A1.

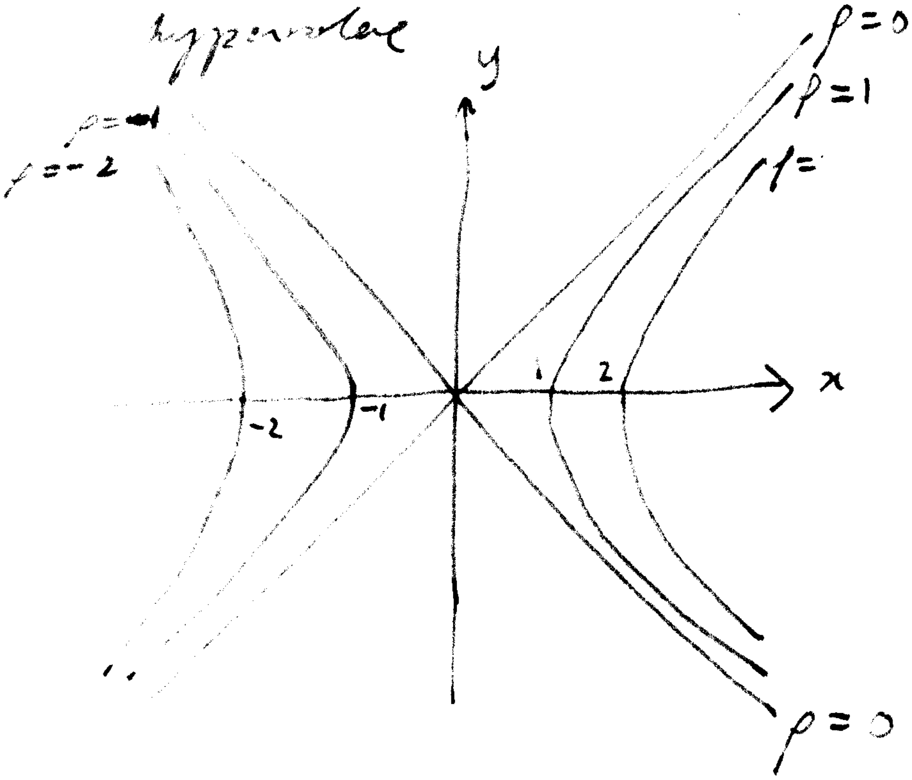
Q1: See lectures

Q2 Recall

$$\cosh^2\eta - \sinh^2\eta = 1$$

$\Rightarrow$ $x^2 - y^2 = \rho^2$ $\Rightarrow$ lines of equal ρ are

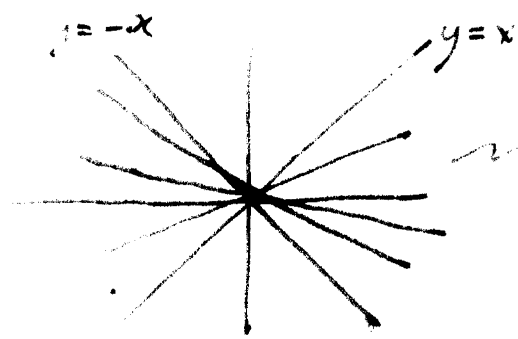
hyperbolae



Also, when η is held fixed, from

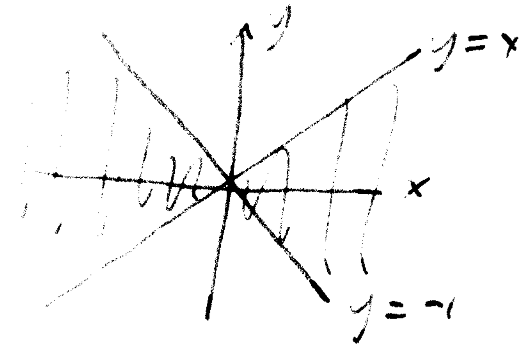
$\frac{y}{x} = \tanh\eta$ we learn that

these are straight lines through 0



note $-1 < \tanh\eta < 1$
so only get these lines.

