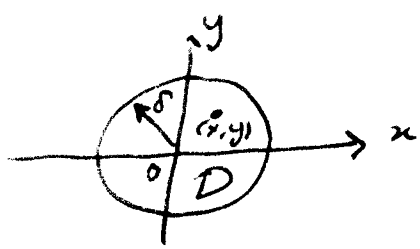
Homework 3

Q1 $\quad f(x,y) = xy$

Clearly $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Proof Let $\epsilon > 0$ and study a disc D about $(0,0)$, radius δ



Obviously, $|x| < \delta$, $|y| < \delta$ whenever $(x,y) \in D$. Thus

$$|xy| = |x||y| < \delta^2$$

so setting $\delta = \sqrt{\epsilon}$ we learn

$$|f(x,y) - 0| = |xy| < \delta^2 = \epsilon \qquad \text{QED}$$

Q2 §14.5

q2 $f = x^2y + 4$, $\dfrac{\partial f}{\partial x} = 2xy$, $\dfrac{\partial f}{\partial y} = x^2$

q4 $f = 6x - 7y$, $\dfrac{\partial f}{\partial x} = 6$, $\dfrac{\partial f}{\partial y} = -7$

q6 $f = \log(x+2y)$, $\dfrac{\partial f}{\partial x} = \dfrac{1}{x+2y}$, $\dfrac{\partial f}{\partial y} = \dfrac{2}{x+2y}$

q8 $f = \tan^{-1}\left(\frac{y}{x}\right)$, $\dfrac{\partial f}{\partial x} = \dfrac{1}{1+\left(\frac{y}{x}\right)^2}\dfrac{\partial\left(\frac{y}{x}\right)}{\partial x} = \dfrac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} = \dfrac{-y}{x^2+y^2}$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} = \frac{x^2}{x^2+y^2}$$

q14 $f = \dfrac{1}{\sqrt{x^2+y^2}}$ $\quad \dfrac{\partial f}{\partial x} = -\dfrac{1}{2}\dfrac{1}{(x^2+y^2)^{3/2}}\dfrac{\partial}{\partial x}(x^2+y^2) = -\dfrac{x}{(x^2+y^2)^{3/2}}$

$$\frac{\partial f}{\partial y} = -\frac{y}{(x^2+y^2)^{3/2}}$$

by symmetry
in x & y

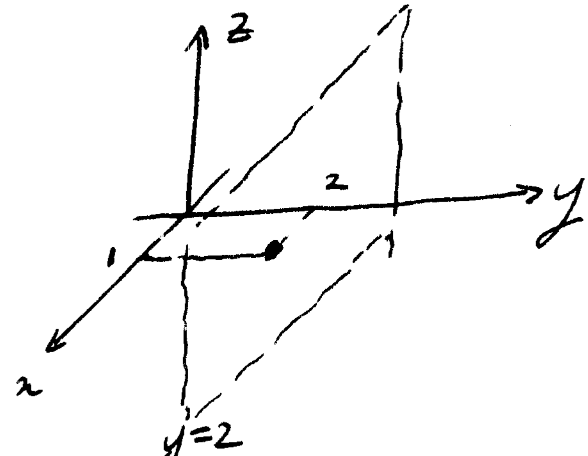
q18 $f = \sin x^2y$, $\dfrac{\partial f}{\partial x} = 2xy\cos x^2y$, $\dfrac{\partial^2 f}{\partial x^2} = 2y\cos x^2y - 4x^2y^2\sin x^2y$

$$\frac{\partial^2 f}{\partial y\partial x} = 2x\cos x^2y - 2x^3y\sin x^2y$$

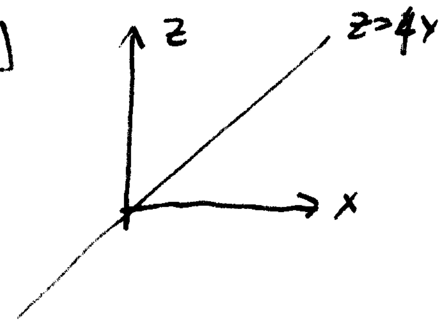
$\|$ ✓ $\quad \dfrac{\partial f}{\partial y} = x^2\cos x^2y$, $\dfrac{\partial^2 f}{\partial y^2} = -x^4\sin x^2y$

$$\frac{\partial^2 f}{\partial x\partial y} = 2x\cos x^2y - 2x^3y\sin x^2y$$

q 21 $z = xy^2$



so in zx-plane $\bar{c}$ $y=2$ get $z = 2^2 x = 4x$.



