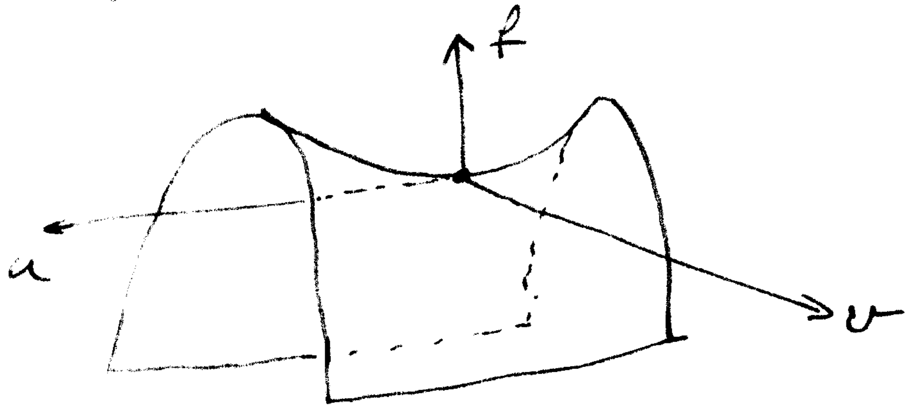
Homework 4

Q1 $g = u^2 - v^2$ is a saddle



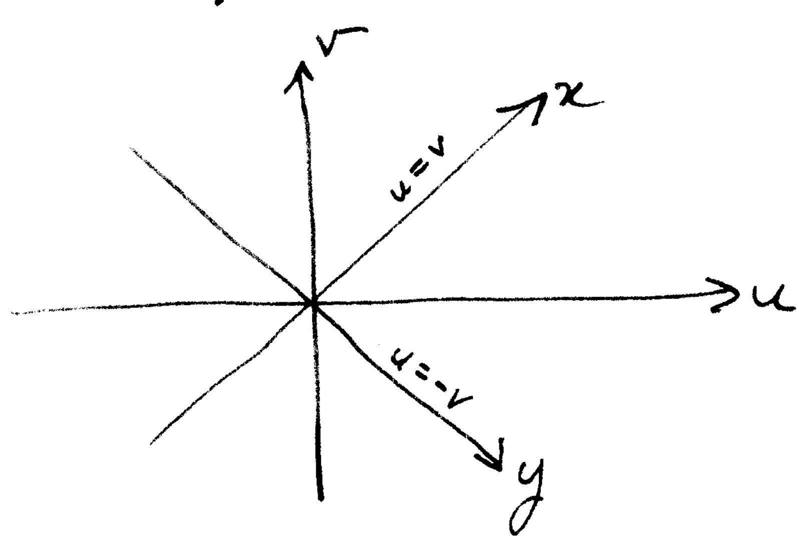
Notice $g = u^2 - v^2 = (u-v)(u+v)$

so calling
$$\left.\begin{aligned} u+v &= \sqrt{2}\,x \\ u-v &= \sqrt{2}\,y \end{aligned}\right\} (*)$$