§2 cr. Coordinates only cover

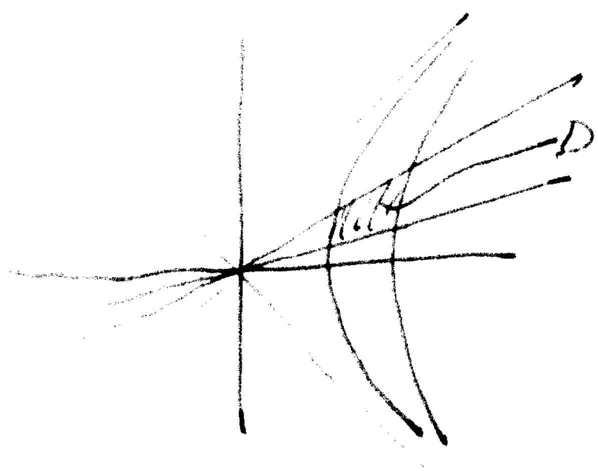


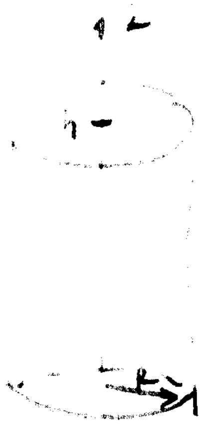
Jac $J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \cosh\eta & \rho\sinh\eta \\ \sinh\eta & \rho\cosh\eta \end{pmatrix}$

$\Rightarrow$ det $J_1 = |\rho\cosh^2\eta - \rho\sinh^2\eta| = |\rho|$

Hence $dA = |\rho|\, d\rho\, d\eta$

A typical natural domain looks like





$$V = \int_0^h dz \int_0^R \rho \, d\rho \underbrace{\int_0^{2\pi} d\theta}_{2\pi}$$

$$= 2\pi h \left[\tfrac{1}{2}\rho^2\right]_0^R$$

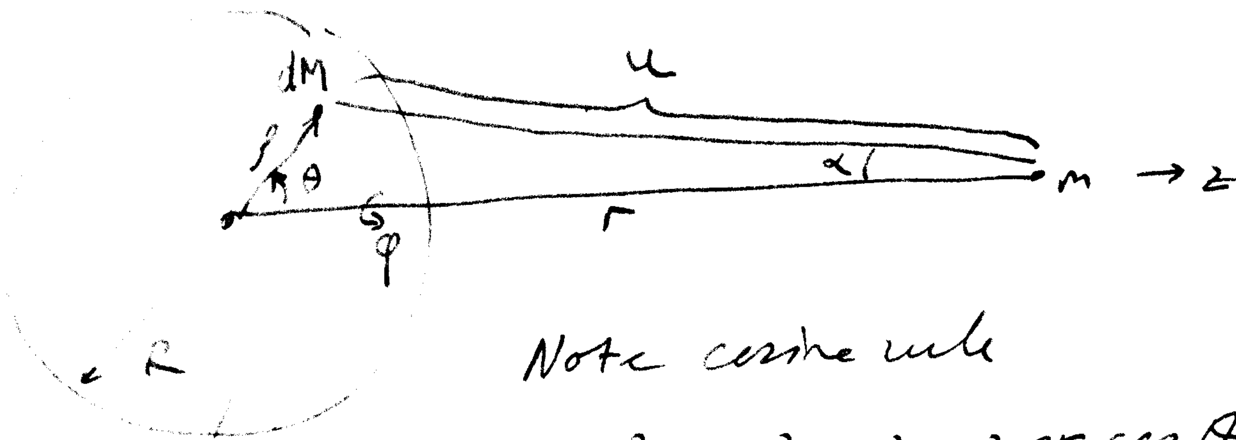
$$= \pi R^2 h \quad = \text{base} \times \text{height}$$

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

$$= \left[\tfrac{1}{3} r^3\right]_0^R \left[-\cos\theta\right]_0^\pi \left[\varphi\right]_0^{2\pi}$$

$$= \frac{4\pi R^3}{3}$$

Q5



Note cosine rule

$$u^2 = \rho^2 + r^2 - 2\rho r\cos\theta \quad \text{(A)}$$

mass density $\dfrac{M}{4\pi R^3/3}$

Now infinitesimal force from dM
IN THE z-direction

$$dF_z = \frac{G\left(\frac{M}{4\pi R^3/3}\, dV\right) m}{u^2}\cos\alpha$$

dM

Note $\cos\alpha = \dfrac{\rho^2 - u^2 - r^2}{2ur}$ by cosine rule.