$\Rightarrow$ slope $\frac{\partial z}{\partial x} = 4x$

but at (1,2) $x=1$

so $\frac{\partial z}{\partial x} = 4$ ✓

slope 4

q 26 $z = e^{x/y}$ need $\frac{\partial z}{\partial y}$ at $x=0, y=1$

$\underbrace{\frac{\partial z}{\partial y} = \frac{-x}{y^2}}_{\text{vanishes at } x=0.} e^{x/2}$ so slope is 0.

2.30 $f(2,3)=1$

$f(1.98,3) = 1.03$

$f(2, 3.04) = 0.98$

$$\Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(2,3)} \simeq \frac{f(2,3)-f(1.98,3)}{2-1.98}$$

$$= \frac{1-1.03}{2-1.98} = \frac{-.03}{.02} = -3/2$$

$$\left.\frac{\partial f}{\partial y}\right|_{(2,3)} \simeq \frac{f(2,3.04)-f(2,3)}{3.04 - 3}$$

$$= \frac{.98 - 1}{3.04 - 3} = \frac{-.02}{.04} = -\frac{1}{2}$$

2.34 NO Later we will show

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

in radial coordinates.

Here $T = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$ so

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) T = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r}\frac{\partial}{\partial r} r\left(-\frac{1}{r^2}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{1}{r}\right) = \frac{1}{r^3} = \frac{1}{(x^2+y^2)^{3/2}} \neq 0$$

WAS THIS YOUR "BRUTE FORCE" ANSWER ?

q3 §14.0

q2 $z = xe^y \quad x = t, \; y = t^3$

$\Rightarrow \dfrac{dx}{dt} = 1 \quad , \quad \dfrac{dy}{dt} = 3t^2$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= e^y . 1 + xe^y . 3t^2$$

$$= e^{t^3} + t\,e^{t^3} . 3t^2$$

$$= \underline{(1+3t^3)e^{t^3}} \quad \text{(WITH)}$$

$\underline{\text{Vs}}$ $\dfrac{dz}{dt} = \dfrac{d}{dt}\left(te^{t^3}\right) = e^{t^3} + t. 3t^2 e^{t^3}$

$$= \underline{(1+3t^3)e^{t^3}} \quad \text{(WITHOUT)}$$

q6 $z = x^2y, \; x = 3t+4u, \; y = 5t-u$

WITHOUT

$\Rightarrow z = (3t+4u)^2(5t-u)$

$$\frac{\partial z}{\partial t} = 2(3t+4u).3(5t-u) + (3t+4u)^2 . 5$$

$$= (3t+4u)(6[5t-u] + 5.(3t+4u))$$

$$= \underline{(3t+4u)(45t+14u)}$$

§38 $f = a + bx + cy + dx^2 + exy + ky^2$

a) When $x = y = 0$

$$f(0,0) = a + b.0 + c.0 + d.0^2 + e.0.0 + k.0^2 = a$$

$\Rightarrow f(0,0) = a$

b) $\frac{\partial f}{\partial x} = b + 2dx + ey \Rightarrow \frac{\partial f}{\partial x}(0,0) = b$

c) $\frac{\partial f}{\partial y} = c + ex + 2ky \Rightarrow \frac{\partial f}{\partial y}(0,0) = c$

etc....

<u>Note</u> the formula you just found

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0) \cdot x + \frac{\partial f}{\partial y}(0,0) \cdot y$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(0,0) \cdot x^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)\, xy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)\, y^2$$

is <u>exact</u> for quadratic polynomials.

It is, in fact approximately true for "small" x and y for ANY nice function $f(x,y)$.

q6 ct.

