sine.

Q.7 When $0 \leq x \leq 1$ we have

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$ (alternating Maclaurin series)

$$\Rightarrow \int_0^1 dx \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}}{x} \leq \int_0^1 dx \frac{1-\cos x}{x} \leq \int_0^1 dx \frac{\frac{x^2}{2!} - \frac{x^4}{4!}}{x}$$

$$\Rightarrow .23958\ldots \leq \int_0^1 dx \frac{1-\cos x}{x} \leq .23981$$

$$\Rightarrow \int_0^1 dx \frac{1-\cos x}{x} = .2397 \pm .0001 \text{ (th.)}$$

9.22

$$\frac{e^{-1/x^2}}{x^n} = \frac{\left(1 - \frac{1}{x^2} + \frac{1}{2!}\cancel{\frac{1}{x^4}} - \frac{1}{4!}\frac{1}{x^r} + \dots\right)}{x^n}$$

$$\xrightarrow{n \to \infty} 0$$

9.26

$$f(x) = f(0) + f'(0)x + \dots + \frac{1}{3!}f'''(0)x^3 + \frac{1}{4!}f''''(x_1)x^4$$

$$\Rightarrow \int_{-1}^{1} f(x) - f\left(\tfrac{1}{\sqrt{3}}\right) - f\left(-\tfrac{1}{\sqrt{3}}\right)$$

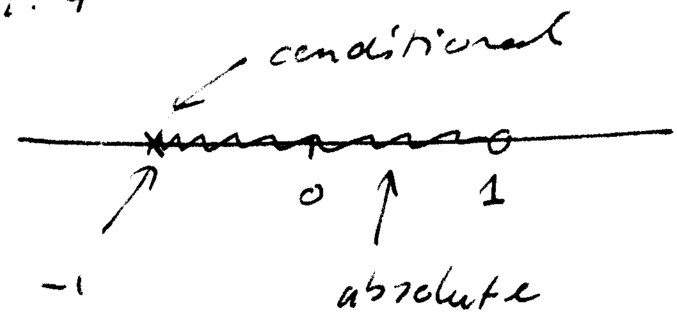
$$= 2f(0) + \frac{1}{3}f''(0) + \int_{-1}^{1}\frac{dx}{4!}f''''(x_1(x))\,x^4$$

$$- (2f(0) + \frac{1}{3}f''(0) + \frac{1}{4!}\left[f''''(\sqrt{3}) + f''''(-\sqrt{3})\right]\cdot\frac{1}{9}$$

$$\leq M\left[\int_{-1}^{1}\frac{dx}{4!}x^4 + \frac{2}{9 \cdot 4!}\right] = \frac{7M}{270}.$$

Q2 § 11.4

q 2



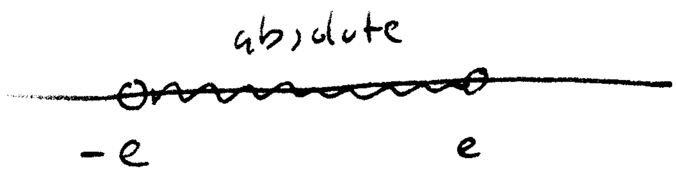

$\boxed{R=1}$

by ratio test $\left| \dfrac{\frac{x^{n+1}}{\sqrt{n+1}}}{\frac{x^n}{\sqrt{n}}} \right| \xrightarrow{n\to\infty} 0$ when $|x|<1$

at $x=-1$ use alternating series test.

$x=1$ diverges by integral test.

q 4




ratio test $\left| \dfrac{(n+1)^2 e^{-(n+1)} x^{n+1}}{n e^{-n} x^n} \right| = \left|\dfrac{n+1}{n}\right| \left|\dfrac{x}{e}\right|^n$

$\xrightarrow{n\to\infty} 0 \quad |x|<e$

$\to 1 \quad |x|=e$

$\to$ DNE else

At endpoints use $n^{th}$ term test.

q12 c.f q6 and think about logarithms.

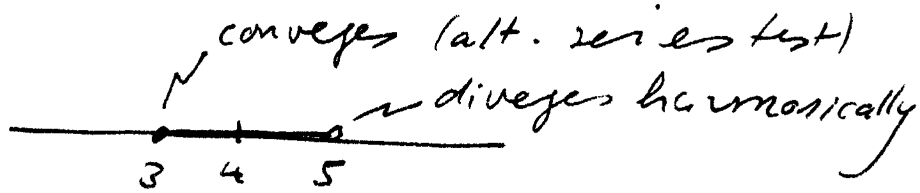
q16 $\sum a_n 6^n$ converges $\Rightarrow$ $R \geq 6$