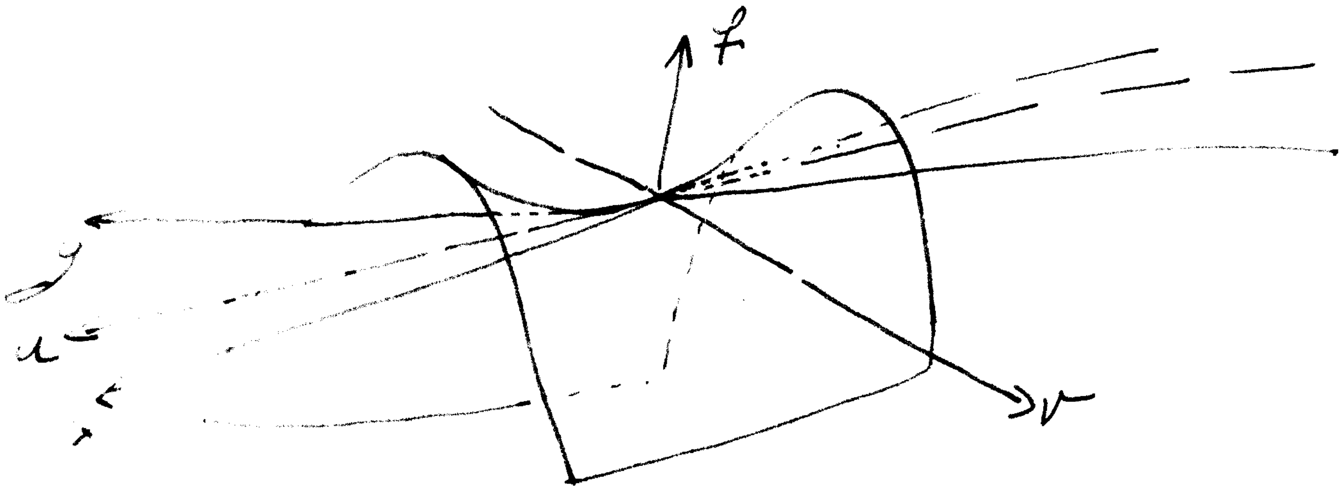
we get

$$g = 2xy = f$$

<u>indeed</u> $(*)$ is just a rotation of the axes



Hence $f = 2xy$ is also a saddle turned around by 45°



Now extrema

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \quad \Rightarrow \quad \text{critical point at } (0,0) = (x,y)$$

$$\frac{\partial g}{\partial u} = 2u, \quad \frac{\partial g}{\partial v} = -2v \quad \Rightarrow \quad \text{critical point at } (0,0) = (u,v)$$

Hessian for f

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \Delta = \det H = -1 < 0 \Rightarrow \text{saddle}$$

Hessian for g

$$H = \begin{pmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial y \partial x} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\Rightarrow \Delta = \det H = -1 < 0 \Rightarrow \text{saddle}$$

Q2 $f = 2xy$

Let $z = f = xy$.

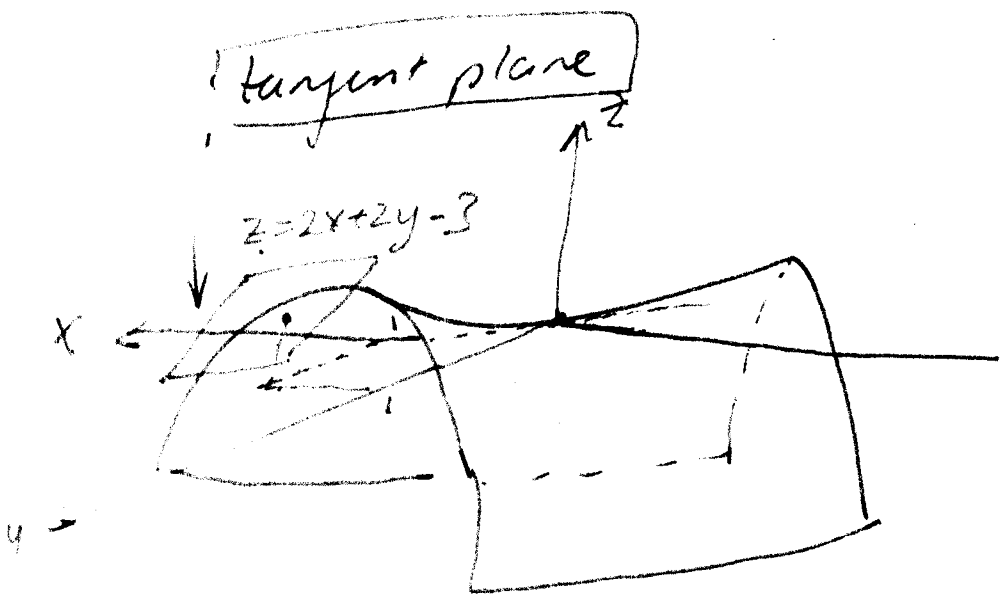
When $(x,y) = (1,1)$ $z = 1$, $\frac{\partial z}{\partial x} = 2y$, $\frac{\partial z}{\partial y} = 2x$