WITH

$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= 2x.y \cdot 3 + x^2 . 5$$

$$= 6\,(3t+4u)(5t-u) + 5\,(3t+4u)^2$$

$$= \underline{(3t+4u)(45t+14u)}$$

q10 $\quad \frac{\partial z}{\partial x} = 3, \quad \frac{\partial z}{\partial y} = 2, \quad \frac{dx}{dt} = 4, \quad \frac{dy}{dt} = -3$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= 3.\ 4 + 2.(-3)$$

$$= 6$$

q12 $\quad z = f(x,y), \quad x = u^2 - v^2, \quad y = v^2 - u^2$

a)

$$\Rightarrow u\frac{\partial z}{\partial v} + v\frac{\partial z}{\partial u} = u.\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right) + v\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right)$$

$$= u\frac{\partial f}{\partial x}.(-2v) + u\frac{\partial f}{\partial y}(2v) + v\frac{\partial f}{\partial x}(2u) + v\frac{\partial f}{\partial y}(-2u) = 0$$

pairwise cancellations!

Q16 $\quad a\dfrac{\partial^2 z}{\partial x^2} + b\dfrac{\partial^2 z}{\partial x \partial y} + c\dfrac{\partial^2 z}{\partial y^2} = 0$

try $z = f(y+mx)$, then [call $\dfrac{d^2 f(x)}{dx^2} \equiv f''$]

$\Rightarrow a f'' m^2 + b f''.m + c f''.1 = 0$

$\Rightarrow (am^2+bm+c) f'' = 0$

Unless $f''=0$ we must have $am^2+bm+c=0$ ✓

Q18 $\quad \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

a) $u = e^{kt} g(x)$

$$\left.\begin{aligned} \Rightarrow \frac{\partial u}{\partial t} &= k e^{kt} g(x) \\ k\frac{\partial^2 u}{\partial x^2} &= k e^{kt} g'' \end{aligned}\right\} \text{equating these says } g''=g$$

Q26 Polar coords $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

a) WARNING $\dfrac{\partial r}{\partial x} \neq \dfrac{\partial x}{\partial r}$!

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\underbrace{\sqrt{x^2+y^2}}_{= r} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$$

26 b) Note a) was $\frac{\partial}{\partial x} r(x,y)\Big|_{y \text{ fixed}}$

whereas we now compute

$$\frac{\partial}{\partial x} r(x,\theta)\Big|_{\theta \text{ fixed}} = \frac{\partial}{\partial x} \frac{x}{\cos\theta} = \frac{1}{\cos\theta}$$

[c)] which is not a contradiction.

30) YES, rewrite the argument on pp 816-817

swapping $x \leftrightarrow y$

28) $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \frac{1}{r}\partial_r r \partial_r u + \frac{1}{r^2}\partial_\theta^2 u$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial\theta}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}r\cos\theta\right)$$

$$= \frac{1}{r}\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\sin\theta$$

$$-\frac{1}{r}\frac{\partial u}{\partial x}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial y}\sin\theta - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial r} \right) \cos\theta$$

$$+ \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial r} \right) \sin\theta$$

$$- \frac{\sin\theta}{r} \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial x \partial y} \frac{\partial y}{\partial \theta} \right)$$

$$+ \frac{\cos\theta}{r} \left(\frac{\partial^2 u}{\partial x \partial y} \frac{\partial x}{\partial \theta} + \frac{\partial^2 u}{\partial y^2} \frac{\partial y}{\partial \theta} \right)$$

$$= \frac{\partial^2 u}{\partial x^2} \cos^2\theta + \cancel{\frac{\partial^2 u}{\partial x \partial y} \sin\theta\cos\theta}$$

$$+ \cancel{\frac{\partial^2 u}{\partial x \partial y} \cos\theta\sin\theta} + \frac{\partial^2 u}{\partial y^2} \sin^2\theta$$

$$+ \sin^2\theta \frac{\partial^2 u}{\partial x^2} - \cancel{\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y}}$$

$$- \cancel{\sin\theta\cos\theta \frac{\partial^2 u}{\partial x \partial y}} + \cos^2\theta \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{\partial^2 u}{\partial x^2} (\cos^2\theta + \sin^2\theta) + \frac{\partial^2 u}{\partial y^2} (\cos^2\theta + \sin^2\theta)$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

QED

& + see notes Q5 The FOO-GOO Fish!!!