Also $dV = \rho^2 d\rho\, d(\cos\theta)\, d\varphi$

But u is a much better variable

than $\cos\theta$.

So using (*) we have

$$2u\,du = -2\rho r\, d(\cos\theta)$$

$$\Rightarrow \quad dV = \rho^2 d\rho \left(\frac{u\,du}{\rho r}\right) d\varphi$$

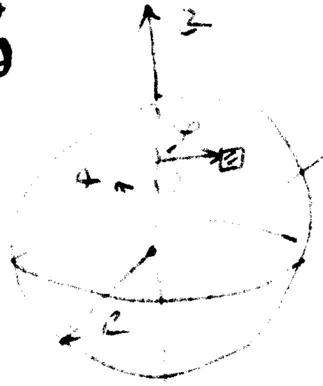
Hence

$$F = \int_0^R \rho\, d\rho \int_{\underbrace{r-\rho}_{u_{min}}}^{\overbrace{r+\rho}^{u_{max}}} \frac{u\,du}{r} \int_0^{2\pi} d\varphi \, \frac{G \frac{M}{4\pi R^3/3} m}{u^2} \, \frac{\rho^2 - u^2 - r^2}{2ur}$$

This iterated integral gives precisely

$$F = \frac{GmM}{r^2}.$$

6b



mass density is constant $= \dfrac{M}{4\pi R^3/3}$

$$\Rightarrow \quad I_z = \frac{M}{4\pi R^3/3} \int_{-R}^{R} dz \int_0^{\sqrt{R^2-z^2}} (\rho\, d\rho) \times \rho^2 \int_0^{2\pi} d\theta$$

2nd moment

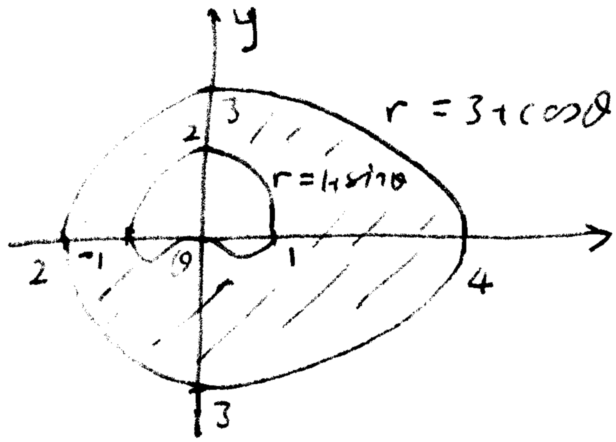
$$= \frac{3M}{4\pi R^3} \int_{-R}^{R} dz \left[\frac{1}{4}\rho^4\right]_0^{\sqrt{R^2-z^2}} \cdot 2\pi$$

$$= \frac{3M}{8R^3} \int_{-R}^{R} dz \left(R^2 - z^2\right)^2$$

$$= \frac{3M}{8R^3} \cdot \frac{16R^5}{15} = \frac{2}{5} MR^2 \checkmark$$

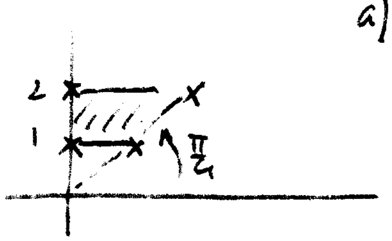
G7 9.5.4

2

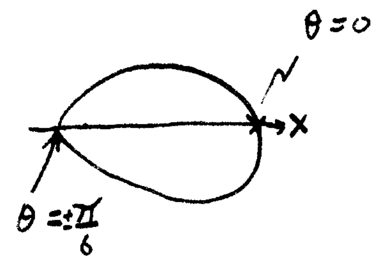


10

a) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \quad \frac{1}{\sin\theta} \le r \le \frac{2}{\sin\theta}$



20 $\bar{y} = 0$ by symmetry



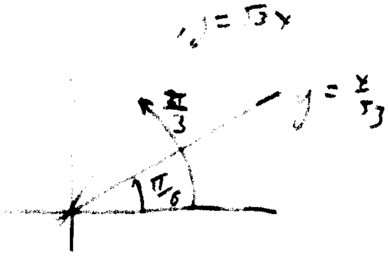
$$\bar{x} = \frac{\int dA\, x}{\int dA} = \frac{\int_{-\pi/6}^{\pi/6} d\theta \int_0^{\cos 3\theta} r\,dr\, r\cos\theta}{\int_{-\pi/6}^{\pi/6} d\theta \int_0^{\cos 3\theta} r\,dr}$$