a) $\sum a_n (-6)^n$ may or may not converge

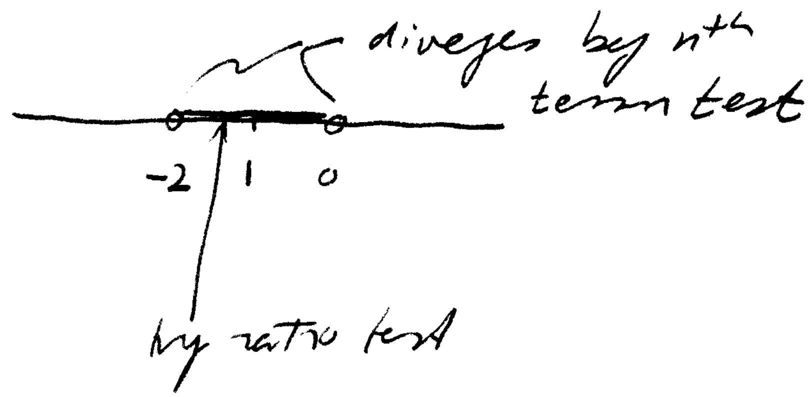
b) $\sum a_n 5^{-n}$ converges

c) $\sum a_n (-5)^n$ converges

q20 $\sum_{n=0}^{\infty} \frac{(x-4)^n}{2n+1}$




q27 $\sum_{n=1}^{\infty} n(x+1)^n$




q38 $f = \left(\frac{1}{1+x}\right)^2$

use $\frac{1}{1+x} \overset{|x|<1}{=} 1 - x + x^2 - x^3 + \dots$

$\Rightarrow \left(\frac{1}{1+x}\right)^2 \overset{|x|<1}{=} (1 - x + x^2 - x^3 + \dots)(1 - x + x^2 - x^3 + \dots)$

$= 1 - 2x + 3x^2 - 4x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

$\left(\frac{1}{1-x}\right)^2 \overset{|x|<1}{=} 1 + 2x + 3x^2 + 4x^3 + \dots$

2 (1) Use

$$\left(\frac{1}{\sqrt{1-x}}\right)^2 = \frac{1}{1-x} \overset{|x|<1}{=} 1 + x + x^2 + \cdots$$

$$\Rightarrow \left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)^2 = 1 + x + x^2 + \cdots \quad \checkmark$$

~~Agrees~~

Thus

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

Now put $x = \frac{v^2}{c^2}$ valid when $v \ll c$.

$\S 3 \quad \S 11.5$

q 1 $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\frac{d}{dx} \sin x = 1 - \frac{3}{3!} x^2 + \frac{5}{5!} x^4 - \cdots$$

$$= 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n}}{(2n)!}$$

$$= \cos x$$

q 2 $\quad e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$

$$\frac{d}{dx} e^x = 0 + 1 + \frac{2}{2!} x + \frac{3}{3!} x^2 + \cdots$$

$$= 1 + x + \frac{1}{2!} x^2 + \cdots = e^x \checkmark$$

q 4 $\quad \frac{1}{1+t^3} \overset{|t^3|<1}{=} 1 - t^3 + t^6 - t^9 \cdots$

$$\Rightarrow \int_0^{.7} \frac{dt}{1+t^3} = \int_0^{.7} dt \, (1 - t^3 + t^6 - t^9 \cdots)$$

$$= \underbrace{\underbrace{\left(t - \frac{1}{4}t^4 + \frac{1}{7}t^7\right.}_{S_3} \left.- \frac{1}{10} t^{10} + \cdots\right)}_{S_4} \Bigg|_0^{.7}$$

$$\int_0^{.7} \frac{dt}{1+t^3} = \frac{S_3 + S_4}{2} \pm \left|\frac{S_3 - S_4}{2}\right| \text{(th.)} = .650 \pm .001 \text{ (th.)}$$

9°

even function

$$\frac{1-\cos x}{1-x^2} = a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + \dots$$

$a_0 = 0$
obviously

$$\Rightarrow 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right)$$

$$= (1-x^2)(a_2 x^2 + a_4 x^4 + a_6 x^6 + a_8 x^8 + \dots)$$

$$\Rightarrow \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!}$$

$$= a_2 x^2 + (a_4 - a_2) x^4 + (a_6 - a_4) x^6 + (a_8 - a_6) x^8 + \dots$$

$$\Rightarrow a_2 = \frac{1}{2!}, \quad a_4 - \frac{1}{2} = -\frac{1}{4!} \Rightarrow a_4 = \frac{11}{24}$$

$$a_6 - \frac{11}{24} = \frac{1}{6!} \Rightarrow a_6 = \frac{331}{720}, \quad a_8 - a_6 = \frac{1}{8!} \Rightarrow a_8 = \frac{3707}{8064}$$

$$\Rightarrow \frac{1-\cos x}{1-x^2} = \frac{1}{2} x^2 + \frac{11}{24} x^4 + \frac{331}{720} x^6 + \frac{3707}{8064} x^8 + \dots$$