$\Rightarrow$ $(z - z_0) = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$

becomes

$$z - 1 = 2\cdot(x - 1) + 2\cdot(y - 1)$$

$\Rightarrow$ $z = 2x + 2y - 3$



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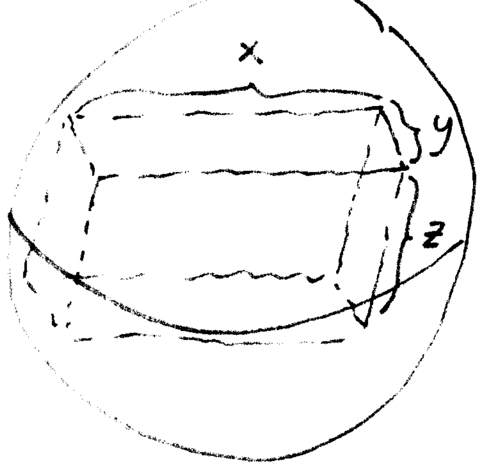
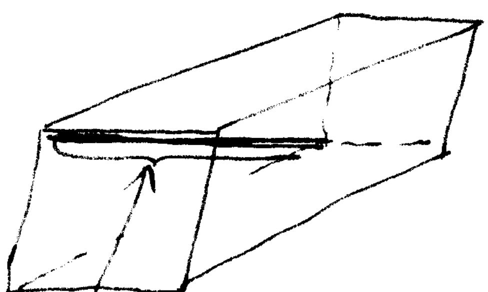


Let the radius be R

The volume of the box $V = xyz$

Notice



Diagonal is $2R$.

Thus $x^2+y^2+z^2=(2R)^2 \Rightarrow z=\sqrt{4R^2-x^2-y^2}$

(the distance from O to (x,y,z)).

Hence
$$V = xy\sqrt{4R^2-x^2-y^2}$$

Look for extrema. <u>Note</u> life is easier studying $V^2 = x^2y^2(4R^2-x^2-y^2)$ (maximizing V or V^2 gives the same answer).

Compute

$$\frac{\partial V^2}{\partial x} = 2xy^2(4R^2-x^2-y^2) - 2x^3y^2$$

$$= 2xy^2(4R^2-2x^2-y^2)$$

$$\frac{\partial V^2}{\partial y} = 2x^2y(4R^2-x^2-2y^2)$$

Solution at $x=y=0$ is boring. So solve

$$\begin{cases} 2x^2+y^2 = 4R^2 \\ x^2+2y^2 = 4R^2 \end{cases}$$

$\Rightarrow x^2-y^2 = 0 \Rightarrow x = \pm y$ only $x=+y$ makes sense

$$\Rightarrow 3x^2 = 4R^2 \Rightarrow x = \frac{2}{\sqrt{3}}R$$

$$y = \frac{2}{\sqrt{3}}R$$

$$z = \sqrt{4R^2 - \frac{4R^2}{3} - \frac{4}{3}R^2} = \frac{2R}{\sqrt{3}}$$

Get a <u>cube</u> with side length $\frac{2R}{\sqrt{3}}$.

To be certain, examine Hessian

$$\Delta = \left(\frac{\partial^2 V^2}{\partial x^2} \frac{\partial^2 V^2}{\partial y^2} - \left(\frac{\partial^2 V^2}{\partial x \partial y} \right)^2 \right)_{x = \frac{2R}{\sqrt{3}} = y}$$

$$\underset{\text{long calculation}}{=} \frac{4096}{27} R^8 > 0$$

and

$$\left(\left(\frac{\partial^2 V^2}{\partial x^2} \right) + \left(\frac{\partial^2 V^2}{\partial y^2} \right) \right)_{x = \frac{2R}{\sqrt{3}} = y} = -\frac{256}{9} R^4$$

$\Rightarrow$ maximum ✓

Q4 14.9

q 4 $\quad f = x^4 + 8x^2 + y^2 - 4y$

$$\frac{\partial f}{\partial x} = 3x^3 + 16x = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 2y - 4 \Rightarrow y = 2$$

extremum at $(0, 2)$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 16 \overset{x=0}{=} 16$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \Rightarrow \begin{cases} \Delta = 32 \\ \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 16 + 2 = 18 > 0 \end{cases}$$

$\Rightarrow$ (RELATIVE) MINIMUM.

q 10 $\quad$ saddle at $(0,0)$, minimum at $(1,-1)$

9.12 $\Delta = \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial f}{\partial x \partial y}\right)^2$

$= 2 \cdot 4 - (-3)^2 = 8 - 9 < 0$

$\Rightarrow$ f has a SADDLE.