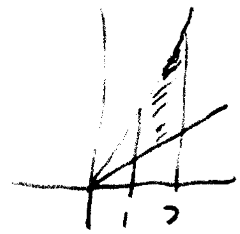
$$= \frac{\int_{-\pi/6}^{\pi/6} d\theta\, \frac{1}{3}\cos^4\theta}{\int_{-\pi/6}^{\pi/6} d\theta\, \frac{1}{2}\cos^2\theta} = \frac{81\sqrt{3}}{80\pi}$$

q 28

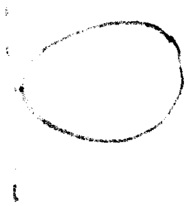
$$\int_1^2 dx \int_{\frac{x}{\sqrt{3}}}^{\sqrt{3}x} dy\,(x^2+y^2)^{3/2} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta \int_{\frac{1}{\cos\theta}}^{\frac{2}{\cos\theta}} r\,dr\,r^{3/2}$$

continue





q 34



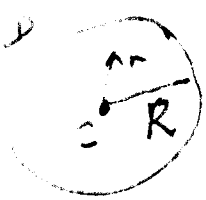
$$(x-a)^2+y^2=a^2$$

$$\Rightarrow r^2-2ar\cos\theta+a^2=a^2$$

$$\Rightarrow r = 2a\cos\theta$$

$$I = \underbrace{\frac{M}{\pi a^2}}_{\text{constant mass density}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2a\cos\theta} r\,dr\,r^2 = \frac{3Ma^2}{2}$$

q 44



$\pi R^2 = A$ area

$$\text{average } \langle r\rangle = \frac{\int_D dA\,r}{\int_D dA}$$

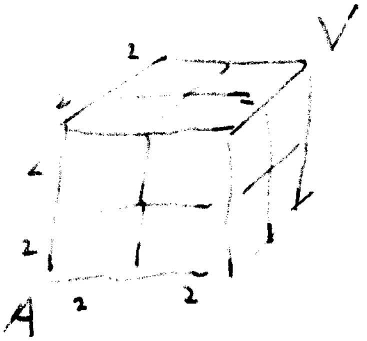
$$= \frac{\int_0^{\sqrt{A/\pi}} r^2 dr \int_0^{2\pi} d\theta}{A}$$

$$= \frac{1}{3}\left(\frac{A}{\pi}\right)^{3/2} \cdot \frac{2\pi}{A}$$

$$= \frac{2}{3}\sqrt{A\pi}$$

xx § 15.5

2

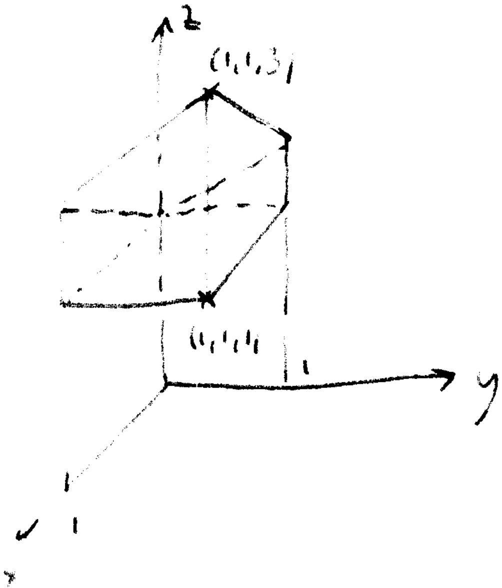


each cube has volume $8\,cm^3$

$M = \int dm = 8$ (add up squares of distances to centers of cubes)

mass density

2b $0 \le x \le 1$, $0 \le y \le 1$, $1 \le z \le 1+x+y$



2c $\displaystyle\int_0^1 dx \int_{x^3}^{x^2} dy \int_0^{x+y} dz\, z = \int_0^1 dx \int_{x^3}^{x^2} dy \left[\tfrac{1}{2} z^2\right]_0^{(x+y)}$