$|x| < 1$

q 22

$$\log 3 = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}}$$

$$= \log\left(1+\tfrac{1}{2}\right) - \log\left(1-\tfrac{1}{2}\right)$$

$$= \left(\tfrac{1}{2}\right) - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} \cdots$$

$$- \left[ \left(\tfrac{-1}{2}\right) - \frac{\left(\frac{-1}{2}\right)^2}{2} + \frac{\left(\frac{-1}{2}\right)^3}{3} \cdots \right]$$

$$= 2\left(\tfrac{1}{2}\right) + \tfrac{2}{3}\left(\tfrac{1}{2}\right)^3 + \tfrac{2}{5}\left(\tfrac{1}{2}\right)^5 + \tfrac{2}{7}\left(\tfrac{1}{2}\right)^7$$

$$+ \tfrac{2}{9}\left(\tfrac{1}{2}\right)^9 + \cdots$$

$$= 1 + \tfrac{1}{3}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{5}\left(\tfrac{1}{2}\right)^4 + \tfrac{1}{7}\left(\tfrac{1}{2}\right)^6$$

$$+ \tfrac{1}{9}\left(\tfrac{1}{2}\right)^8 + R$$

where $R \leq \frac{1}{11}\left(\underbrace{\left(\tfrac{1}{2}\right)^{10} + \left(\tfrac{1}{2}\right)^{12} + \left(\tfrac{1}{2}\right)^{14} + \cdots}_{\frac{1}{2^{10}}\frac{1}{1-\frac{1}{4}} = \frac{4}{3.2^{10}}}\right)$

$\Rightarrow \quad 1.09849 < \log 3 < 1.09849 + .0001$

$\Rightarrow \quad \log 3 = 1.09855 \pm .00005$ (th.)

true value: $\log 3 = 1.098612289\ldots$

#26

$$f = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$g = b_0 + b_1 x + b_2 x^2 + \cdots$$

Let $b_0 = a_0 = 0$

then
$$\frac{f}{g} = \frac{a_1 x + a_2 x^2 + \cdots}{b_1 x + b_2 x^2 + \cdots}$$

$$= \frac{a_1 + a_2 x + a_3 x^2 + \cdots}{b_1 + b_2 x + b_3 x^2 + \cdots}$$

$$\xrightarrow{x \to 0} \frac{a_1}{b_1} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

Taylor's theorem

Pt 3 1100

2.2 $(2+3i)^2 = 4 + 12i + 9i^2 = -5+12i$

$$\frac{4}{3-i} = \frac{4(3+i)}{(3-i)(3+i)} = \frac{4(3+i)}{9+1} = \frac{6}{5} + \frac{2i}{5}$$

2.4 $z_1 = 2e^{i\pi/4}$, $z_2 = e^{\frac{3i\pi}{4}}$

$$z_1 z_2 = 2e^{i\pi(\frac{1}{4}+\frac{3}{4})} = 2e^{i\pi} = -2$$

2.6 $i^3 = i^2 i = -i$, $i^4 = 1$

$$i^{73} = i^{4\cdot 18+1} = (i^4)^{18} i = i$$

2.7 $z = .9\, e^{\frac{i\pi}{4}}$

$$\lim_{n\to\infty} (.9)^n = 0 \Rightarrow \lim_{n\to\infty} z^n = 0$$

2.9 $z^4 = r^4 e^{4i\theta}$, $8+8\sqrt{3}i = 16\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$

$\quad z = re^{i\theta}$ $\qquad = 2^4 e^{\frac{i\pi}{3}}$

$$\Rightarrow z = 2e^{\frac{i\pi}{12}},\ 2e^{\frac{7i\pi}{12}},\ 2e^{\frac{13i\pi}{12}},\ 2e^{\frac{19i\pi}{12}}.$$

q 20 d) $\left(\cos\frac{\pi}{12} + i\frac{\pi}{12}\right)^{20} = \left(e^{\frac{i\pi}{12}}\right)^{20} = e^{\frac{5i\pi}{3}}$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

q 26 $z = re^{i\theta} = r\cos\theta + ir\sin\theta$

$\Rightarrow \bar{z} = r\cos\theta - ir\sin\theta = re^{-i\theta}$

$\arg(\bar{z}) = -\arg(z) = -\theta$

$\frac{1}{z} = \frac{1}{r}e^{-i\theta} \Rightarrow \arg\frac{1}{z} = -\theta$

q 28 $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} = e^{\frac{i\pi}{4}}$

$\Rightarrow z^2 = e^{\frac{2i\pi}{4}} = i$

q 38 $az^2+bz+c$ has roots $z_1, z_2$

hence $az^2+bz+c = a\left[\left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$

$= a\left(z+\frac{b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\right)\left(z+\frac{b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}\right)$

$= a(z-z_1)(z-z_2)$

Hence $z_{\substack{1\\2}} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$

$$\Rightarrow \quad z_1 + z_2 = -\frac{b}{a} \quad \checkmark$$

$$z_1 z_2 = \frac{b^2}{4a^2} - \frac{b^2-4ac}{4a^2} = \frac{c}{a} \quad \checkmark$$

The rest is easy.