9.16 SADDLE

9.24 $f = x + y + \dfrac{8}{xy}$

$\dfrac{\partial f}{\partial x} = 1 - \dfrac{8}{x^2 y} = 0 \Rightarrow x^2 y = 8$

solution
$(x,y) = (2,2)$

$\dfrac{\partial f}{\partial y} = 1 - \dfrac{8}{xy^2} = 0 \Rightarrow xy^2 = 8$

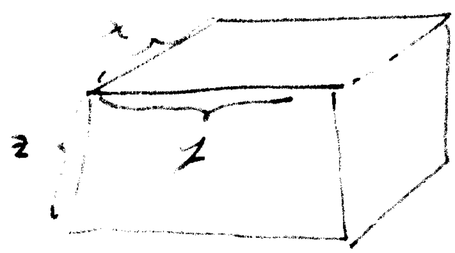
$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{16}{x^3 y} \qquad \dfrac{\partial^2 f}{\partial y^2} = \dfrac{16}{xy^3} \qquad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{+8}{x^2 y^2}$

$\Delta = \dfrac{16^2}{x^4 y^4} - \dfrac{8^2}{x^4 y^4} \Rightarrow \Delta(2,2) = \dfrac{16^2}{2^8} - \dfrac{8^2}{2^8}$

$= 1 - \frac{1}{4} = \frac{3}{4} > 0$

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = \dfrac{16}{2^4} + \dfrac{16}{2^4} > 0 \Rightarrow$ MINIMUM.

7.2.



Let c be the cost

$$c = 3(2xy) + 2(2zx) + 2(2zy)$$

but the volume $xyz = 1 \Rightarrow z = \frac{1}{xy}$

$$\Rightarrow c = 6xy + \frac{4}{y} + \frac{4}{x}$$

look for extrema

$$\frac{\partial c}{\partial x} = 6y - \frac{4}{x^2} = 0$$

$$\frac{\partial c}{\partial y} = 6x - \frac{4}{y^2} = 0$$

$\Rightarrow$ $yx^2 = \frac{2}{3}$, $xy^2 = \frac{2}{3}$

only solution is $x = y = \sqrt[3]{\frac{2}{3}}$

so cost is extremized at

$$c = 6\left(\tfrac{2}{3}\right)^{2/3} + 8\left(\tfrac{2}{3}\right)^{-\frac{1}{3}}$$

$5/3$

2 $4/3$

3

$$= 2^{5/3}3^{2/3} + 2^{8/3}3^{-\frac{1}{3}}$$

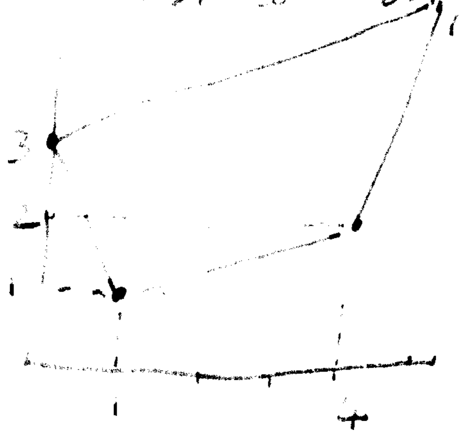
$$= 6 \cdot 2^{2/3}3^{1/3} \doteq 13.7 \text{ minimum cost}$$

no use Hessian to verify this is a minimum.

38) $f = -x + 3y + 6$ is a plane

$\Rightarrow$ has no critical pts ($\frac{\partial f}{\partial x} = -1 = 0$ has no solution!)