$\displaystyle = \frac{1}{2}\int_0^1 dx \int_{x^3}^{x^2} (x+y)^2 dy = \frac{1}{2}\int_0^1 dx \left[\frac{1}{3}(x+y)^3\right]_{x^3}^{x^2}$

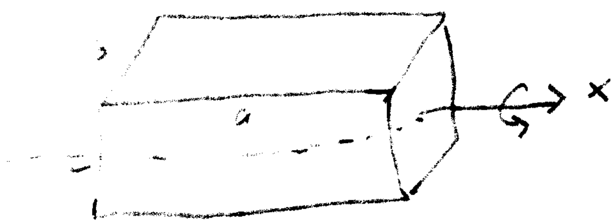
$\displaystyle = \frac{1}{2}\int_0^1 dx \left[\frac{1}{3}(x+x^2)^3 - \frac{1}{3}(x+x^3)^2\right] = \frac{5}{112}$

Q 2c) a) $0 \le x \le 1$

$0 \le y \le 1$

$1-y \le z \le y+1$

Q 3a)



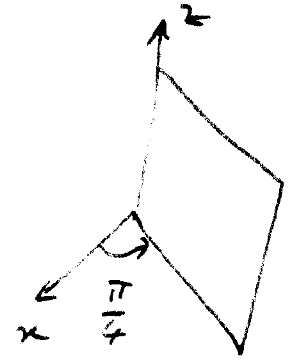
$$I = \frac{M}{abc} \int_0^a dx \int_{-\frac{b}{2}}^{\frac{b}{2}} dy \int_{-\frac{c}{2}}^{\frac{c}{2}} dz \, (y^2+z^2) = \frac{M}{12}(b^2+c^2)$$

↑ constant mass density

Q 3b) 0 by symmetry

Q

p 2 $\quad \theta = \frac{\pi}{4}$ is a plane

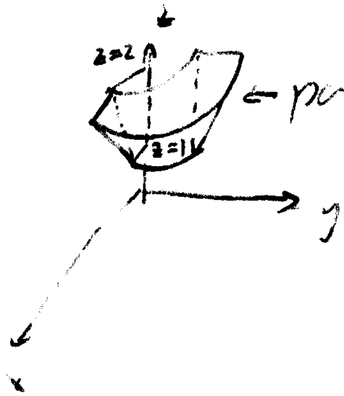


p 12 $\quad 0 \le \theta < 2\pi$

$0 \le r \le 3$

$r \le z \le 3$

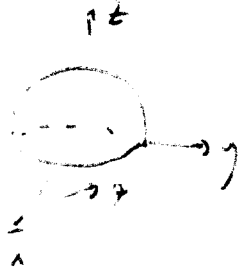
p 16



p 22 $\quad M = \frac{\pi}{8}$ so $\bar{z} = \frac{8}{\pi} \int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} dr \int_r^{\sqrt{1-r^2}} dz\, z^2 r = \frac{16}{15}\left(1 - \frac{1}{2\sqrt{2}}\right)$

p 32 $\quad I = \frac{3Ma^2}{5}$

p 38



$\bar{x} = \bar{y} = 0$, compute $\int_V z\, dV$

$= \int_0^{2\pi} d\theta \int_0^a dr \int_0^{\sqrt{a^2-r^2}} dz\, rz$

$= \frac{3a}{8}$

q 1R 3, 5, 7

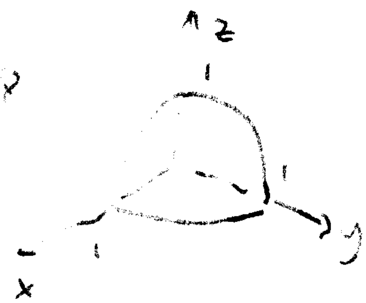
q 2 see lectures

q 15 $0 \le \theta \le \frac{\pi}{2}$

$0 \le \varphi \le 2\pi$

$0 \le r \le R$

q 18



q 32

$$\langle z^2 \rangle = \left(\frac{1}{4\pi a^3/3}\right) \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^a dr \; r^2 \sin\theta \;\; r^2 \cos^2\theta$$

$$= \frac{a^2}{5}.$$

q 38

mass density $\dfrac{M}{\frac{1}{3}\pi a^2 h}$

easiest in cylindrical coords!