Q5 9.1.7

q4 $e^{2+3i} = e^2(\cos 3 + i\sin 3)$

$\mathrm{Re}\, e^{2+3i} = e^2\cos 3$

$\mathrm{Im}\, e^{2+3i} = e^2\sin 3$

q2 $5e^{\pi i/4} = \frac{5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$

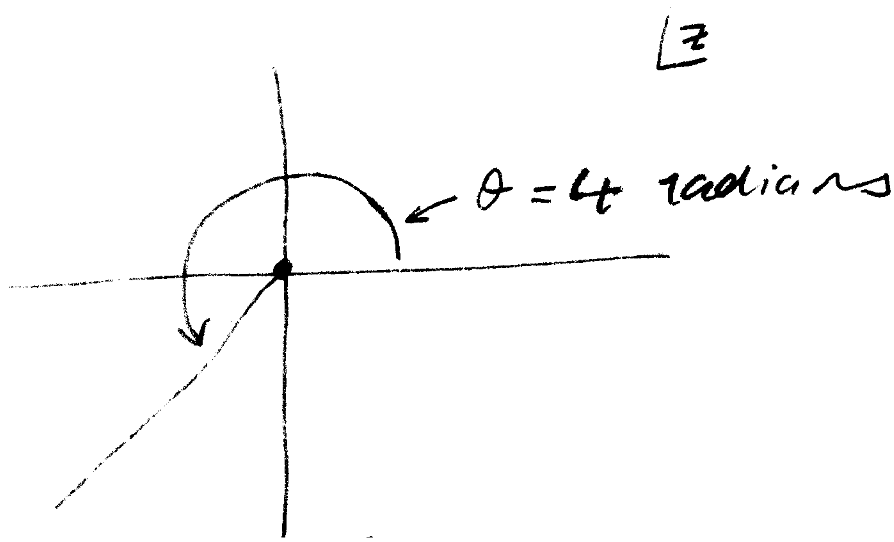
q6 $2e^{\pi i}\cdot 3e^{-\pi i/3} = -2\cdot 3\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$

$= -6\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -3 + 3\sqrt{3}\,i$

q18 $e^{z} = -1 \Rightarrow z = \ldots -3i\pi, -i\pi, i\pi, 3i\pi, 5i\pi, \ldots$

q20 $e^{x+4i} = e^{x}e^{4i}$



9.22

$$1+x+x^2+\dots x^{n-1} = \frac{1-x^n}{1-x}$$

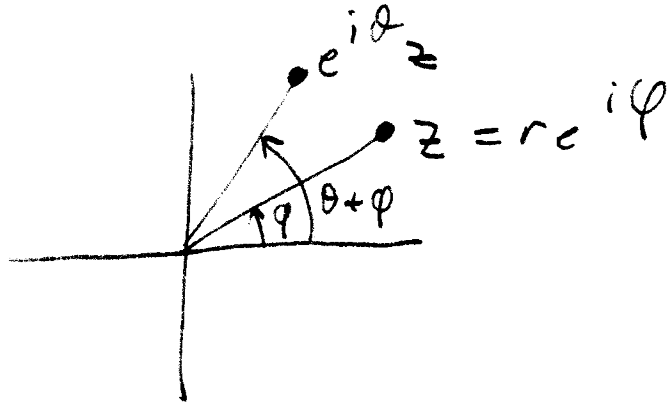
$$\Rightarrow \mathrm{Re}\left(1+e^{\theta i}+(e^{\theta i})^2+\dots+(e^{\theta i})^{n-1}\right)$$

$$= \mathrm{Re}\,\frac{1-e^{n\theta i}}{1-e^{\theta i}}$$

$$= \mathrm{Re}\,\frac{(1-e^{n\theta i})(1-\cos n\theta + i\sin n\theta)}{(1-\cos n\theta - i\sin n\theta)(1-\cos n\theta + i\sin n\theta)}$$

$$= \frac{1-\cos\theta-\cos n\theta+\cos(n-1)\theta}{2-2\cos\theta}$$

9.24

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