Must look at boundary



Along edges $y = \alpha x$ and f is linear in x only so can only have extrema at endpoints

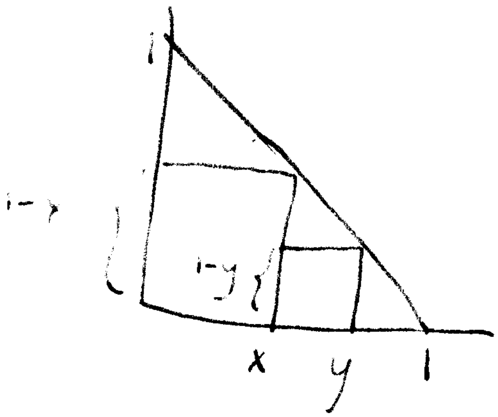
$\Rightarrow$ check only (1,1), (4,2), (0,3) & (5,6)

$\Rightarrow$ the maximum is at (5,6) where $f = 19$.

42) a) Think about various lines through the origin!

c) $f = -(x+y)^2$

46)



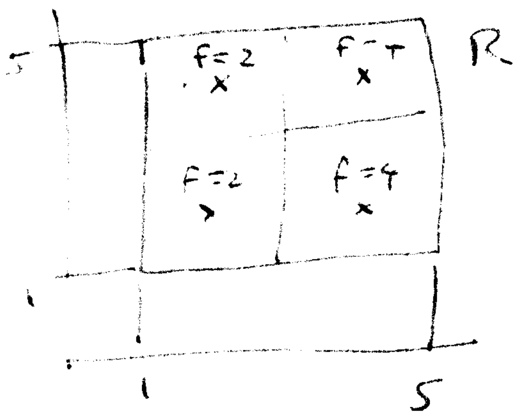
$$A = x(1-x) + (y-x)(1-y)$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial A}{\partial x} &= 1 - 2x - (1-y) = y - 2x \\ \frac{\partial A}{\partial y} &= (1-y) - (y-x) = 1 + x - 2y \end{aligned}\right\} \Rightarrow (x,y) = \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

$\Rightarrow A = \frac{1}{3}$ can check this is a maximum via the hessian.

Q5) § 15.1

9+) a)



$\int_R f(P)\,dA \approx 4(2+2+4+4) = 48$

$\uparrow$ area of each square

b) $f_{MIN} = 1$ on R $\Rightarrow$ $\int_R f_{MIN}\,dA = 1.16 = 16$ (area of R)

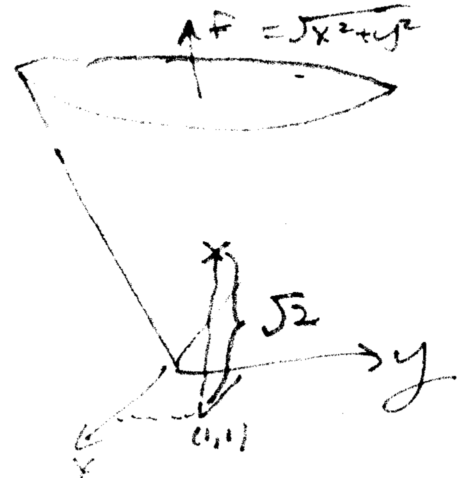
$f_{MAX} = 5$ on R $\Rightarrow$ $\int_R f_{MAX}\,dA = 5.16 = 80$

9) $\sum_{\text{each } i} A_i f(P_i) \le M \sum_i A_i = MA$

similarly for minimum.

$\therefore mA \le \int_R dA\, f \le MA$

7 b) a)



b) $V = \int_R z\,dA \le \int_R \sqrt{2}\,dA = \sqrt{2}$

9.12) $$f_{av} = \frac{f_{av_1} A_1 + f_{av_2} A_2}{A_1 + A_2}$$

9.18) b) disc

c) yes

d) arbitrarily small

Q6) use

$$|x-1| < \delta$$

$$|y-1| < \delta$$

$$\Rightarrow 1-\delta < x < 1+\delta$$

$$1-\delta < y < 1+\delta \underset{x-1}{\Rightarrow} -1-\delta < -y < -1+\delta$$

adding $\Rightarrow$

$$0-2\delta < x-y < 0+2\delta$$

choose $\delta = \varepsilon/2$ $\Rightarrow$